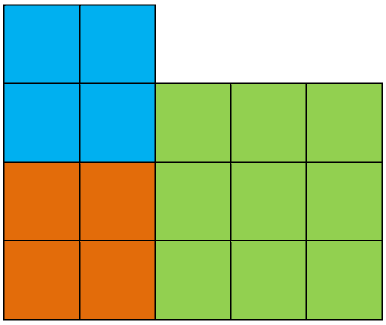
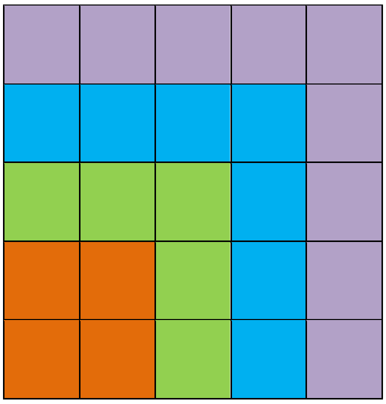
BundleBundle Math
on a BundleBundle Board

Math with the Child's own Numbers
*A paradigm-shift from **MatheMatism** to **ManyMath***

Existence before **Essence** means **Counting** before **Adding**

4 **2s**, 2**BB** **2s**, 1**BBB** **2s** 2 **2s**, 1**BB** **2s** 1 **2s**, 1**B** **2s**	3 **3s**, 1**BB** **3s** 1 **3s**, 1**B** **3s**	1**BB** **5s** = 1**BB**2**B**1 **4s** 1**BB** **4s** = 1**BB**2**B**1 **3s** 1**BB** **3s** = 1**BB**2**B**1 **2s**

4 **2s** plus 3 **3s** add next-to as 3**B**2 **5s** or 3 2/5 **5s** or 4**B**-3 **5s**
as an example of Integral Calculus adding areas

A 10x10 **B**undle-**B**undle **B**oard, a **BBB**oard with

- 6 **7s**
- 4 **tens**
- ten **3s**
- 4 **3s**

6*7 = (**B**-4)*(**B**-3)
= 10**B** – top4**B** – side3**B** + 4*3
= 3**B**12 = 4**B**2 = 42

Allan.Tarp@MATHeCADEMY.net, Denmark, September 2024, beta 02

Contents

Abstract

Looking at four fingers held together two by two, we see four fingers, the essence. But, before school, children see what exists, bundles of twos in space, and two of them when counted in time. So, we may ask how mathematics may be taught to children if using their own two-dimensional bundle-numbers with units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by working with existence instead of listening to essence.

Here we use the two core concepts of philosophical Existentialism holding that existence precedes essence. This will mean that counting precedes adding since outside totals must first be counted to be added later inside. In this 'Many-math' approach, mathematical concepts are re-rooted in outside existing examples instead of being defined as examples itself inside. Now tens, hundreds and thousands become bundles, bundle-bundles, and bundle-bundle-bundles, as does 2, 4 and 8 when counting in twos instead.

Here one-dimensional lines on a ruler are replaced by two-dimensional rectangles on a ten-by-ten Bundle-Bundle Board, a BBBoard, containing the outside existing subjects that is linked to inside essence predicates in a number-language sentence as in a word-language sentence. Here units are always included in counting sequences as 0Bundle1, 0B2, …, 1B0. Here digits become icons with as many sticks as they represent. Here also operations become icons created in the counting process. Division is a broom to push-away bundles to be stacked by a multiplication lift before pulled-away by a subtraction rope to find the unbundled that are included on-top as decimals, fractions, or negative numbers. Here recounting in another unit creates the recount formula, T = (T/B) x B, saying that T contains T/B Bundles.

Here recounting tens in icons creates equations solved when recounting moves a number to the 'opposite side with opposite sign'. Here recounting icons in tens leads to early algebra when 6*7 becomes (B-4) * (B-3) placed on a BBBoard and found by pulling-away the top 4B and side 3B, and adding the 4*3 pulled away twice. Here bundle-bundles allow rectangular stacks to be recounted in squares with the square root as the side.

Here recounting in a different physical unit creates per-numbers as 4$/5kg bridging the units by recounting, and with like units becoming fractions. Here mutual recounting the sides in a stack halved by its diagonal leads to trigonometry before geometry.

And now, after being counted and recounted, stacks may finally add on-top after recounted has made the units like, or next-to as areas, i.e., as integral calculus becoming differential calculus when reversed, also used to add per-numbers and fractions that must be multiplied to unit-numbers to add. Squares add as the square formed by their mutual Bottom-Top line.

All in all, there are four ways to unite the world's four number-types. Addition and multiplication add unlike and like unit-numbers, where integration and power add unlike and like per-numbers. Together with their opposite operations, subtraction, division, differentiation, and the factor-finding root or the factor-counting logarithm they form an 'Algebra square' that is named after the Arabic word 'Algebra', meaning to reunite.

And that now is the number-language allowing us to tell inside number-tales about outside totals using the same three genres, fact, and fiction and fake, as does the word-language.

Can mathematics be decolonized? Well of course, since mathematics is a socially constructed essence that will always be a colonization of the natural existence it came from and reduces.

Allan.Tarp@gmail.com, Aarhus, Denmark, April 2024

Introduction

This is a tale about finding, reflecting, and spreading a new paradigm in mathematics education where existence precedes essence, which allows mastery of Many to precede mastery of Math. This leads to discovering and accepting the two-dimensional bundle-numbers with units that children develop when adapting to the physical fact Many before school.

Precedence to existence will also end the Math Word War fighting over if the words should essence or existence by allowing 23 to be worded ‘2 Bundle 3’, and 234 to be worded ‘2BB 3B 4’, and differentiable and continuous to be worded ‘locally linear’ and ‘locally constant’, etc. etc. etc.

As to this new paradigm, the first section is about finding it, the next is about reflecting on its contents and consequences, and the third is about spreading the news about it in different places.

● SECTION I, FINDING a new Paradigm, BundleBundle Math

In ‘Grade one Class one in a Decolonized Future’ a teacher wants to teach the class that one plus one is two always by pointing to a number line. However, the teacher is stopped by a child showing a V-sign and saying: “Mister teacher, here is one 1s in space, and here is also one 1s. If we count them in time, we can see how many 1s we have by saying ‘one, two’. So, we have two 1s. But only until we add them as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. The thumb is also one 1s. They cannot be counted since they are not the same. But they can be added to one 3s. So, again one plus one is one. Here is another 3s on the other hand. They are the same, so we can count them as two 3s. And we can add them as one 6s. Or, we can split the two 3s into six 1s and see that two times three is six. So, the counting numbers two and three can be multiplied, but they cannot be added. Therefore, please forget adding your line-numbers without units. Instead, help us adding the bundle-numbers with units we bring to school, as 2 3s and 4 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on a peg board.” Then the child guides the teacher through the new paradigm step by step.

‘Valid Always or Sometimes? Mathema-tics or -tism?’ introduces the names ‘mathema-tism’ and ‘Manymath’. Mathematics adds numbers without units so it is true inside but seldom outside the school, whereas mathematics that add numbers with units may be called ‘Many-math’,

‘From Many to Bundle-numbers with Units, for Teachers’ let the teacher reflect on what the child told him in chapter one. First he validates the child’s claim that children use bundle-number with units before school by asking a3year old child “How many years next time?” and by observing that the child says “That is not four, that is two twos” when seeing four fingers held together two by two.

‘Micro Curricula, for Learners’ designs 23 micro curricula that may be tested as action learning or action research using, e.g., design research. The chapter ends with an overview over the differences between Essence- math and Existence-math.

‘Many before Math may Decolonize Math’ is a video presentation of a folder and a workshop. The folder is called “Decolonize Mathematics with the child’s own Bundle-numbers with units, and the workshop is called “Flexible Bundle Numbers Develop the Childs Innate Mastery of Many”, both available on YouTube. They build upon four core papers.

‘Math Dislike Cured with BundleBundle Math’ is a booklet with 20 chapters with hands-on exercises that allows schools and parents to choose an education that accepts and develops the 2D number blocks that the children bring to school instead of forcing a 1D number line upon them. Also, the booklet allows the children to practice ‘counting before adding’ and to include bundle-counting and re-counting to different units. The booklet thus is an answer to the question ‘How to Save and Develop a Child’s Math Potential?’

‘Bundle-counting and Next-to Addition Roots Linearity and Integration’ is a short introduction to the research project found in the following chapter.

‘Research Project in Bundle-counting and Next-to Addition’ raises the research question “What kind of mathematical learning takes places when children count in bundles less than ten?” It contains eight micro-curricula: Creating Icon-numbers, Counting in Bundles, Re-counting Bundle-numbers in the Same Bundle, Re-counting Bundle-numbers in a Different Bundle, Adding Bundle-numbers OnTop, Adding Bundle-numbers NextTo, Reversing Adding Bundle-numbers OnTop, Reversing Adding Bundle-numbers NextTo.

‘CATS: Learning Mathematics through Counting & Adding Many in Time & Space’ is a web-based teacher education that contains 2*4 study units in ‘mathematics from below, the LAB-approach’, organised as lab-activities where the learner learns ‘CATS’, i.e. learns to Count and Add in Time and Space. The study units CATS1 are for primary school, and the study units CATS2 are for secondary school. The units were developed for a web-based teacher-training course in mathematics at a Danish teacher college. The online education uses PYRAMID**e**DUCATION where 8 teachers are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. The coach teaches the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In each pair each student corrects the other student’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers. The MATHeCADEMY.net thus teaches teachers teach mathematics as ‘many-math’, a natural science about Many. It is a virus academy saying: To learn mathematics, don’t ask the instructor, ask Many. The material is question-based. Details on www.mathecademy.net, and on the video introducing the MATHeCADEMY dot net, https://youtu.be/CbRDG64onKA.

10. The ‘KomMod Report’, a Counter-report to the Ministry’s Competence Report is an opponent view on the official response to a dozen question raised by the Danish Ministry of Education, thus showing that only two competences is needed to master Many, to count and to add, in opposition to the report eight competences.

11. Word Problems shows how traditional word problems are solved with bundle-bundle math.

● SECTION II, REFLECTING on the New Paradigm

12. ‘A short History of Mathematics’ shows how trade between the far East and Europe has been instrumental in developing the mastery of Many that the Greeks called Mathematics.

13. ‘What is Math - and why Learn it?’ is a letter from an uncle to a nephew trying to give him an understanding of what mathematics is good for.

14. ‘Fifty Years of Research without Improving Mathematics Education, why?’ At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is “Only one: to improve, mathematics education should ask, not what to do, but what to do differently.” Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. The essay contains the following chapters: The Outside Roots of Mathematics, Rethinking Mathematics from Below, How School Teaches Mathematics, How School Could Teach Mathematics, Conclusion.

15. ‘Postmodern Enlightenment, Schools, and Learning’ is an essay discussing how postmodern thinking may be useful when rethinking mathematics education.

16. ‘Can Postmodern Thinking Contribute to Mathematics Education Research’ is addressing the same question but now giving the answer in the form of an auto interview.

● SECTION III, SPREADING the New Paradigm

17. ‘The ICME Conferences 1976, 1996-2024’ contains a list of papers presented at the conferences called ‘International Congress on Mathematical Education’, first at the ICME 3 in Karlsruhe, then at the ICME 9 2000 in Tokyo, then at the ICME 10 & 11 & 12, in 2004 & 2008 & 2012 in Copenhagen & Monterrey & Seoul, then at the ICME 13 in Hamburg, then at the ICME 14 in Shanghai, an finally at the ICME 15 in Sydney.

18. ‘The Swedish MADIF papers 2000-2020’ contains a list of papers presented at the conferences called ‘International Congress on Mathematical Education’. The first three paper were accepted, the rest were rejected. The papers are called, Killer-equations, job threats and syntax errors, 2000. Student-mathematics versus teacher-metamatics, 2002. Mathematism and the irrelevance of the research industry, 2004. The 12 math-blunders of killer-mathematics, 2006. Mathematics: grounded enlightenment - or pastoral salvation, 2008. Discourse protection in mathematics education, 2010. Post-constructivism, 2012. Golden learning opportunities in preschool, 2014. Calculators and icon-counting and cup-writing in preschool and in special needs education, 2016. Grounding conflicting theories, 2016. The simplicity of mathematics designing a stem-based core mathematics curriculum for young male migrants, 2018. Math competenc(i)es - catholic or protestant? 2018. Sustainable adaption to quantity: from number sense to many sense, 2020. Per-numbers connect fractions and proportionality and calculus and equations, 2020. Sustainable adaption to double-quantity: from pre-calculus to per-number calculations, 2020. A Lyotardian dissension to the early childhood consensus on numbers and operations, 2020. Salon des refusés, a way to assure quality in the peer review caused replication crisis? 2020.

19. ‘The Swedish Mathematics Biennale’ is arranged the days after the MADIF conference. Here are my proposals for the Mathematics Biennale 2016, and the proposals for the Mathematics Biennale 2018 and 2020, and the proposals for the Mathematics Biennale 2024. In the period from 2000 to 2024 I had two talks accepted, but except for 2020 I was always allowed to arrange a table with ideas.

20. ‘The MES Conferences’ focus on Mathematics Education and Sociology. Except for the first, my papers ware always rejected. They were MES 1 1998, What if Mathematics is a Social Construction. MES 2 2001, Searching for Hidden Contingency. MES 4 2007, Modern or Postmodern Critical Research. MES 6 2011, Anti-bildung Enlightenment Education in Berlin. MES 8 2015, Invitation to a Duel on how to Improve Math Education. MES 9 2017, Paper, project, symposium & poster.

21. ‘CERME Conferences’ focus on European Research in Mathematics Education (CERME13) with many different topic study groups. My proposals were all rejected. CERME 10 Irland: Cup-counting and Calculus in Preschool and in Special Needs Education. CERME 11 Holland: Addition-free Migrant-Math Rooted in STEM Re-Counting Formulas. CERME 12 Italy: Bundle-Numbers Bring Back Brains from Special Education. CERME 13 Hungary: To Master or not to Master Math Before Many, that is the Question.

22. ‘The Catania Trilogy 2015’ was written for The Mathematics Education for the Future Project held on Sicily in a friendly atmosphere where they were all accepted. Decreased PISA Performance in spite of Increased Research. Count in Icons before Tens, then Add NextTo before OnTop. Truth, Beauty, and Goodness in Mathematics Education. PerNumbers replace Proportionality, Fractions & Calculus

23. ‘CTRAS Conferences’ focuses on Classroom Teaching Research for All Students. Hre my papers were warmly received and I was asked to give a keynote presentation in 2017. CTRAS 2016 in Germany, CTRAS 2017 in China, CTRAS 2018 in China, CTRAS 2019 in China, CTRAS 2020 in China, CTRAS 2021 online, CTRAS 2022 online, CTRAS 2023 online, CTRAS 2024 online.

24. ‘The 8th ICMI-East Asia Conference on Mathematics Education 2018’, here I was an invited lecturer.

25. ‘ICMI Study 24, School Mathematics Curriculum Reforms 2018’ accepted my paper and I was asked to be one of two editors on a chapter, that was canceled a month before deadline. Still, I finished it and published it, (http://mathecademy.net/appendix-to-curriculum-study-icmi-24/)

26. ‘NORMA 24’ is a Nordic Conference on Mathematics Education where the focus was ‘Interplay between research and teaching practice in mathematics education’. Here all proposal were rejected even the poster. From a colonized to a decolonized mathematics, from 8 to 2 competences, from non-unit to unit-numbers. Respecting the child’s innate number sense, is that Woke-math? To master or not to master math before Many, that is the question. A proposal for a symposium celebrating three Danish math educators about to turn 80, Niss and Skovsmose and Tarp.

27. ‘Curriculum Proposal at a South African teacher college’ is a curriculum proposal written after a two month stay that was arranged after the MES 1 conference.

28. ‘Celebrating the Luther year 1517 with some Theses on Mathematics and Education’. Two essays written for Danish newspapers but not printed. The were called, Mathematics, Banality or Evilness, and Does Europe really need Compulsory School Classes?

29. ‘Invitation to a Dialogue on Mathematics Education and its Research’ was an invitation to persons that was rejecting my papers. Noone accepted. The issues were called 01. Mathematics Itself. 02. Education in General. 03. Mathematics Education. 04. The Learner. 05. The Teacher. 6. The Political System. 07. Research. 08. Conflicting Theories. 09. Me and Mathematics Education and Research. 10. How to Improve Mathematics Education.

30. ‘MrAlTarp YouTube videos’ is the list of my YouTube videos. They are called Online math opens for a communicative turn in number language education. AI and Difference Research in Math Education. Continuous means locally constant. From STEAM to STEEM part II. Flexible Bundle Numbers Develop the Childs Innate Mastery of Many. Children's innate Mastery of Many developed by flexible bundle-numbers. To master Many Recount before Adding. Bring Back Brains from Special Education in Mathematics. From STEAM to STEEM. Trigonometry Before Geometry Probably Makes Every Other Boy an Excited Engineer. Introducing the MATHeCADEMY.net. Mathematics language or grammar. The two infection formulas, part 1. The two infection formulas, part 2. CupCount and ReCount before you Add. Preschoolers learn Linearity & Integration by Icon-Counting & NextTo-Addition. Deconstructing Calculus. Deconstructing PreSchool Mathematics. Deconstructing PreCalculus Mathematics. Deconstructing Fractions. A Postmodern Deconstruction of World History. 8 Missing Links of Mandarin Math I. 8 Missing Links of Mandarin Math II. A Postmodern Mathematics Education.

Finally, a two-level table of contents is included.

● SECTION I, FINDING a new Paradigm, BundleBundle Math

01. Grade one Class one in a Decolonized Future

The teacher: Welcome children, I am your teacher in math, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So …

Showing a V-sign a child says: “Mister teacher, here is one 1s in space, and here is also one 1s. Now we count them in time to see how many 1s we have by saying ‘one, two’. So, we have two 1s. But only until we add them as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. And together with this neighbor V-sign the total is one 2s plus two 1s which is one 4s, and not three 3s.

And, if I two times show you three 1s I have shown you six 1s. So, the counting numbers two and three can be multiplied, but they cannot be added.

Therefore, please forget adding your line-numbers without units. Instead, help us adding the bundle-numbers with units we bring to school, as 2 3s and 4 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on a peg board. If we add them next-to, we add plates, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says it is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school, instead of being colonized with your line-numbers without units.

We know that you want to bundle in tens, and in ten-tens, and in ten-ten tens, but we like to bundle also in 2s, in 3s, in 4s, in half-tens, etc. We know that you have not been taught this and that the textbook doesn’t teach it. But don’t worry, we will teach you what we found out in preschool. Or better, instead of you colonizing our ways let us find out together what math may grow from our bundle-numbers with units. My uncle is a philosopher, and he calls it existentialism if we let existence come before essence.

And, when existence comes before essence, we must count the totals before we can add them. We know you say that 8 divided by 2 is 8 split in 2 parts, but to us 8 divided by 2 is 8 counted in 2s. You cannot split 9 in 2 parts, but we can easily count 9 in 2s as 4 bundles and 1 unbundled that becomes a decimal, 9 = 4B1 2s, or a fraction if we count it in 2s also, 9 = 4 ½ 2s. Or, with negative less-numbers we get 5 bundles less 1, 9 = 5B-1 2s.

You only write 47 and say ‘4ten7’ when you should write 4B7, and say ‘4 bundles with ten per bundle and 7 unbundled’. And, you only see the essence, five, where we see all the ways the five fingers may exist.

Now, let us begin with the fingers on a hand. You only see the essence, five, but we see all the ways the five fingers may exist.

F01. A total of fingers many exist as five ones, T = 5 1s, or as one bundle of fives, T = 1B0 5s. Also, the fingers may be bundled in 4s as T = 0B5 = 1B1 4s or as two bundles less 3, T = 2B-3 4s. And the fingers may be bundled in 3s as T = 0B5 = 1B2 = 2B-1 3s. And the fingers may be bundled in 2s as T = 0B5 = 1B3 = 2B1 = 3B-1 2s. But 2 2s is also one bundle of bundles, 1 bundle-bundle, 1 BB, so we also have that T = 1BB0B1 2s. Putting two hands together we see, that eight is one bundle-bundle-bundle, 1 BBB, so that ten is 1BBB0BB1B0 2s. And, if we count ten fingers in 3s, T = 3B1 3s = 1BB0B1 3s. Likewise if we count in tens, twelve is 1B2, and forty-seven is 4B7, and 345 is 3BB4B5.

F02. Here we counted in space, but we also use bundle as the unit when we count in time. If we count our finger in 3s we cannot say ‘1, 2’, and so on since 1 is not 1 3s. Instead, it is 0 bundle 1 3s, so we count ‘0B1, 0B2, 0B3 or 1B0, 1B1, 1B2 or 2B-1’. Or we may count ‘1B-2, 1B-1, 1B0, 2B-2, 2B-1.’

F03. With sticks we see that 5 1s may be bundled as 1 5s that may be rearranged as one icon with 5 sticks. The other digits may also be seen as icons with the number of sticks they repents, where zero is a looking glass finding nothing. We don’t need an icon for ten since here the total is 1B0 if we count in tens.

F04. The calculations are icons also. If we reduce 8 by 2, subtraction is a ‘pull-away icon’ for a rope so that 8-2 means ‘from 8 pull-away 2’. Now a calculator can predict the result, 8 – 2 = 6. And this creates a split formula ‘8 = (8-2) + 2’ telling that 8 remains if the pulled-away is placed next to, or ‘T = (T-B)+B’ with T and B for the total and the bundle.

If we recount 8 in 2s, division is a ‘push-away icon’ for a broom so that 8/2 means ‘from 8 push-away 2s’. Now a calculator can predict the result, 8/2 = 4. If we stack the 4 2s, multiplication becomes a ‘lift icon’ predicting the result, 8 = 4x2. This creates a recount formula ‘8 = (8/2) x 2’ telling that 8 contains 8/2 of 2s, or ‘T = (T/B) x B’ or ‘T = (T/B)*B’ used to change units all over math, economy, and science.

Finally, addition is a ‘two-ways icon’ showing that two stacks as 2 3s and 4 5s may be added horizontally next-to as areas using integral calculus, or vertically on-top after recounting has made the units like.

F05. A reversed calculation is called an equation using the letter *u* for the original unknown number. The split and recount formulas may be used to solve equations.

The reverse calculation or equation ‘*u*+2 = 8’ asks ‘8 is split in 2 and what?’. The answer, *u*, is of course if found by the splitting 8 = (8-2) + 2. So, *u*+2 = 8 = (8-2) + 2 predicts that *u* = 8-2, which is also found by simply pulling-away 2 from 8, *u* = 8-2.

The reverse calculation or equation ‘*u**2 = 8’ asks ‘8 is how many 2s?’. The answer, *u*, is of course if found by the recounting 8 = (8/2)*2. So, *u**2 = 8 = (8/2)*2 predicts that *u* = 8/2, which is also found by simply recounting 8 in 2s, *u* = 8/2.

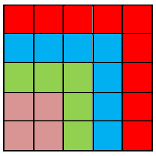
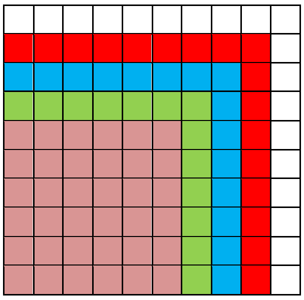
In both cases we see that we find the solution by moving to opposite side with opposite sign. My uncle says that this follows the official definitions. 8-2 is the number *u* that added to 2 gives 8, so if *u* = 8-2 then *u*+2 = 8. And, 8/2 is the number *u* that multiplied with to 2 gives 8, so if *u* = 8/2 then *u**2 = 8. And he warned us against a ‘same on both sides’ method you might want to teach us.

A combined equation as ‘3**u* + 2 = 14’ may be solved by a song:

3**u* + 2 = 14	Equations are the best we know; they’re solved by isolation.
(3**u*) + 2 = 14	But first the bracket must be placed, around multiplication.
3**u* = 14 – 2	We change the sign and take away, so only u itself will stay.
u = (14 – 2)/3	We just keep on moving, we never give up.
u = 4	So feed us equations, we don’t want to stop.

F06. On a ten-by-ten Bundle-Bundle-Board, a BBBoard we see that all the bundle-bundles are squares. And we see that 2 2s need 2 more 2s and 1 corner to become 3 3s, which need 2 more 3s and 1 corner to become 4 4s, which need 2 more 4s and 1 corner to become 5 5s, and so on. So, 1BB 3s = 4B1 2s, and 1BB 4s = 5B1 3s, and 1BB 5s = 6B1 4s, and so on.

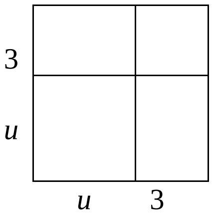
So, 1BB 3s = 1BB2B1 2s, and 1BB 4s = 1BB2B1 3s. And that 1BB 4s =1BB-2B1 5s where the 1 is added since it is pulled away twice. Thus, 1, 4, 9, 16, and 25 are the first squares.

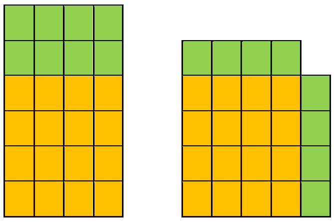
  

From ten, 1 BB 9s =1BB-2B1 or 8B1 tens. And 1 BB 8s = 1BB-4B4 or 6B4 tens. And 1 BB 7s = 1BB-6B9 or 4B9 tens. And 1 BB 6s = 1BB-8B16 tens or 2B16 tens or 3B6 tens. So, 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100 are the squares. We see that of course 1 and 9 have the same last digit, as has 2 and 8, and 3 and 7, and 4 and 6. With 4BB 5s = 10B tens, 1BB 5s = 2B5 tens = 25.

F07. We see 6 7s as 1BB-(4+3)B(4*3) = 3B12 = 4B2 = 42. Or as (B-4)*(B-3) that is left if we pull-away 3B from the side and 4B from the top and add the 4 3s that is pulled away twice, which gives 3B12 tens or 4B2 tens. The calculator gives the answer directly when entering ‘6*7’. Also, we see what my uncle calls the FOIL-method in algebra: (B – 4)*(B – 3) = B*B – 3*B – 4*B + 4*3.

F08. Splitting a BBBoard in two squares and two stacks solves square-equations. If the sides are split in an unknown number *u* plus 3 then the board contains the a *u*-square and a 3-square and two 3*u* stacks, so if *u* squared + 6*u* + 8 is zero, then the (*u*+3) square only contains 9-8 = 1 which is a 1-square, so *u* can be -2 or -4 since both (-2+3) and (-4+3) give 1 when squared.





F09. A rectangular stack as 6 4s may be changed into a square by moving half the surplus from the top to the side to make it close to the so-called square root:

T = 6*4 = (6-1)*(4+1) – 1 = 5*5 – 1.

We fill the 1 with a 4*t* stack from the side and *t*4 stack from the top. This gives 2*4*t* = 1, or 8*t* = 1 = (1/8)*8, so *t* = 1/8, so T = (5-1/8)*(5-1/8) = 4.88*4.88. Guessing 4.88 as the square root is close to what the calculator says, √(6*4) = 4.89….

F10. On the BBBoard we see that totals may be counted with many different units, 2s, 3s and tens. So, we sometimes need to recount the total in another unit and ask 2 3s is how many 4s, or 2 tens is how many 7s, or 6 7s is how many tens. We can do the recounting on the board, or we can predict it with calculations using the recount formula. The calculator says that 2*3/4 is ‘1.more’, and that 2*3 – 1*4 is ‘2’, so the prediction is that 2 3s is 2B2 4s. Likewise, the calculator says that 20/7 is ‘2.more’, and that 20 – 2*7 is 6, so the prediction is that 2 tens is 2B6 7s or 3B-1 7s.

F11. We see that in shops you can exchange apples with money. If the sign says 4$ per 5kg then we have the per-number 4$/5kgm or 4$ for each 5kg. So, if we want to have 20kg we simple recount them in 5s to see how many times we have 4$: 20kg = (20/5)*5kg = (20/5)*4$ = 16$. Likewise, 20$ may be recounted in 4s to see how many times we get 5kg.

If my share of a 5$ bet is 2$ then I will get 2$ per 5$ of a 30$ gain, that I then recount in 5s. Here 2$ per 5$ is a fraction 2/5. To find 2/5 of 30 I simply recount 30 in 5s to see how many times I get 2$: 30$ = (30/5)*5$ gives me (30/5)*2$ = 12$. The fraction 2/5 can occur with other units as (2 3s)/(5 3s) = (2*3)/(5*3) = 6/15. Likewise with the fraction 8/20 = (2*4)/(5*4) = (2 4s)/(5 4s) = 2/5.

F12. A stack has an internal line called a diagonal. To find out how steep it is we can recount the up- in the out-number: up = (up/out)*out. Here the per-number, up/out, is called the line's slope or gradient or tan(Angle)-number. So, with a 3/5 gradient a total of 20 meter out will give (20/5)*5out = (20/5)*3up = 12up meter. And, in a circle with radius 1, half the perimeter is p = *n**(tan(180/*n*)) for *n* large.

F13. Once counted and recounted, totals may be added next-to or on-top. Adding 2 3s and 4 5s next-to as 8s means adding areas, which my uncle calls integral calculus where we multiply before we add. On a BBBoard we see that the sum is 3B2 8s as predicted by recounting on a calculator. Adding 2 3s and 4 5s on-top as 5s, 2 3s must first be recounted to 1B1 5s, so here the sum is 5B1 5s, or 8B2 3s if we add them as 3s after recounting 4 5s to 6B2 3s.

We turn the question around if we ask '2 3s and how many 5s total 4 8s?'. On a BBBoard or with cubes we see that we must pull away the 2 3s before we recount the rest in 5s. So here subtraction comes before division, which my uncle calls differential calculus used to calculate changes as per-numbers. And where the n-square grows with 2*n almost as we saw above.

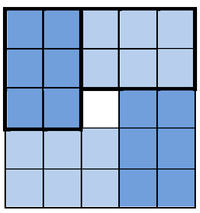
F14. In a shop we add both kg numbers and per-numbers in mixtures as 2kg at 3$/kg plus 4kg at 5$/kg. Here we can add the unit-numbers 2 and 4 to 6 kg, but before we add the per-numbers they must be multiplied to unit-numbers also, so they add as 2*3$ and 4*5$, which is the same as adding 2 3s and 4 5s by using integral calculus. My uncle says that we later will learn to use integral calculus to add piecewise or locally constant per-numbers as the area under their per-number curve, which is easy to find with differential calculus where the areas are written as differences so that all the middle terms cancel out when added.

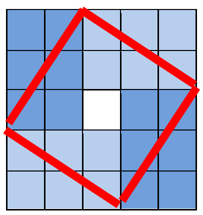
F15. Fractions also occur in mixtures as 2 apples with 1/2 red and 3 apples with 2/3 reds. Again, we add the unit numbers directly to 5 apples, whereas the per-numbers must be multiplied first to ½*2 and 2/3*3 that add to 3. So here the sum is 5 apples with 3/5 reds. My uncle say that you will try to teach us that ½ plus 2/3 is 7/6, but I think he jokes because there cannot be 7 red among 6 apples.

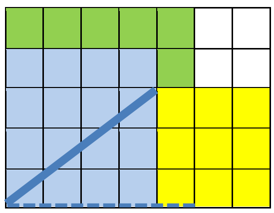
If we 3 times double 1, 1 becomes 1*2 = 2, and 2 becomes 2*2 = 4, and 4 becomes 4*2 = 2*2*2 = 8. We can now turn it around by 3 times taking the fraction ½ of 8. This will reduce 8 to 4, and 4 to 2, and 2 to 1, which is the same as taking 1/8 of 8. So here the fraction ½ is added 3 times by the same repeated multiplication as we use for bundle-bundles and bundle-bundle-bundles. My uncle uses the word 'power' for this calculation.

F16. To add squares, we take four copies of a 2 3s stack and place them as a 5-square. Inside this square we see two stacks and a 2-square and a 3-square. But we also see a square formed by four diagonals that is surrounded by four half stacks. So, in a 2 3s stack the two side-squares add as the diagonal square.

To see how much a 4-square grows when a 3-square is added, the bottom-top line shows the side of the new square.







F17. We see that the textbook wants to teach us that 6+8 is 14. But 6 and 8 cannot exist without units. If we count in 6s, then 6+8 = 1B0 + 1B2 = 2B2 6s or 1B4 tens. If we count in 8s, then 6+8 = 1B-2+ 1B0 = 2B-2 8s or 1B6 8s or 1B4 tens.

If we count in tens we can use half-bundles, ½B, so here 6+8 = 1 ½B1 + 1 ½B3 = 1B4 tens. On a BBBoard we place the 6 and 8 on-top instead of next-to see the sum, and to see that 8-6 = 2, and that 6-8 = -2.

F18. We see that the textbook wants to teach us to carry and to borrow. But we don't need that since bundle-numbers are flexible with both overloads and underloads.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

The two-digit numbers 23 and 47 can be multiplied, and 1081 can be divided by 23 on a BBBoard:

	4B	7	
3	~~12B~~	21	↓↓
2B	8BB	~~14B~~	
	8BB	26B	21
	10BB	8B	1

	? 5B	? -3	
3	~~15B~~	-9	↑↑
2B	10BB	~~-6B~~	
	10BB	9B	-9
	10BB	8B	1

So, 23*47 = 1081, and 1081/23 = 4B7 = 47

F19. Counting ten fingers in 3s we get T = 1BB0B1 3s = 1*B^2 + 0*B + 1. My uncle calls this an example of a general bundle-formula called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration.

With units we see there can be only four ways to unite numbers: addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers. If we go backwards, we split a total in unlike or like unit-numbers or per-numbers. Here subtraction and division split a total into unlike and like unit-numbers. Differentiation splits a total into unlike per-numbers. And finally, a total is split into like factors by the factor-finding root and the factor-counting logarithm. My uncle calls this 'the Algebra square' where the Arabic word Algebra means to reunite.

Calculations unite/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\,dx$ $dT/dx = f$	$T = a^b$ $b\sqrt{T} = a$ $\log a(T) = b$

F20. With the Algebra square we have a number-language we can use to predict numbers with calculation sentences that my uncle says countians the same as the word-language sentences, a subject, a verb and a predicate, the total is three fours, T = 3*4.

So now we can number-talk about things in space, geometry, and things in the past, statistics, and things in the future, probability. And we can number-talk about money and meter and motion and discuss if what we say is fact or fiction or fake.

My uncle says that if we are allowed to keep and develop our natural bundle-numbers with units then we will all be numerate as asked for by the UN Sustainable Development Goal 4 about quality education. So, allowed to keep and develop our natural intelligence we don't need any AI.

F21. Finally, to protect our own number-language and innate number-sense, we now want to slightly reformulate the American Declaration of Independence:

"We, the children, declare unanimously that when it becomes necessary for us to dissolve the educational bands which have connected us with you, and to assume among the separate and equal station to which our human nature entitle us, a decent respect requires that we above have declared the causes which impel us to the separation of an education that prohibits this."

F22. And, mister teacher, if you want to learn more about how to teach mathematics as Many-math with bundle-bundle numbers you can do as my uncle did, you can see MrAlTarp's many YouTube videos. And you can also take a small group online PYRAMIDe DUCATION at the MATHeCADEMY.net to learn the CATS method, Count&Add in Time&Space.

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02. Valid Always or Sometimes? Mathema-tics or -tism?

The school teaches that ‘2+3 = 5’ and that ‘2x3 = 6’. But are they both valid outside? 2 bundles of 3s may always be recounted as 6 1s, but 2weeks + 3days is 17days. So, even if both hold inside the school, outside ‘multiplication holds, but addition folds.’

Mathematics that adds numbers without units may be called ‘mathematism’, true inside but seldom outside the school, whereas mathematics that add numbers with units may be called ‘Many-math’, using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units are called ‘calculus’ and ‘proportionality’, the core of mathematics. Where normally they come very late, here they occur in the first lesson.

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that 8 = (8/2)x2, or T = (T/B)xB with T and B for Total and Bundle. This linear ‘proportionality’ recounting-formula to shift units now occurs in the first lesson when we bundle-count with units.

Also, it solves equations where ‘*u* x 2 = 8’ is asking ‘How many 2s in 8?’, which is answered by ‘*u* = 8/2’ since 8 is recounted in 2s as above, thus simply moving ‘to opposite side with opposite sign’.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, 8 = 2B2 3s, or as a fraction when also counted in 3s as 2 = (2/3)x3, 8 = 2 2/3 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, 8 = 3B-1 3s. Here we may even see the 3 bundles of 3s as one bundle of bundles, one bundle-bundle, 1BB, so that 8 = 1BB 0B -1 3s, where the bundle-bundle is a square.

Which makes you wonder if any rectangular bundle-number may be recounted in squares with the square root as its side, and if squares can add as squares, e.g., as the square created by their mutual Bottom-Top line. In that case, calculus will be easy since areas then can add as squares.

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equation by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas that again may be added to one square.

So, with Many-math’s ‘counting before adding’ we have learned most mathematics almost before we begin.

This will please the fourth of the 17 UN Sustainable Development Goals that defines quality education as ‘ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.’ And where the subgoal 4.6 wants to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”.

To meet this goal, we therefore replace ‘mathematism’ with ‘Many-math’. To make the difference between the two more clear we may use the basic philosophical question: What comes first, existence or essence, what is in the world or what we think about it?

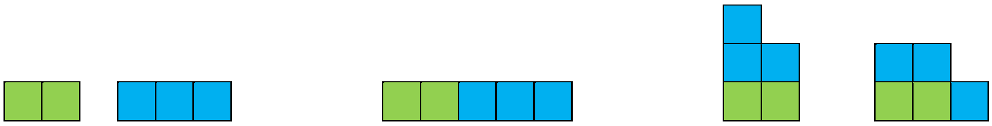
So, we ask: “What makes children learn mathematics, listening to essence, or working with existence?” Where ‘mathematism’ chooses the former, Many-math chooses the latter to develop a number-language by working with Many as it exists in time and space as repetition and multiplicity, and in the word-language as plurals.

So finally, the number-language may have its communicative turn to be learned by telling tales about things and actions in space and time, just as the word-language had around the 1970’s as described in H.G. Widdowson’s book ‘Teaching Language as Communication’.

03. From Many to Bundle-numbers with Units, for Teachers

“No, that is not four, that is two twos”. Said a 3year old child when asked “How many years next time?”; and when seeing four fingers held together two by two. This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time. The number ‘two’ thus exists both in space and in time.

In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Either horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle.



Here 1 plus 1 does not add up to 2 since the units are not the same, whereas one 2-bundle + one 2-bundle may both be two 2-bundles or one 4-bundle, but not 2 4-bundles.

In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit.

So, as space-numbers, 2+3 is 5, but as time-numbers 2+3 is undecided until their units are known.

On a hand, a collapsed V-sign shows that 1 1s + 1 1s = 1 2s, which together with a V-sign’s 2 1s total 1 2s + 2 1s = 1 4s, and not 3 3s as may be expected if 1+1 = 2 always. Adding with units we get 1*2 + 2*1 = 4. Here we multiply before adding, thus adding areas, also called integral calculus.

To get a more precise definition of space-numbers as 2s and time-numbers as 2 we observe the following when using the time-number sequence ‘1, 2, 3, 4, 5’ to count the fingers on a hand:

After in time saying ‘1’ when pulling away one finger we now in space have 1 1s. And, after in time saying ‘2’ when pulling away one more finger we now in space have 1 2s that originally was 2 1s. And, after in time saying ‘3’ when pulling away one more finger we now in space have 1 3s that originally was 3 1s. And, after in time saying ‘4’ when pulling away one more finger we now in space have 1 4s that originally was 4 1s. And, after in time saying ‘5’ when pulling away one more finger we now in space have 1 5s that originally was 5 1s. Now there is no more fingers to pull away so we may also call 1 5s for 1 bundle with 5 per bundle written as 1B0 5s with B for bundle.

With the remaining fingers we may use the time-number sequence ‘6, 7, 8, 9, ten’. In space we then get ‘1B1, 1B2, 1B3, 1B4, 1B5 or 2B0’ if we count with 5s as the unit. If we count with tens as the unit, we instead get ‘6s, 7s, 8s, 9s and 1B0’ tens.

We see that reaching the unit-number means changing from one counting of singles to two countings, a counting of bundles and a counting of unbundled singles. Later we have three countings of a total T in bundle-bundles, bundles and unbundled singles.

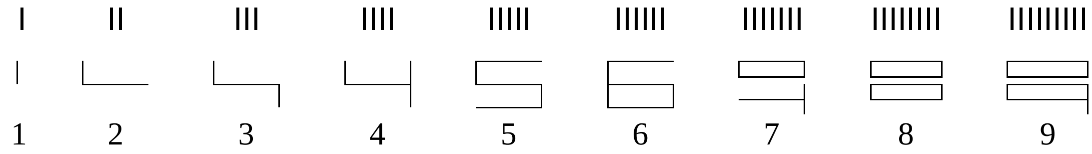
T = 3tens & 4 = 3B4 tens = 34

T = 6 hundreds & 7 tens 8 = 6BB 7B 8 tens = 678

Instead of saying ‘3ten4’ we should really say ‘3 bundles with ten per bundle, and 4 unbundled’ to show that the bundle-size is a per-number that also add by their areas: 2 3s + 4 5s = (2*3+4*5)/8 8s.

Likewise, instead of saying ‘6hundreds7ten8’ we should really say ‘6 bundle-bundles and 7 bundles with ten per bundle, and 8 unbundled’.

The bundles may be rearranged as icons with as many sticks or strokes as the represent. As the unit, ten needs not one but two icons for the bundles and unbundled, ten = 1B0 = 10.



With their bundle-numbers with units, children thus open our eyes for a different mathematics that, freed from its present essence-bounds, may return to its original identity as a natural science about the natural fact Many existing in space and time. This decolonization may create a natural number-language to communicate about outside existence instead of about inside essence only.

If existence comes before essence, then counting comes before adding, which is new since normally numbers are given to add. So, let now the adult educator be educated by the uneducated child to see how much math comes from counting. These discoveries will be transformed into a sequence of micro-curricula in mathematics where counting outside existence comes before adding inside essence.

Looking at five fingers we see that the inside essence ‘five’ outside may exist in different forms with each their label.

In space, five may exist as five ones, or as a bundle of one fives that can be rearranged as an icon with the number of sticks it represents. Or, if counted in twos, five may exist as one bundle and three, 1B3, as 2B1, or as 3B-1 needing 1 to become an extra bundle, or even as 1BB 0B 1 since two twos held together exist as a bundle of bundles, one bundle-bundle, 1BB, or 1B-square becoming a square with centi-cubes.

In time, the five fingers are counted in 5s one by one with the bundles included as a unit: 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0 5s. Counting only ‘1, 2, …’ would mean that the first single should be 1 5s instead of 0B1 5s. Digits without units are not numbers but follower-names as the names of the weekdays or the months.

We count by bundling, so a number has at least two digits counting the unbundled and the bundles; and perhaps the bundle-bundles. We cannot count in 1s since, as a single, 1 is not a bundle, and since a bundle of bundles of 1s will stay a 1s, 1 1s = 1 1s, instead of becoming a new unit as, e.g., 2 2s becoming a new unit, 4s, as seen when counting five fingers in 2s: 0B1, 0B2 or 1B0, 0B3 or 1B1, 0B4 or 1B2 or 2B0 or 1BB 0B 0, and finally 0B5 or 1B3 or 2B1 or 1BB 0B 1.

Therefore, when counting and adding Many, we will no longer use a ruler’s 1dimensional line-numbers without units as 5 and 42. Instead we now will use two-dimensional bundle-numbers with units as 0B5, and 4B2 that exist as rectangular or squared totals on a ten-by-ten bundle-bundle pegboard, a ‘BBBoard’. Which allows learning mathematics indirectly when formulating inside tales about outside totals as, e.g., 6 7s existing on the BBBoard limited by two rubber bands, and that may be recounted in tens as a total of four tens and two, shortened to ‘T = 4B2’ tens. This number-language sentence or formula contains an outside subject linked to an inside predicate, just as does a word-language sentences as ‘This is a glass’.

Flexible bundle-numbers with units allow the same total, forty-two, to be recounted with over-load or under-load so that T = 42 = 4B2 = 3B12 = 5B-8, which makes unneeded the place value system. As well as carrying and borrowing since now 17 + 28 = 1B7 + 2B8 = 3B15 = 4B5 = 45, and 57 – 28 = 5B7 – 2B8 = 3B-1 = 2B9 = 29.

With units, 2digit numbers without may be postponed since with 6s as the unit, 6+9 = (1B0 + 1B3) 6s = 2B3 6s, and with 9s as the unit, 6+9 = (1B-3 + 1B0) 9s = 2B-3 9s = 1B6 9s.

When bundle-counting outside totals we find that not only digits but also the operations are icons, but in the reverse order.

Power now is the first operation we meet as a bundle-bundle hat when counting in 3s will change 9 into 3 3s, a bundle of bundles, a bundle-bundle, a BB, or a B^2, that on a BBBoard is a square where 2 3s is a rectangle that may be transformed into almost a square with the rectangle’s square root as the side by moving half the excess from the top to the side.

Division and multiplication then follow as a broom and a lift to push-away and stack bundles. Here, *recounting* in 2s will change 8 into (8/2)x2, or T = (T/B)xB telling that the total T contains T/B Bundles. This proportionality ‘recount-formula’ is used all over to shift units. Also, it solves multiplication equations as ‘u*2 = 8’ asking “How many 2s in 8?” which of course is found by recounting 8 in 2s as 8 = (8/2)*2, so that the solution is u = 8/2 found by moving ‘to opposite side with opposite sign’. This follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, so if u*2 = 8 then u = 8/2. So, now disappears the balancing method solving equations by doing the same to both sides.

Subtraction now follows as a rope to pull-away the stack to locate unbundled singles, thus *splitting* the total in two, T = (T–B)+B, the ‘split-formula’. Finaly in the end, addition is a cross showing the two ways to unite stacks, next-to and on-top.

The split-formula solves addition equations as ‘u+2 = 8’ asking “What is the number that with 2 added becomes 8?” which of course is found when splitting 8 by pulling-away the 2 that was added, 8 = (8-2)+2, so that the solution is u = 8-2, again found by moving ‘to opposite side with opposite sign’. Also, this follows the formal definition: 8-2 is the number u that with 2 added 2 gives 8, so if u+2 = 8 then u = 8-2. So here also the balancing method is not needed.

Recounting 8 in 3s, a calculator may inside predict the outside result. Entering ‘8/3’ gives ‘2.more’, and entering ‘8-2x3’ gives ‘2’ unbundled. This prediction is validated when outside pushing-away 3s from 8 twice. Included on-top of the bundles, the unbundled becomes decimals if writing T = 2B2 3s; or fractions if counted in bundles also, T = 2 2/3 3s, or replaced by a negative number, T = 3B-1 3s, telling what is missing in space for an extra bundle, or what was pulled-away in time from it. So, counting the unbundled leads to decimals, fractions, and negative numbers, 8 = 2B2 = 2 2/3 B = 3B-1 3s.

Recounting from one icon-unit to another may be predicted by a calculator. To inside predict the answer to the outside question “2 3s = ? 4s”, entering ‘2*3/4’ gives ‘1.more’. So, the unbundled are found when pulling-away 1 4s predicted by entering ‘2*3 – 1*4’ giving ‘2’. The calculator thus predicts that 2 3s = 1B2 4s, which may be validated outside.

Recounting from tens to icons when asking ‘How many 6s in 24?’ leads to the equations ‘u*6 = 24’ solved by u = 24/6 since 24 recounts in 6s as ‘24 = (24/6)*6’, so the solution again follows the ‘opposite side & sign’ rule.

Recounting from icons to tens when asking ‘How many tens in 6 7s?’ leads to early algebra when placed on a BBBoard as (B-4)*(B-3), which is left when we pull-away the top 4B and the side 3B and add the 4 3s pulled away twice, so (B-4)*(B-3) = B*B – 4B – 3B -- 4*3 = (10-4-3)B + 4*3 = 3B0 + 1B2 = 4B2 = 42, which clearly shows that minus times minus must be plus. Also, the four rectangles on the BBBoard illustrates the FOIL-method, First, Outside, Inside, Last; only here, Outside and Inside has changed place.

Recounting rectangles in squares, we may ask “How to square 6 4s by finding its square root as its side?” Moving half the excess from the top to the side gives the first guess as 5 5s. Then, to fill out the 1x1 top right corner we pull-away a slice, u, from the top and side. Here, 2*4*u = 1, or u = 1/8, so 4 7/8 = 4.88 is our second guess, which is close to the calculator’s answer, 4.90. Since two squares add as the square created by their Bottom-Top BT line we now have a way to add rectangular areas.

A split square may also be used to solve quadratics: In an (u+3)-square, a vertical and horizontal dividing line will split a BBBoard into two squares, u^2 and 3^2, as well as two 3*u rectangles. So (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6 *u + 9. Now, with the quadratic u^2 + 6*u + 8 = 0 they all disappear except 9-8 = 1. So (u+3)^2 = 1, which gives -2 and -4 as the two solutions. If, on the other hand, u^2 + 6*u + 10 = 0, then 9-10 = -1, and then the equation (u+3)^2 = -1 has no solution.

Also, a split square with $y = x$^2 may show that, with dy and dx as small changes of y and x, dy will be 2*x*dx if we neglect the tiny-tiny upper right corner. So, with $y = x$^2, dy/dx = 2*x.

Recounting in different physical units, e.g., from $ to kg, $ = ($/kg)*kg, we get a 'per-number', $/kg, to connect the units: with 3$/5kg, 12$ = (12/3)*3$ = (12/3)*5kg = 20kg. With like units, per-numbers become fractions or percentages, 3$/5$ = 3/5, and 3$/100$ = 3/100 = 3%.

Recounting mutually the sides in a stack split by its diagonal, the per-numbers gives trigonometry that connects the sides and the angles., e.g., rise = (rise/run)*run = tangent-Angle*run. Likewise,

rise = (rise/diagonal)*diagonal = sine-Angle*diagonal, and

run = (run/diagonal)*diagonal = cosine-Angle*diagonal.

Here the tangent-Angle describes the steepness or the rise-ability of the diagonal. In an x-y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is almost a diagonal in a rectangle, and since the run and the rise here are changes in x and in y, Δx and Δy, the tangent-Angle here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

Once counted and recounted, totals may finally be added on-top or next-to.

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like. Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process asking, e.g., "2 3s and how many 5s total 4 8s" is called differential calculus since you must find the difference between the two known totals before recounting it in 5s, (T2-T1)/5, or ΔT/5.

Adding 2kg at 3$/kg and 4 kg at 5$/kg, the unit numbers 2kg and 4kg add directly whereas the per-numbers 3$/kg and 5$/kg first must be multiplied to unit-numbers before adding, thus adding as areas, i.e., as integral calculus. Likewise with fractions where 1red of 2 apples plus 2red of 3 gives 3red of 5, and of course not 7red of 6 apples as taught by 'mathematism'. Per-numbers and fractions and digits thus are not numbers, but operators needing a number to become a number.

Here the per-numbers are piecewise constant, but they may also be locally constant as in the case of a falling object with an increasing meter/second number. Calculus thus occurs three times, as next-to addition of stacks in primary school, as adding piecewise constant per-numbers in middle school's mixture problems, and as adding locally constant per-numbers in high school where the tiny area-strips are written as changes, p*dx = dA, to profit from the fact that adding many changes makes all middle changes disappear leaving only the total change of A from the start to the end.

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives 106%^10 or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

Looking inside my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bend the two outer fingers. So, 1/3 of the Ls are bent, and ½ of the Rs. Does that mean that 1/3 of the bent are Ls? No, ½ is. So, in cross-tables we must also see fractions as operators needing numbers to become numbers.

The first months the children thus meet the core of mathematics: functions, equations, proportionality, trigonometry, and calculus.

As well as the four operations that unite unlike and like unit-numbers and per-numbers: addition, multiplication, integration, and power, seen in an 'Algebra-square' that is named after the Arabic word 'algebra' meaning 'to reunite'.

And that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations **unite**/ *split Totals into*	Unlike	Like
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a * n}$ $T/n = a$
Per-numbers m/s, $/100$ = %	$\boldsymbol{T = \int f\, dx}$ $dT/dx = f$	$\boldsymbol{T = a\wedge b}$ $b\sqrt{T} = a$ $\quad$ $loga(T) = b$

Figure 01. The Algebra square shows how to unite and split our four number-types, and how to solve equations by moving 'to opposite side with opposite sign'.

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling. As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models. Fact models talk about the past and present and only needs to have the units checked. Fiction models talk about the future and needs to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that '2+3 = 5' always despite 2weeks + 3days = 17days, thus transforming mathematics to 'mathematism'.

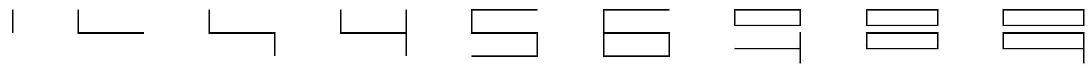
04. Micro Curricula, for Learners

Based on our investigation we may now design a sequence of micro-curricula, MC, for developing a number-language by working with totals existing as outside things and actions on a BBBoard that may be supplied with centimeter cubes placed on-top of the BBBoard.

MC01. Digits as icons in space, IIIII = 5

The total here exists as sticks to be rearranged at a table and reported by a drawing on paper. The ‘T=?’ question is answered in two ways, as a collection of single ones, I I I I I, or as one bundle of ones, ~~IIII~~, that may be rearranged into an icon, 5, called a digit containing the number of sticks that it represents if written in a less sloppy way, and thus somewhat like the digits on a calculator. Each time the folding ruler is folded to look like the icon.

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

Figure 02. A digit as an icon with the number of sticks as it represents, four in the 4-icon, etc.

If we count in tens, ten sticks are replaced by one stick in a different color or material to allow more bundles to be rearranged as icons also so that 67 means 6-Bundle-7, 6B7, called 6-ten-7.Ten thus needs no icon since it becomes ‘one bundle and no unbundled’, written as T = 1B0 tens, or T = 1.0 tens, or T = 10 if leaving out the unit and the decimal point.

Example. One stick is one single. An extra stick added to 1 stick gives two singles that may unite to one 2-icon. And so on. An extra stick added to 8 sticks gives nine singles that may unite to one 9-icon. And an extra stick added to 9 sticks gives ten singles that instead of uniting to one ten-icon is bundled together as one bundle replaced by one stick of a different color or a different material and written as 1B0 since there are no singles left.

An extra stick added to ten sticks gives eleven singles that may unite to one bundle and 1 single left, which made the Vikings call eleven ‘one left’ and written as 1B1.

Likewise with twelve that the Vikings called ‘two left’. There is no ‘three left’ because of the ancient counting method “one, two, many”. So, from 3 we specify both the bundles and the singles.

Zero is iconized as a looking glass not finding anything. The name ‘twenty’ comes from the Vikings’ ‘twende ti’.

Skill building. Roll some dice twice (physically or virtually) to get the number of bundles and unbundled singles. Then phrase and report the number.

So, with 3 and 5, say three-bundle-five, three-ten-five, and thirty-five; and finally write T = 3B5 = 35.

End test. Roll some dice twice an extra time.

MC02. Tally-counting in time, **** = ~~IIII~~ I**

The total here exists as sticks to be moved one by one on a table and reported by strokes on paper, as well as some dice. The ‘T=?’ question is answered by tally-counting the total in time reported as some 5-bundles and some unbundles singles, e.g., T = 2B1 5s.

Example. In a sentence, count the e’s and the a’s.

Skill building. Some dice is rolled a dozen times to show Even (1 2s, 2 2s or 3 2s) or Odd (1, 3, 5). The tally counting in 5s is reported with two totals e.g., W = 1B4, and L = 0B3, giving a total T = 2B2, or 1B7, or 3B-3. And giving the difference D = 1B1.

End test. Roll some dice an extra time.

MC03. Bundle-counting in time with units: 0B1, …, 0B5 or 1B0, 3 3s = 1BB

The total here exists as lines or stacks on a BBBoard. The ‘T=?’ question is answered in time by moving the finger along the pegs with a counting sequence that by including the bundle as a unit makes the place value system unneeded. First, we count in lines, then in stacks marked by a vertical rubber band.

Counting 5 fingers in 3s we ask, “What do we have here?” to emphasize that we focus on existence instead of essence. We cannot count one finger as ‘one’ since 1 3s is 3 1s and we only have one. Instead, we count ‘0 bundle 1, 0B2, 0B3 or 1B0’ since 3 1s is 1 bundle with no unbundled left.

Counting 5 fingers in 2s we notice that four fingers are 1B2, but also 2B0, and 1BB0B0 since 2 2s is a bundle of bundles, a bundle-bundle, a BB, that is a square, as is 3 3s, 4 4s etc.

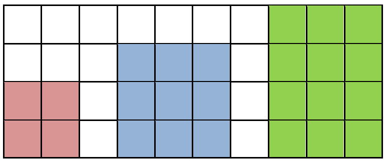


Figure 03. 2 2s, and 3 3s, and 4 4s as bundle-bundle squares

Counting the five fingers on a hand, their essence ‘5’ may exist in various ways:

T = 1 5s = 1B1 4s = 1B2 3s = 1B3 2s = 2B1 2s = 1BB 0B 1 2s = 5 1s

Counting ten fingers in 3s, we get 0B1, 0B2, 0B3 or 1B0, 1B1, … , 2B3 or 3B0, 3B1.

T = 3B1 = 1BB0B1 3s, since 3 3s are a bundle of bundles, a bundle-bundle, a BB, a square.

Counting ten fingers in 2s we notice that 8 as 2BB 0B 0 is 1BBB 0BB 0B 0. So, we may also write ten as 1BBB 0BB 1B 0 2s, or as 1010 if leaving out the units.

Finally, when counting hundred on the BBBoard we finish with 1BB 0B 0:

0B1, 0B2, … ,0B9, 0Bten or 1B0, 1B1, … , 9B8, 9B9, 9Bten or tenB0 or 1BB0B0.

1BB0B0	1BB0B1	1BB0B2	1BB0B3	1BB0B4	1BB0B5	1BB0B6	1BB0B7	1BB0B8	1BB0B9	1BB0B10
10B0	10B1	10B2	10B3	10B4	10B5	10B6	10B7	10B8	10B9	10B10
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	9B10
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	8B10
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	7B10
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	6B10
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	5B10
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	4B10
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	3B10
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	2B10
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	1B10
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	0B10

Figure 04. Counting from zero to 109 with units.

Skill building. Count a dozen and a score in 5s, 4s, 3s, and 2s.

End test. Count 30 in 3s.

MC04. Bundles counted in space with over- and underloads, 5 = 1B3 = 2B1 = 3B-1 2s

The total here exists as fingers and sticks. The ‘T=?’ question is answered in space by ‘flexible bundle counting’ that allows unbundled to stay unbundled as an overload, and that allows borrowing extra sticks to fill up an extra bundle with an underload. Using flexible bundle-numbers with units makes carrying and borrowing unneeded.

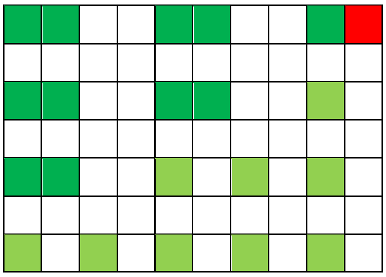


Figure 05. Five fingers may be recounted in 2s as 0B5, or 1B3, or 2B1, or 3B-1

Five fingers may be recounted in 5s as 0B5 (an overload) or 1B0 or 2B-5 (an underload).

Five fingers may be recounted in 4s as 0B5 or 1B1 or 2B-3.

Five fingers may be recounted in 3s as 0B5 or 1B2 or 2B-1.

Ten fingers may be recounted in tens as ½B from 1 to ten:

½B-4, ½B-3, ½B-2, ½B-1, ½B0, ½B1, ½B2, ½B3, ½B4, ½B5 or 1B0 as ten.

This may ease standard calculations.

T = 6+3 = ½B1 + ½B-2 = 1B-1 = 0B9 = 9

T = 6 + 7 = ½B1 + ½B2 = 1B3 =13

T = 8-3 = ½B3 – ½B-2 = 0B5 = 5, thus showing that -(-2) = +2

T = 4*7 = 4*½B2 = 2B8 = 28

Skill building. The action is repeated with nine fingers arranged counted in 5s, 4s, 3s, and 2s. And with 2-digit numbers, e.g., 67 = 6B7 = 7B-3 = 5B17. Then with cubes, and with sticks.

End test. The action is repeated on a BBBoard, or on an abacus.

MC05. Splitting, 8 = (8-2)+2

The total here exists as a line of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pulling-away’ a bundle hidden under cubes.

A total of 1 8s is split by pulling-away 2. To pull-away just once may be iconized by a rope, -, so that ‘8-2’ means ‘from 8 pull-away 2’ in time, or ‘from 8 pulled-away 2’ in space.

The original 8 now is split in 8-2 and 2 so that 8 = (8-2) + 2. Here addition is iconized by a cross showing the two directions we can add, next-to or on-top so that ‘4+2’ means ‘4 with 2 added’. With T for the total and B for the bundle this ‘split-formula’ may be written as T = (T–B)+B.

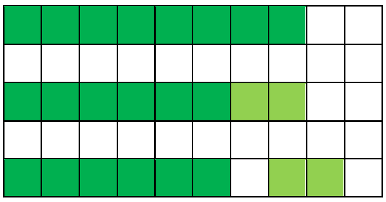


Figure 06. A total of 8 split in two parts by pulling-away 2, so 8 = (8 – 2) + 2

Splitting may be used to solve equations coming from reversed actions.

The question "What is the number that with 2 added gives 8" may be shortened to an equation with a letter for the unknown number, '*u*+2 = 8'. Of course, the number is found by reversing the action and pull-away the number that was originally added, so *u* = 8-2, which also comes from splitting 8, '*u*+2 = 8 = (8-2) + 2'. So, we see that the solution is found by moving "to the opposite side with the opposite sign". Also, it follows the formal definition of subtraction: 8-2 is the number *u* that added to 2 gives 8, or if *u*+2 = 8 then *u* = 8-2.

Skill building. The action is repeated with fingers, sticks, cubes, and an abacus. The action is repeated with other numbers, e.g., 9 = (9 – 3) + 3.

End test. Pick two numbers.

MC06. Recounting, 8 = (8/2)x2

The total here exists as lines of pegs on a BBBoard. The 'T=?' question is answered by 'pushing-away' bundles hidden under cubes.

A total of 1 8s is recounted in 2s by 4 times pushing-away 2s. To pull-away more times may be iconized by a broom, /, so that '8/2' means 'from 8 push-away 2s' in time, and 'from 8 pushed-away 2s' in space.

With the pushed-away 2s arranged in a stack, 8 contains 2s 4 times or 8/2 times so that 8 = 4*2, or 8 = (8/2)x2. Here multiplication is iconized by a lift so that '4x2' or '4*2' means '4 times stacking 2s'. With T for the total and B for the bundle this 'recount-formula' may be written as T = (T/B)xB.

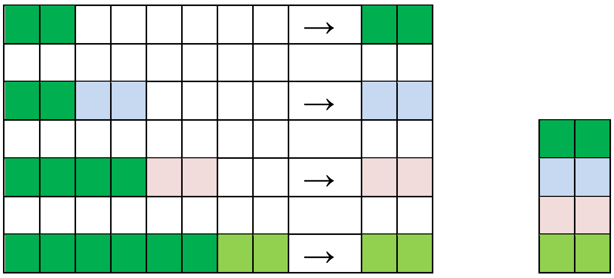


Figure 07. A total of 8 is recounted by pushing-away 2s and lifted into a stack, so 8 = (8/2)x2

Recounting may be used to solve equations coming from reversing actions: The question "What is the number of 2s in 8" may be shortened to an equation with a letter for the unknown number, "*u**2 = 8". Of course, the number is found be reversing the action and push-away the 2s that was originally united, so *u* = 8/2, which also comes from recounting 8, 'u*2 = 8 = (8/2)*2'. So, we see that the solution is found by moving "to the opposite side with the opposite sign". Also, it follows the formal definition of division: 8/2 is the number *u* that multiplied with 2 gives 8, or if *u**2 = 8 then *u* = 8/2.

Skill building. The action is repeated with 12 counted in 2s and 3s using a finger to hide a bundle.

End test. 18 counted in 2s, and in 3s.

MC07. Including the unbundled, 8 = (8/3)*3 = 2B2 = 2 2/3 = 3B-1 3s

The total here exists as cubes. The 'T=?' question is answered by pushing-away bundles to a stack that is then pulled-away to find the unbundles that then are included on-top of the stack.

Recounting 8 in 3s, first 2 times we push-away 3s, then we pull-away the stack of 2 3s and find 2 unbundled that are placed on-top of the stack. Here they may be seen as singles in a bundle described by a decimal number, 8 = 2B2 3s, or as a fraction part when also recounted in 3s as 2 = (2/3)*3, 8 = 2 2/3 3s. Or we may write 8 = 3B-1 to show that in space 1 is missing in the next bundle, or that in time 1 is pulled-away from it.

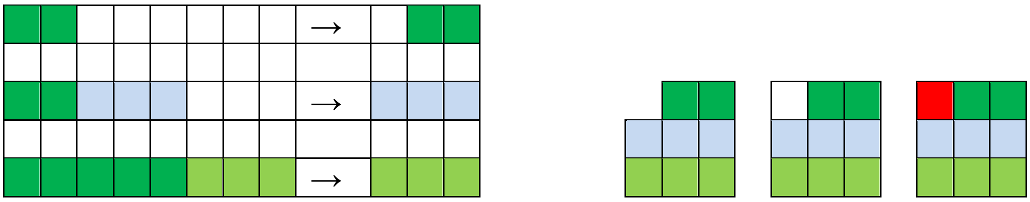


Figure 08. Unbundled become decimals, fractions or less-numbers, 8 = 2B2 = 2 2/3 = 3B-1 3s

With ten as the bundle-number, unbundled occur in the same way: T = 4B3 = 4 3/10 = 5B-7 tens.

Skill building. The action is repeated on a BBBoard with 11 counted in 3s and 4s using cubes or fingers to hide a bundle.

End test. Recount 8 in 5s, and in 3s.

MC08. Recounting in squares, 6 4s = 1 BB ?s

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by working on the upper right corner occurring when half the excess is move from the top to the side to give a first guess about the square root. The rectangle may be shown with rubber bands or with cubes.

If we want to square the total T = 6 4s we move half of the excessing 2 4s from the top to the side to get a 5 x 5 square, and an unfilled square in the upper right corner that we try to fill with a rectangular 4*u slice of the top and the side.

Here u is found by the equation 2*u*4 = 1, or 8*u = 1, giving u = 1/8 = 0.125, and 5-0.125 = 4.875 as our next guess.

However, now there is too much in the corner, so we repeat the process, or consult a calculator showing that the correct answer is √(6*4) = 4.90, which is very close to our third guess.

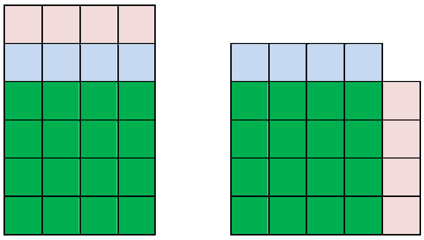


Figure 09. Recounting 6 4s by moving half the excess to the side to try to get a 5 x 5 square

To find the quadratic numbers we see that 5 5s comes from 4 4s by adding 4 twice and 1 for the top right corner. So, with 4 4s as 16, 5 5s is 16 + 4 + 4 + 1= 25.

In this way we may predict the square numbers to be 1, 4, 9, 16, 25, 36, 49, 84, 91 and 100.

And we see that a BB square increases with 2B+1 when B increases with 1.

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. Square 9 5s. Solve the quadratic equation x^2 + 8*x + 12 = 0

MC09. Recounting in another icon, 3 4s = ?5

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered on a BBBoard and predicted on a calculator.

With rubber bands on a BBBoard we see that 3 4s may be recounted as 2B2 5s. This may be predicted by a calculator. To find how many 5s there is in 3 4s we enter "3*4/5". The answer is '2.more'.

To find them we pull-away the stack of 2 5s by entering ‘3*4-2*5’ that gives the answer ‘2’. So, the calculator predicts that 3 4s recount as 2B2 5s, which is validated on the BBBoard.

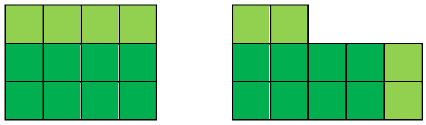


Figure 10. A total of 3 4s recounted in 5s manually. and predicted by a calculator

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. 4 5s = ?6s.

MC10. Recounting from tens to icons, 2 tens = ? 7s

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by recounting.

With rubber bands on a BBBoard we see that 2 tens may be recounted as 2B6 7s. This may be predicted by a calculator. To find how many 7s there is in 2 tens by recounting we enter ‘20/7’.

The answer is ‘2.more’ found by pulling away the stack predicted by ’20-2*7’ giving ‘6’. So, the calculator predicts that 2 tens recount as 2B6 7s, which is validated on the BBBoard.

Alternatively, the question “How many 7s in 20?” leads to the equation u*7 = 20 that is solved moving to opposite side with opposite sign so that again u = 20/7, or u = 2 6/7.

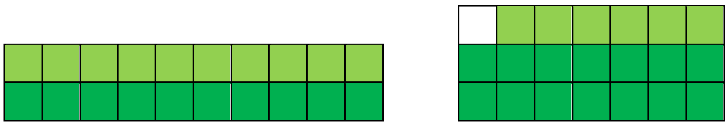


Figure 11. A total of 2 tens recounted in 7s manually, and predicted by a calculator

We notice that decreasing the bundle will increase the height. To study this closer we recount 1 dozen in 6s, 4s, 3s, 2s, and 1s and place a dot in the upper right corner each time.

The points then form a curve called a hyperbola.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 4 tens = ?8s.

MC11. Recounting from icons to tens, 6 7s = ? tens

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by finding what we must pull-away from the bundle-bundle.

Recounting from icons to tens apparently is another name for the multiplication tables.

With rubber bands on a BBBoard we see that 6 7s is left if from the ten bundles we pull-away 4 top and 3 side bundles, and add the upper right 4 3s that we pull-away twice:

T = 6 7s = 6*7 = (10 – 4 – 3)*B + 4 3s = 3B + 1B2 = 4B2 = 42.

This leads to early algebra if instead we write:

T = 6 7s = 6*7 = (B – 4)*(B – 3) = BB – 4*B – 3*B + 3*4

Here we see that minus times minus must be plus.

So, a quick way to find the answer is to add and multiply the less-numbers and subtract the first and add the latter. With 4 and 3 as the less-numbers here, we quickly learn to say:

“Less (4+3) bundle (4*3)”, or “Less 7 bundle 12”, or “3 bundle 12”, or “4 bundle 2”, or “42”.

We may also write B – 4 and B – 3 on top of each other and then multiply down and across. Or we may use the FOIL method: First, Outside, Inside, Last.

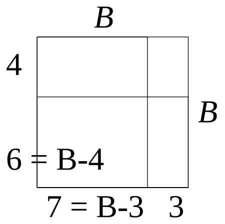
	T = 6 * 7 = (B-4) * (B-3) = BB – 4B – 3B + 4*3 = 3B12 = 4B2 = 42	$T = \begin{pmatrix} 1B & -4 \\ 1B & -3 \end{pmatrix}$ = 1BB – 4B – 3B + 4*3 = 10B – 7B + 1B2 = 3B12 = 42	$T = \begin{pmatrix} 2B & +3 \\ 4B & +6 \end{pmatrix}$ = 8BB+12B+12B+18 = 8BB+24B+18 = 10BB 5B 8 = 1058

Figure 12. 6*7 is left when pulling-away 4B and 3B and adding the 4 3s pulled away twice

Multiplying the two-digit numbers 23*46 as 2B3 4B6s, a vertical and a horizontal rubber band between the bundles and the singles allows a BBBoard to show the four stacks 2B 4Bs, and 2B 6s, below the 3 4Bs, and the 3 6s. With overloads, they add up to 8BB 24 B 18, or to 10BB 5B 8 = 1058 without.

This process may be reversed when asking ‘1058 = ? 46s’. First 1058 is written with an overload as 10BB 5B 8 = 8BB 25B 8. Since 4B*2B = 8BB, the 2B contributes 2B*6 to the 25 Bs. The rest 13B 8 may be rewritten as 12B 18, which recounted in 3s gives 4B6. So, the answer is 1058 = 23 46s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 7 8s = ? tens.

MC12. Recounting in another physical unit creates per-numbers, 3$/5kg

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

Recounting a physical total T as 3$ and 5 kg gives a ‘per-number’ 3$/5kg called the price and marked as a 3x5 rectangle on a BBBoard.

The question “20kg = ?$” then is answered by recounting in the per-number:

20kg = (20/5)*5kg = (20/5)*3$ = 12 $.

On a BBBoard the counting sequences now are 5, 10, 15, 20kg, and 3, 6, 9, 12$ since the per-number here has changed from 3/5 to 12/20.

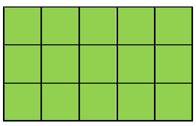


Figure 13. The per-number 3$/5kg show as 3 5s, or 6 10s, or 9 15s, or 12 20s, etc.

Or we may introduce a new unit to make the digits like: 3$ = (3/5)*5$ = n*5$ = 5n$ with the new *n* = 3/5. So, 20 kg = 20*n*$ = 20*3/5$ = 12$.

Alternatively, the units may be recounted:

$ = ($/kg)*kg = (3/5)*20 = 12

Or we may equate the per-numbers: $/kg = *u*/20 = 3/5. Moving to opposite side with opposite sign we then get 5**u* = 3*20, or *u* = 3*20/5 = 12.

Skill building. The action is repeated with other numbers.

End test. With 5$/2kg, 12kg = ?$, and ?kg = 12$.

MC13. With the same unit, per-numbers become fractions, 3$/5$ = 3/5

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

If a whole contains a part, they have the same unit. In this case the per-number becomes a fraction without units. Still, we may use the units ‘p’ and ‘w’ for the part and the whole.

To get the fraction 3/5 of 20$ thus means to get 3p/5w of a 20$ whole. Recounting in the per-number thus gives 20w = (20/5)*5w = (20/5)*3p = 12p, or 12$ of 20$.

To get the fraction 3/5 of 100 thus means to get 3p/5w of a 100 whole. Recounting in the per-number thus gives 100w = (100/5)*5w = (100/5)*3p = 60p, or 60 of 100, written as 60%.

To ask “20$ is what percentage of 80$” means asking about the fraction 20/80 of 100. Or we may introduce a new unit 80$ = 100% to see that 20$ = (20/80)*80$ = (20/80)*100% = 40%.

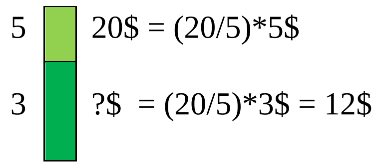


Figure 14. A fraction column with per-numbers to the left and unit-numbers to the right

To add 10% to 200$ we introduce the per-number 200$/100%. After the addition the total is

T = 100% + 10% = 110% = (110/100)*100% = (110/100)*200$ = 220$.

So, adding 10% means multiplying with 110%, and adding 10% 5 times means multiplying with 110%^5 = 161.1% thus giving 50% plus 11.1% extra, also called compound interest.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. With 2p/5w, 10p = ?w, and ?p = 20w, and 2p/5w = ?%.

MC14. Recounting a stack’s sides gives trigonometry, rise = (rise/run)*run = tanA*run

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

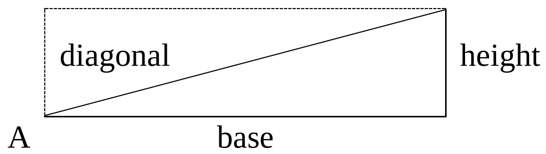


Figure 15. A stack with a base and a height and a diagonal

On a BBBoard, we mark a 3x4stack as a rectangle with height 3 and base 4. If we recount the height and the base in the diagonal, we get the per-numbers sine and cosine:

height = (height/diagonal) * diagonal = sine Angle * diagonal, shortened to

$h = (h/d) * d = \sin A * d = \sin A\ d$s,

This gives the formula sin A = height / diagonal, or sin A = h/d, or sin A = 3/5 in our case.

Likewise, cos A = base / diagonal, or sin A = b/d, or cos A = 4/5 in our case.

height = (height/base) * base = tangent Angle * base, shortened to

$h = (h/b) * b = \tan A * b = \tan A\ b$s,

This gives the formula tangent A = height / base, or tan A = h/b, or tan A = 5/10 in our case.

A protractor shows that the angle *A* is a little above 25 degrees. Testing this we get tan 25 = 0.466. The reverse tan-button 'tan^-1' gives the precise result, tan^-1(0.5) = 26.6 degrees.

Using the words 'run' and 'rise' instead of 'base' and 'height', we get the diagonal's slope-formula: tan *A* = rise/run. Here the tangent-number describes the steepness of the diagonal.

In a x-y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is a diagonal in a rectangle, and since the run and the rise are changes in x, Δx, and in y, Δy, the tangent-number here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

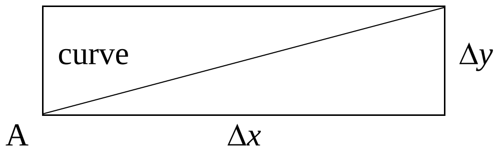


Figure 16. A curving curve is linear locally with small changes in *x* and *y*

The word 'tangent' is used since the height will be a tangent in a circle with center in A, and with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, *h* = tan *A*.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical.

Half the circumference in a circle with radius 1 is called π, and *π* = 100 * *h* = 100 * tan 1.8 = 100 * tan (180/100) = 3.1416. This gives a formula for the number π: *π* = tan (180/*n*) * *n*, for *n* large enough. We also see that in a circle with radius r, the circumference is 2**π***r*, and the area is *π***r*^2, or *π*/4* *d*^2 where *d* is the circle's diameter. So, a *d*-circle takes up almost 80% of the space inside the surrounding *d*-square.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. Add a 4-square and a 6-square as a square.

MC15. Adding next-to or on-top, T = 2 3s + 4 5s = ? 8s; T = ? 3s; T = ? 5s

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by recounting.

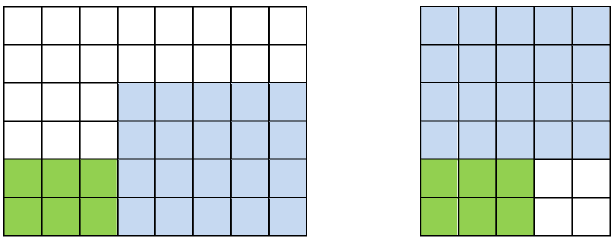


Figure 17. Two totals 2 3s and 4 5s added next-to, and added on-top

Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process by asking '2 3s and how many 5s total 4 8s' is called differential calculus because you find the difference between the known totals before recounting it in 5s, (T2-T1)/5, or ΔT/5.

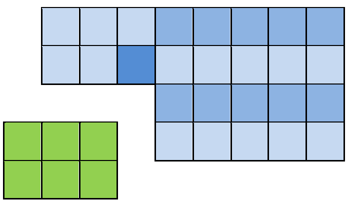


Figure 18. Next-to addition is reversed when asking 2 3s + ? 5s = 4 8s

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like To recount 2 3s in 5s, first we enter ‘2*3/5’ giving ‘1.more’, then we enter ‘2*3-1*5’ giving ‘1’, so 2 3s is 1B1 5s, which give the total 1B1 5s + 4B0 5s = (1B1 + 4B0) 5s = 5B1 5s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 3 4s + 6 5s = ? 9s. And 3 4s + 6 5s = ? 4s. And 3 4s + 6 5s = ? 5s. And 3 2s + ? 5s = 4 6s.

MC16. Subtracting and adding single digit numbers, 8+6 = 1B2 + 1B0 = 2B2 6s

The total here exists as lines of pegs on a BBBoard. The ‘T=?’ question is answered by using rubber bands to mark the bundles.

With a subtraction as ‘8 – 6 =?’, a rubber band marks 8 on a BBBoard, and fingers hide the 6 that is pulled-away, so 8 – 6 = 2.

With an addition as ‘8+6=?’, two rubber bands marks 8 and 6 on two BBBoard parallel lines to show that the sum may exist in two ways, as 2B2 6s, or as 2B-2 8s.

Figure 19. Adding 6 and 8 as 2B2 6s, or as 2B-2 8s, or as 2*½B + 1 + 3

Here, using half bundles, 5s, will easy recounting in tens since

6+8 = ½B1 + ½B3 = 1B4 = 14.

Multidigit numbers may be added and subtracted with an over- or an under-load, which makes carrying and borrowing unneeded.

T = 36 + 47 = 3B6 + 4B7 = 7B13 = 8B3 = 83

T = 86 – 37 = 8B6 – 3B7 = 5B-1 = 4B9 = 49

T = 4*67 = 4*6B7 = 24B28 = 26B8 = 268

T = 268 /4 = 26B8 /4 = 24B28 /4 = 6B7 = 67

Skill building. The action is repeated with other one-digit and two-digit numbers.

End test. 9 – 7 = ?, 9+7 = ?, T = 38+46 = ?; T = 82 – 54 = ?

MC17. Adding per-numbers and fractions by integral calculus

The total here exists as a rectangular bundle-number on a BBBoard. The ‘T=?’ question is answered by using rubber bands to mark the bundles.

Asking “2kg at 3$/kg and 4 kg at 5$/kg total what?” the unit numbers 2kg and 4kg add directly whereas the per-numbers 3$/kg and 5$/kg first must be multiplied to unit-numbers before adding, thus added as areas, i.e., as integral calculus. Here the per-numbers are piecewise constant, but they may also be locally constant as in the case with a falling object having an increasing meter/second number.

Before adding, fractions must also be multiplied to unit, numbers. So, with apples, 1red of 2 plus 2red of 3 gives 3red of 5, and of course not 7red of 6 as taught by ‘mathematism’.

Figure 20. Per-numbers add as areas and fractions add with units, both as integral calculus

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives 106%^10 or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

	B	~~B~~	
Left	1	2	3
Right	1	1	2
Total	2	3	5

	B	~~B~~	
Left	1/3	2/3	1
Right	1/2	1/2	1
Total	-	-	-

	B	~~B~~	
Left	1/2	2/3	-
Right	1/2	1/3	-
Total	1	1	-

Figure 21. In cross tables, per-numbers must pass through the unit-numbers

Looking at my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bent the two outer fingers. So, 1/3 of the Ls are bent, and ½ of the Rs. Does '1/3 of the Ls are bent' mean that '1/3 of the bent are Ls'? No, ½ is.

So, in a cross table we cannot go from the per-numbers in one direction to those in the other direction without going through the unit-number table. This is called the Bayes-principle.

Skill building. The action is repeated with other numbers.

End test. 3kg at 4$/kg and 5 kg at 6$/kg total what?"

MC18. Adding and subtracting Bundle-Bundle squares

The total here exists as bundle-bundle number squares on a BBBoard. The 'T=?' question is answered by using rubber bands or cubes to mark the bundles.

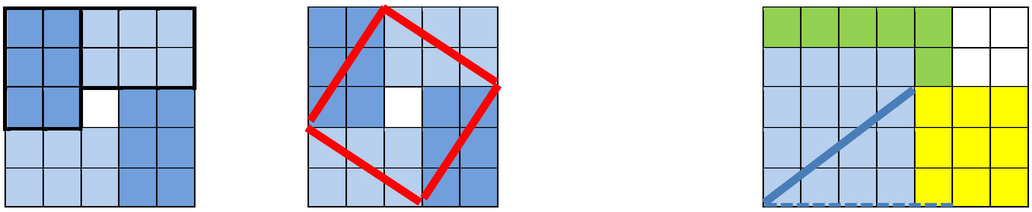


Figure 22. The two squares add as the square formed by the mutual Bottom-Top BT line

On a BBBoard we place four 7 3s so they form a ten-by-ten square that inside contains two squares, 7 7s and 3 3s as well as two stacks. But it also contains one square formed by the diagonals in the stacks as well as four half stacks.

So, the two squares add as the square formed by the mutual Bottom-Top BT line thus having the length as the square-root of the sum, i.e., √(7^2 + 3^2) = 7.62.

So, in this stack, adding the height and the bundle as squares gives the square of the diagonal. This rule is named by the ancient Greek thinker, Pythagoras.

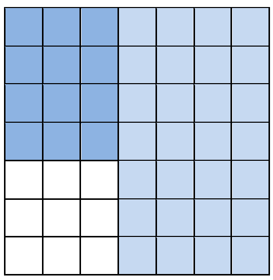


Figure 23. On a BBBoard we see that 7^2 – 3^2 = 7*(7-3) + (7-3)*3 = (7+3)*(7 – 3)

On a BBBoard we pull-away the a 3-by-3 square from a 7-by-7 square. This leaves 7 (7-3)s and (7-3) 3s that may be turned to 3 (7-3)s totaling (7+3) (7-3)s.

So 7^2 – 3^2 = (7+3)*(7 – 3).

Adding squares may also be involved when solving a quadratic equation. On a BBBoard we see that T = (x+3)*(x+3) is a square with four parts, two squares x^2 and 3^2, and two stacks 2*3*x, so that T = x^2 + 6*x + 9. The quadratic equation x^2 + 6*x + 8 = 0 then makes the whoe square go away except for 9-8 = 1. So (x+3)^2 = 1, which gives two solutions , x= -2 and x = -4 that hold when tested:

(-2)^2 + 6*(-2)+ 8 = 4-12+8 = 0, and (-4)^2 + 6*(-4) + 8 = 16-24+8 = 0

The quadratic equation ‘x^2 + 6*x + 10 = 0’ has no solutions since here ‘(x + 3)^2 = -1’.

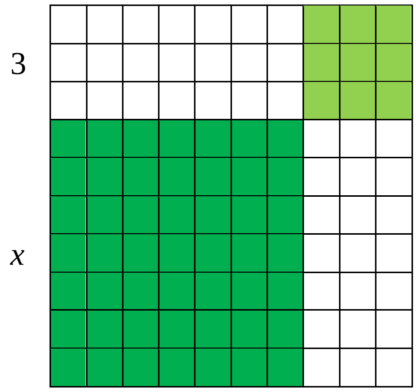


Figure 24. An (x+3)-square contains an x-square and a 3-square and two 3*x stacks

Alternatively, we can rewrite the equation x^2 + 6*x + 8 = 0, first as

x^2 + 2*3*x + 3^3 – 3^2 + 8 = 0, then as

(x+3)^2 – 9 + 8 = 0, then as

(x+3)^2 = 9 – 8 = 1 again with the solutions x= -2 and x = -4 that hold when tested:

(-2)^2 + 6*(-2)+ 8 = 4-12+8 = 0   (-4)^2 + 6*(-4) + 8 = 16-24+8 = 0

Skill building. The action is repeated with other numbers.

End test. Add 3 3s and 4 4s as a bubndle-bundle square. Solve the equation x^2+8*x+12 = 0

MC19. Adding unspecified letter-numbers

The total here exists as a rectangular bundle-number on a BBBoard. The ‘T=?’ question is answered by using rubber bands to mark the bundles.

In the letter-number T = 3ab the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as 3 abs, or as (3a) bs, or as (3b) as. To avoid being confused by the ‘s’ we will omit it, so T = 3ab = 3 * ab = 3a * b or 3b * a.

Since totals need a common unit to add, this must be first found:

T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a

T = 2ab^2 + 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b

Skill building. The action is repeated with other numbers and letters.

End test. T = 4ab^2d + 8bcd

MC20. Change in time

The total here exists as dots on a BBBoard. The ‘T=?’ question is answered by transferring the results to a squared paper and connect the dots with a curve.

In time, a total grows by being added or multiplied by a number, called addition-growth and multiplying-growth, or linear and exponential growth.

Addition-growth:

Final number = Initial number + growth-number * growth times, or shortly, $T = B + a*n$.

The number a is also called the slope.

Multiplying-growth:

Final number = Initial number * growth-factor ^ growth times, or shortly, $T = B * a\text{^}n$, since 200$+5% = (200*105%) $, so here *a* is 1+interest rate = 100% + 5% = 105%.

Combined growth (savings in a bank):

Here we have that $A/a = R/r$, where *A* is end-dollars, *a* is the period-dollars, *R* is the end-rate, *r* is the period-rate, and $1+R = (1+r)\text{^}n$, where *n* is the number of periods.

100% split in *n* parts will give the Euler number $e = (1+1/n)\text{^}n = 2.718$ for *n* large enough.

Changing the growth-number constantly will give a quadratic growth with a parabola curving upwards or downwards if the number increases or decreases.

Changing the curvature constantly will give cubic growth with a double parabola with curvature and counter-curvature.

Decreasing the growth-factor constantly give logistic saturation growth with a hill-curve in infections. Confusing exponential and saturation growth can cause unnecessary damage.

MC21. Bundle-numbers in a coordinate system

The total here exists as dots on a BBBoard. The '*T*=?' question is answered by rubber bands as lines on the BBBoard.

The bundle-number '*y xs*' with a height y and a width *x* may be called a 'changing bundle-number'. Here $y = 2*x$ gives a rising and $y = 9\text{-}x$ a falling bundle-number.

Marking the top right corners we get two lines. To inside predict the outside intersection point we equate the two heights, $2*x = 9 – x$. Moving to opposite side with opposite sign we get $3*x = 9$, and $x = 9/3 = 3$, which makes $y = 2*3 = 6$.

So, the prediction is that the two bundle-numbers become like as 3 6s, which is validated on the BBBoard where the first dot now is 0 instead of 1.

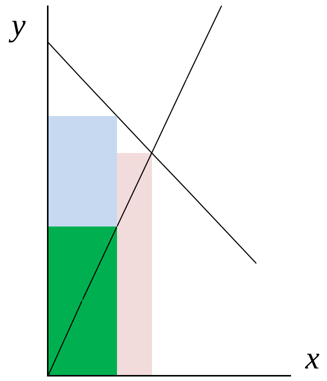


Figure 25. In an *x-y* coordinate system bundle-number stacks may rise and fall

In a rising bundle-number its total T will increase since here the height increases with increasing width. In the falling bundle-number this is not the case since the height decreases with increasing width. Here the total is $T = y*x = (9\text{-}x)*x = 9x – x\text{^}2$.

Setting up a table with x = 0, 1, 2, …, 9 we see that first T increases and then T decreases; and that T tops as 20 for x = 4 and x = 5; and that T = 20.25 for x = 4.5.

x	0	1	2	3	4	5	6	7	8	9
y	0	8	14	18	20	20	18	14	8	0

In general, even if a bundle-number is rising, its rise may be falling, so its marked corners will lay on a bending line called a parabola where $y = b*x + a*x^2$.

Passing through the points (x,y) = (1,6) and (2,10) we find that 10 = b*2+ a*4, and 6 = b*1 + a*1, or 12 = b*2 +a*2. We now equate the two equations for b*2: 10 – a*4 = 12 – a*2.

Moving to opposite side with opposite sign we get 10–12 = a*4–a*2, or -2 = a*2, or -1 = a. With 6 = b+a, this gives b = 7.

So on the parabola, the points (x,y) are connected by the formula $y = 7*x – x^2$. It thus passes through the points (0,0), (1,6), (2,10), (3,12), (4,12), (5,10), (6,6), and (7,0).

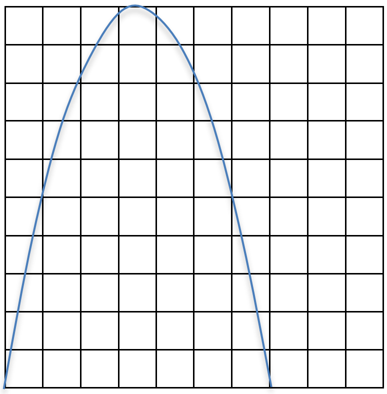


Figure 26. Passing through (0,0) and (1,6) and (2,10), the parabola formula is $y = 7*x – x^2$

Can the intersection points be predicted between the parabola and the two line above?

Can it be predicted that a falling bundle-number b-a*x will have its maximum at the width b/(2*a)?

Skill building. The action is repeated with other rising and falling bundle-numbers to find when they are like and when the falling bundle-numbers tops.

End test. y = 9-2*x and y = x.

MC22. Games Theory and damage control

The total here exists as towers of cubes and dots on a BBBoard. The 'T=?' question is answered by rubber bands as lines on the BBBoard.

In a Game Theory 2x2 zero-sum game two players A and B each have 2 strategies resulting in four different payments from B to A. It is called a zero-sum game since one player's gain is the other player's loss. In some game, if B chooses strategy B1 then the payment to A is 8$ or 2$ if A chooses strategy A1 or A2. And if B chooses strategy B2 then the payment to A is 4$ or 6$ and A chooses strategy A1 or A2. We may show this game by building four towers with cubes.

First, we assume that B chooses strategy B1. If A now p times chooses A2 and n-p times A1 then A's outcome after n rounds will total

T = 2*p + 8*(n-p) = 8*n – 6*p = (8*n – 6*p)/n * n = (8 – 6*p/n) * n.

Which is T1 = 8 – 6*p/n per round, shown on a BBBoard as a line connecting 8 to the left where p is 0, to 2 to the right where p is n.

Next, we assume that B chooses strategy B2. If A now *p* times chooses A2 and *n-p* times A1 then A's outcome after *n* rounds will total

T = 6**p* + 4*(*n-p*) = 4**n* + 2**p* = (4**n* + 2**p*)/*n***n* = (4 + 2**p*/*n*)**n*.

Which is T2 = 4 + 2**p*/*n* per round, shown on a BBBoard as a line connecting 4 to the left where *p* is 0, to 6 to the right where *p* is *n*.

With *p*/*n* as *u* we find the intersection point by equating the two totals: T1 = T2, or 8 – 6**u* = 4 + 2**u*, or 8**u* = 4 = (4/8)*8, or *u* = 4/8 = ½ giving T1 = T2 = 5$.

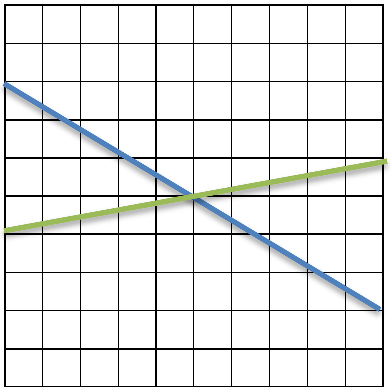


Figure 27. If B choses strategy B1, A will receive between 8 and 2$, else between 4 and 6$

So, if A mixes the strategies 1-to-1by flipping a coin, the average result will be 5$ per round.

Seen from B's side we also get the two lines S1 = 4 + 4**u*, and S2 = 6 – 4**u* that intersect where 4 + 4**u* = 6 – 4**u*, or 8*u* = 2, or *u* = 2/8, or *u* = 1/4 giving S1 = S2 = 5$.

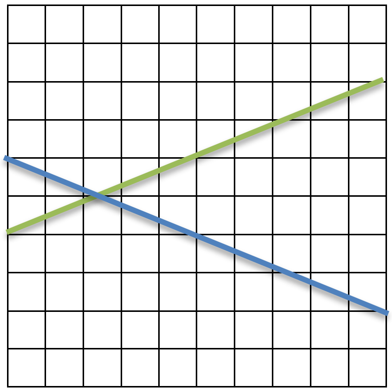


Figure 28. If A choses strategy A1, A will receive between 4 and 8$, else between 6 and 4$

So, if B mixes the strategies 1-to-3 by flipping two coins then the average result again will be 5$ per round. 5$ then is called the value of the game, i.e., the amount B must receive per round to make the game fair with no winner or loser in the long run.

From A's side the 5$ is called the 'maxi-min' value since deviating from it will decrease the value. From B's side the 5$ is called the 'mini-max' value since deviating from it will increase the value.

In a similar game 4$ is changed to 8$. Here the strategy A1 dominates A2 that will always be lower to A. Likewise, the strategy B1 dominates B2 that will always be higher to B. So, here the value of the game is 6. This point is called a saddle point since the payment goes up one way and down the other.

Skill building. The game is repeated with other payments found, e.g., by rolling some dice.

End test. Replace the four payments 8,2,4,6 with 9,3,5,8.

MC23. Simple board games

The goal is to experience how different games may take place on a BBBoard.

• A racetrack. A 4x4 lawn is placed in the middle of a BBBoard. The start- and end-line goes from (5,0) to (5,3). A trip may change zero or one unit in the horizontal and vertical direction. You may touch but not cross the interior or exterior boundaries. If you do so you restart with 0 speed as close to the crossing point as possible. You may cross your opponent's track, but not end in the same point. The race may be repeated with different lawn shapes.

• Survival. You begin in (5,5). You roll a dice and move the number to the right if the number is even, else to the left. You roll again and now move the number up if the number is even, else down. You may touch but not cross the boundary. How many steps can you survive?

• Vertical race. A rubber band splits a BBBoard vertically in the middle. Two players each have three bricks placed at level one. They roll a dice and pick a brick to move the number upwards, and down again if there is a surplus. The winner is the first to have all three bricks at level ten.

The Algebra Square

There are two kinds of numbers in the world, unit-numbers, and per-numbers, which may be unlike or like and which may be reunited. The aim of mathematics is therefore not to 'math', because you can't do that, but to act: 'Re-Unite Un-like & Like Unit-Numbers & Per-Numbers'.

The four operations that unite unlike and like unit-and per-numbers are: addition, multiplication, integration, and power as shown in the Algebra-square above that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations **unite**/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	***T = a + n*** *T – n = a*	***T = a * n*** *T/n = a*
Per-numbers m/s, $/100$ = %	***T* = ∫ *f* dx** *dT/dx = f*	***T = a^b*** *b√T = a* *loga(T) = b*

Figure 29. The Algebra square shows how to unite and split our four number-types, and how to solve equations by moving 'to opposite side with opposite sign'.

Fact and fiction and fake, the three genres of number-models

Once we know how to count and recount totals, and how to unite and split the four number-types, unlike and like unit-numbers and per-numbers, we can actively use this number-language to produce inside tales about outside totals existing in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then and what-then models, or room, rate, and risk models.

Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas. Typically, they model the past and the present. They need to be checked for correctness and units.

Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically, they model the future. They need to be supplied with scenarios building on alternative assumptions.

Fake stories are ‘what-then’ stories that quantify and predict unpredictable qualities by using fake numbers and formulas. Typically, they add without units or hide alternatives. Here, number stories need to be replaced by word stories.

Modeling and de-modeling

The goal is to experience how formulas calculating y from x form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

Modeling means solving an outside problem inside with four steps. First an outside problem is translated to an inside problem. Then the inside problem leads to an inside solution that then is translated to an outside solution, that finally is evaluated to see if another cycle is needed.

A typical example is mixture problems. the outside problems here may ask “2kg at 3$ per kg and 4kg at 5$ per kg total what?” The inside problem places the second information under the first ready to add. The inside solution my then simply add all numbers, which leads to the outside solution “2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 8$/kg”.

This model is not accepted, so another cycle is needed. This time the per-numbers are multiplied to unit numbers before adding, which leads to the outside solution “2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 26$/6kg”. This model is accepted.

De-modeling is the opposite process: It means solving an inside problem outside with four steps. First an inside problem is translated to an outside problem, then the outside problem leads to an outside solution that then is translated to an inside solution, that finally is evaluated to see if another cycle is needed.

A typical example is uniting fractions.

Adding fractions as 1/2 + 2/3 only has meaning when taken of the same unit,

$u = (u/6)*6 = k*6$, where $k = u/6$, and 6 = 2*3

$T = (½ + 2/3)*u = (½ + 2/3)*6*k = (3+4)*k = 7*k = 7*u/6 = 7/6*u$,

So, in this case, 1/2 + 2/3 = 7/6.

• We now model the orbit of a ball sent away with an angle. A constant up-number will give a line that goes up or down or horizontal.

But here gravity makes the up-number decrease so the line curves down as a bended line called a parabola.

We choose the initial angle A determined by tan A = 6.

From (0,0) we assume that the ball takes a ‘1 out, 5 up’ step followed by a ‘1 out, 3 up’ and a ‘1 out, 1 up’, etc., to reach the points (1,5), (2,8), (3, 9), (4,8), (5, 5), (6,0).

Since $y = 0$ for $x = 0$ and for $x = 6$, the formula may contain the two factors (x-0) and (6-x), so a guess could be $y = a*x*(6-x)$.

In the point (1,5) this formula becomes an equation,

$5 = a*1*(6-1)$, or $5 = a*5$, solved by $a = 1$.

So, the parabola formula may be $y = 1*x*(6-x)$, or

$y = -x^2 + 6*x$.

This formula holds when tested on the other points:

8 = -2^2 + 6*2, or 8 = -4 + 12, or 8 = 8, etc.

We find that with 4 as the first up-number, the orbit formula will be $y = -x^2 + 5*x$, etc.

The height after 5 steps is found by the equation *y* = -5^2+6*5 = -25+30 = 5.

The height 8 is reached after *x* steps found by the equation

8 = -*x*^2+6**x*, or *x*^2-6**x*+8 = 0, solved by *x* = 2 and *x* = 4.

The height 10 is never reached since there are no solutions to the equation:

10 = -*x*^2+6**x*, or *x*^2-6**x*+10 = 0.

Instead, that top-point is found in the middle at *x* = 6/2 = 3, giving *y* = -3^2+6*3, or *y* = 9.

To see if it breaks through a roof with the formula *y* = 12-*x*, we equate the two *y*s and get the equation 12 – *x* = -*x*^2 + 6**x*, or *x*^2 - 7*x* + 12 = 0, that is solved for *x* = 3 and *x* = 4.

• We now model the beginning monthly income of a business trying to establish itself at a market. We use the formula y= *x*^2-6*x*+9 where the steps form a parabola curving up when passing the points (0,9), (1,4), (2, 1), (3,0), (4,1), (5,4), and (6,9).

Later the monthly income will change its curvature from up to down until it reaches a maximum level.

So, from *x* = 3 we use a different model that contains the up-numbers 0, 1, 2, 3, 2, 1, 0, 0. This gives a 'logistical' s-shaped curve describing growth with saturation.

When prompted, AI may give a formula for this curve as *y* = 9/(1+25*2^(-1.9**x*))

We see that the up-numbers form a hill. When prompted, AI may give a formula for this curve as *y* = 3/(2^(0,44*(*x*-3)^2))

• A cats and mice cohabitation on an island is an example of a predator-prey model where cats eat mice. We expect a cycle in time since many cats and many mice leads to many cats and few mice, which leads to few cats and few mice, which leads to few cats and many mice, which leads to many cats and many mice once again.

In a model we assume that a mice-population at 7 and 2 will make the cat-population change with 7-5 and 2-5 respectively. Likewise, a cat-population at 7 and 2 will make the mice-population change with 5-7 and 5-2 respectively. We see that initial populations at the level 5 will give a stable model. Here we assume that the initial populations for the cats and the mice are 8 and 1 respectively. The following period the two populations will then be 8 + (1-5) = 4, and 1 + (5-4) = 2 respectively.

Continuing, we see that the cat population will change as 8, 4, 1, 2, 6, 9, 8; and that the mice population will change as 1, 2, 6, 9, 8, 4, 1. This allows the points (8,1), (4,2), etc., to be marked on a BBBoard, showing a cycle continuing again and again. Different initial numbers will give different cycles.

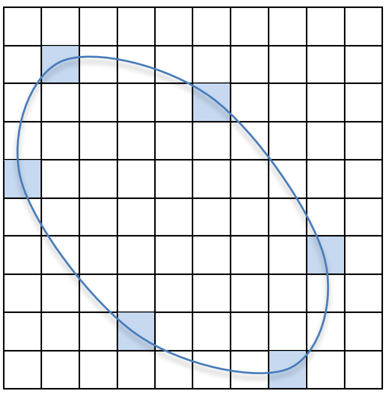


Figure 30. Cats eat mice, so the mice-number decreases, so the cat-number decreases, so the mice-number increases, so the cat number increases, so the mice-number decreases, etc.

• We now model cyclic movements up and down as observed in nature with day and night, with summer and winter, and with tide in an ocean. A cyclic movement may be created by the up-numbers +2, +1, +0, -1, -2, -2, -1, +0, +1 +2.

Beginning at the point (0,5), AI may be prompted to give a formula for this curve as

y = 5+3*sin (0.63*x*).

• Saving money may take place at home or in a bank. At home the terminal capital *c* after *n* months will be *c = b + a*n*, where *b* is the initial capital, and *a* is the change-number per month. In a bank, the terminal capital after *n* months will be *c* = *b**(1+*r*)^*n* where *r* is the change-percent per month. Combining the two in a bank, the terminal capital *C* may be found by the formula *C*/*a* = *R*/*r* where *R* is the total interest rate including the compound interest, 1+*R* = (1+*r*)^*n*. This capital may be used as an installment plan to pay out a debt *D* that has grown to *E* = *D**(1+*R*) in the same period.

• If an interest rate at 100% is split in 12 portions the total interest is found from the equation 1+*R* = (1+1/12)^12 = 2.613 so that *R* = 1.613 = 161.3% = 100% plus 61.3% as additional compound interest. This leads to the Euler number *e* = (1+1/*n*)^*n* = 2.7183 for n large, which shows that the additional compound interest cannot surpass 71.8% when splitting up 100%.

• Biological populations typically grow exponentially with a constant periodical rate giving a constant doubling time. This may be shown on a BBBoard where the vertical numbers are in tens. Beginning with ¼, a doubling sequence will be ¼, ½, 1, 2, 4, 8, 16, 32, 64, 128.

Alternatively, a capital may decrease by always taking away half of what is left. Beginning with 8 this gives a halving sequence 8, 4, 2, 1, ½, ¼ .

This exponential decay may be recognized as a mirror of the exponential growth.

• On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angels and its three sides.

We see that these 7 numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle the angle is predicted by the formula tan *A* = 6/10, which on a calculator gives *A* = 31.0 degrees. And the area is ½*6*10 = 30. And that the diagonal d is found by squaring: *d*^2 = 10^2 + 6^2 = 136, giving *d* = √136 = 11.7.

• On a BBBoard, twice rolling 2 dices may suggest we go to the two points (3,6) and (4,3) that then constitute one side in a square. We now may find the area of the square and the intersection point of the two diagonals. We notice that in the slopes of the sides the out- and up-number change places, and one changes the sign also.

• Optimizing income under constraints (also called 'Linear Programming'). At a fair, a class sells caps and shirts. They may buy at most 6 boxes with caps and 4 boxes with shirts that each cost 1unit. Their budget is 8units, and their income is 1unit per shirt-box and 2 units per cap-box. How can they optimize the income?

On a BBBoard a horizontal and a vertical rubber band shows the limit on the shirts and on the caps. A line connecting (0,8) and (8,0) shows the budget-line not to be passed. A line connecting (0,10) and (5,0) shows the 10unit income-line that is moved to the right until (6,2) where the first constraint will be violated. So the calls should buy 6 boxes with caps and 2 boxes with shirts, which will give them an income at 2*6 + 1*2 or 14 units.

Three footnotes

The total here exists as rectangular bundle-number on a BBBoard.

The goal is to experience the content of three calculation laws.

The commutative law: The order does not matter, $a*b = b*a$

The distributive law: When adding, like units may be bracket out, $a*c + b*c = (a + b)*c$

The associative law: Bracket may be moved at will, $a*(b*c) = (a*b)*c$

On a BBBoard two rubber bands mark 6 3s. Turning the board a quarter round we have 3 6s thus illustrating that 6*3 = 3*6.

A third rubber band split the 6 3s in 4 3s and 2 3s to illustrate that 4*3 + 2*3 = (4+2)*3.

With cubes 2 3s 4 times gives a Total of (2*3)*4. Turning it over, twice we have 3 4s, thus illustrating that (2*3)*4 = 2*(3*4),

Teacher education

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating internal predicates by asking the outside subject itself instead of an instructor.

This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for the primary school, and C2, A2, T2 and S2 is for the secondary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team.

Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation.

The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

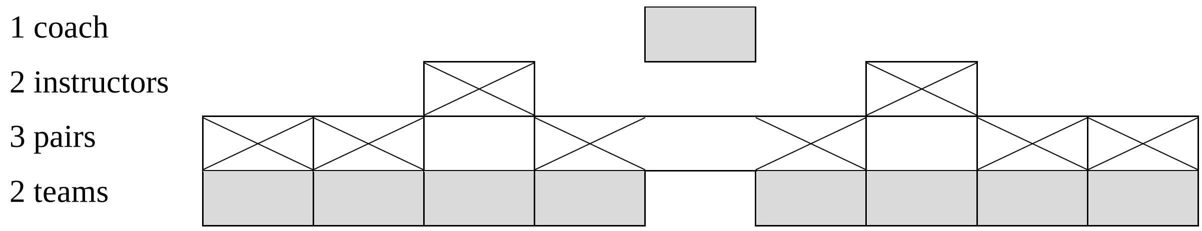


Figure 31. PYRAMIDeDUCATION with 2 teams as 3 pairs and 2 instructors, plus a coach

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles?

The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2B2 3s.

Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Additional material may be found as MrAlTarp YouTube videos.

How different is the difference?

Digits now are no longer symbols as letters, but icons with as many sticks as they represent. 3 now is called '1B0 3s' or '0B3 tens'. Ten, eleven end twelve now are also called 'one-bundle-zero', 'one-bundle-one', and 'one-bundle two'. And hundred and thousand are also called 'bundle-bundle' and 'bundle-bundle-bundle'.

Multidigit numbers no longer occur without units since with units, 23 now is 2B3 thus making the place value system unneeded. Calculations with overloads and underloads give bundle-numbers with units a flexibility that makes carrying and borrowing unneeded, e.g., 46+37 = 4B6+3B7 = 7B13 = 8B3 = 83. And 86 – 37 = 8B6 – 3B7 = 5B-1 = 4B9 = 49.

Addition now depends on the units so 2+3 is not 5 by necessity. 2weeks + 3weeks = 5weeks, but 2weeks + 3days = 17days. So, without a unit, 3 does not exist, only with a bundle-unit as, e.g., 0B3 tens, or 1B0 3s, or 1B1 2s, or 1B-1 4s, or 1B-2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., '2+3' = (1B0 + 1B1) 2s = 2B1 2s, or '2+3' = (1B-1 + 1B0) 3s = 2B-1 3s = 1B2 3s. Likewise with subtraction '9-6' = (1B3 -1B0) 6s = 0B3 6s = '3', or '9-6' = (1B0 -1B-3) 9s = 0B--3 9s = 0B3 9s, showing that minus times minus must be plus.

Also, addition now is not well-defined since 2 3s and 4 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and 6*8 is not 48 by necessity. Instead, 6*8 exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or in tens: 6 8s is 5B3 9s, and 4B8 tens.

Division now is different, since 8/2 has different meanings in time and space by meaning '8 split in 2 in time', but '8 split in 2s in space' when recounting 8 in 2s.

Solving equations now is different. The equation '*u**2 = 8' asks "How many 2s in 8?" which of course is found by recounting 8 in 2s as 8 = (8/2)*2, so that the solution is *u* = 8/2 that is found by 'moving to opposite side with opposite sign', which follows the formal definition: 8/2 is the number *u* that multiplied with 2 gives 8, if *u**2 = 8 then *u* = 8/2. So, the balancing method now is unneeded.

Thus, no longer equations are seen as two equivalent numbers-names that remain equivalent if the same operation is performed on both. And no longer are they transformed by using the communicative, associative, and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements.

And we no longer use the neutralizing 'do the same to both sides' weight-method to solve the equation 2**x* = 8 saying: 2**x* = 8; (2**x*)*½ = 8*½; (*x**2)*½ = 4; *x**(2*½) = 4; *x**1 = 4; *x* = 4

The multiple calculation 2+3*4 no longer is 14 by definition or by the 'PEMDAS' rule. With units, 2+3*4 exits as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation '2**a* + 3**a* = (2+3)**a*' no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer 'go over one', instead a per-number links the two units by recounting: with 4$ per 5kg, or 4$/5kg, 16$ = (16/4)*4$ = (16/4)*5kg = 20 kg.

Fractions no longer are numbers by themselves, instead they are per-number with like units, 3meter/4meter = ¾, 3 meter/100meter = 3/100 = 3%. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two apples plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., ½ + 2/3 = (1+2)/(2+3) = 3/5 in this case, and not 7 red of 6 apples as mathematism teaches.

Trigonometry no longer must wait to after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal.

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait to secondary school since Bundle-Bundles are squares that lead directly to the question ‘how to square a rectangle’ that provides a double split square containing the three parts of a quadratic equation.

The simplicity of the Algebra Square will no longer be hidden. And no longer will models be seen as mere approximations but as tales with three genres, fact, and fiction, and fake.

Overview over the differences between Essence- math and Existence-math

	Essence-math, mathematism	**Existence-math, Many-math**
Digits	Symbols	Icons
345	Place value system	T = 3BB 4B 5, BB = B^2, BBB = B^3
Operations	Functions, order: + – x / ^	Icons, opposite order: ^ / x – +
3 + 4	3 + 4 = 7	Meaningless without units
3 * 4	3 * 4 = 12	3*4 = 3 4s, may be recounted to 1.2 tens
9 = ? 2s	Meaningless, only ten-counting	9 = 3B3 = 5B-2 = 4B1 = 4½ 2s
8 = ? 2s	Meaningless, only ten-counting	8 = (8/2)*2, T = (T/B)*B, proportionality
2*u = 8	2*u = 8 so, (2*u)*½ = 8*½ so, (u*2)*½ = 4 so, u*(2*½) = 4 so, u*1 = 4, so, u = 4	2*u = 8 = (8/2)*2 so, u = 8/2
6*7 = ?	eh 44? eh 52? eh 42? OK	6*7 = (B-4)*(B-3) = (10-4-3)*B + 4*3 = 3B12 = 4B2 = 42
4kg =5$, 6kg =?	1kg = 5/4$, 6kg = 6*5/4$	6kg = (6/4)*4kg = (6/4)*5$
1/2 + 2/3 = ?	1/2 + 2/3 = 3/6 + 4/6 = 7/6	1/2*2 + 2/3*3 = 3/5*5
2*3 + 4*5	2*3+4*5 = 6+4*5 = 10*5 ?	2*3+4*5 = 3B2 8s (next-to) or 5B1 5s, or 8B2 3s (on-top)
6 + 9 = ?	6 + 9 = 15 5 – 6	2B3 6s or 2B-3 9s or 2 ½B(1+4) = 1B5 tens
Tangent = ?	tan = sine/cosine	raise = (raise/run)*run, tan = raise/run

Conclusion

We began by observing the difference between ‘mathematism’ adding without units, which is true inside but seldom outside, and ‘Many-math’ instead using bundle-numbers with units inspired by how the uneducated child sees the outside existing fact Many. We then explored the consequences of letting existence come before essence by letting counting and recounting come before adding. Finaly, we formulated a sequence of micro-curricula on how Many-math may be learned in school by working with things and actions on a 2dimensional Bundle-Bundle-Board. But, will this allow the learner to learn mathematics or even to be numerate as wished by the UN Sustainable Development Goals?

Apparently, different definitions of 'numerate' exist where existence and essence have different order. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

In their common history, England once colonized America. So, the difference in the definitions is interesting. The former uses the passive term 'being' where the latter uses the active term 'having'. The former connects the definition to the inside essence of mathematics while the latter connects it directly to the outside existence of numbers. The choice thus is: shall existence precede essence as philosophical Existentialism holds, or shall essence be allowed to colonize existence with a 'no-unit regime' to use a Foucault-phrase? Maybe it is time to see if children stay numerate if their own 2D bundle-numbers with units are not colonized by 1D line-numbers without units. Maybe it is finally time for a Kuhnian paradigm shift in number-language education. Therefore, think things. Or, in the Viking version: "Derfor, tænk ting".

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05. Many before Math may Decolonize Math, a Video

A presentation of a folder and a workshop

Folder: “Decolonize Mathematics with the child’s own Bundle-numbers with units,

http://mathecademy.net/bundle-bundle-numbers-with-units/

Workshop: “Flexible Bundle Numbers Develop the Childs Innate Mastery of Many”

https://youtu.be/z_FM3Mm5RmE

Based on the papers

Tarp, A. (2004). Pastoral Power in Mathematics Education. Paper accepted for presentation at the Topic Study Group 25. The 10th Int. Conference on Mathematics Education, ICME, 2004.

Tarp, A. (2016). From Essence to Existence in Mathematics Education. *Philosophy of Mathematics Education Journal No. 31* (November 2016)

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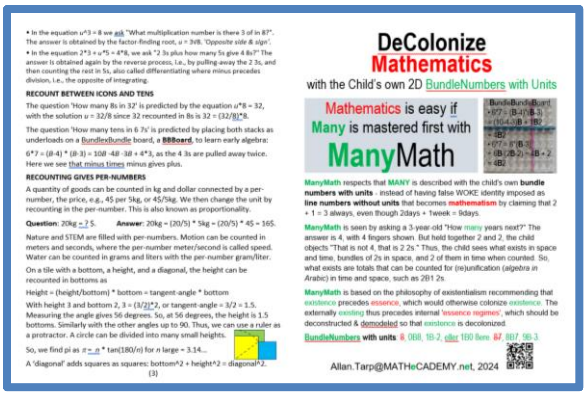
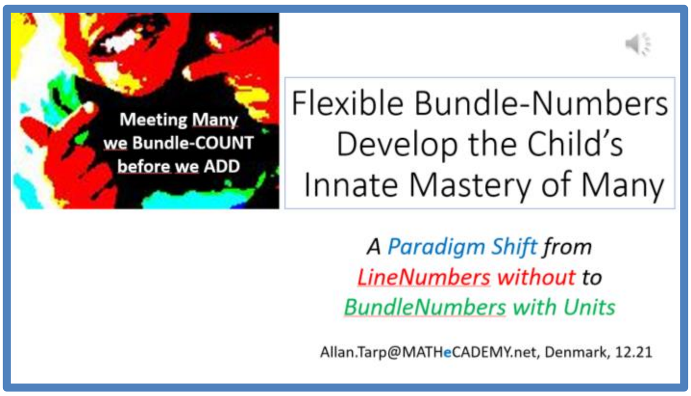
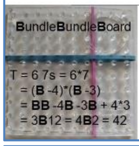




Many before Math, Math decolonized by the child’s own BundleBundle-Numbers with units https://youtu.be/uV_SW5JPWGs

Many before Math!

Mathematics DeColonized
by the Child's own 2D
BundleBundle Numbers



Make Math Easy with Many-Math
where Existence precedes Essence

Allan.Tarp@gmail.com
MATHeCADEMY.net
Spring 2024

My first math lesson

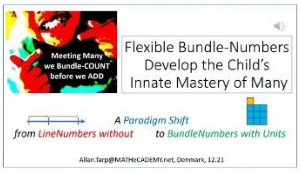
Hello, my name is Allan. In my first lesson I said "Welcome children, I am your teacher in mathematics, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So ..."

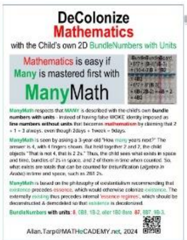
Then a child stopped me, holding four fingers together two by two: "Mister teacher, here is one, and here is one. You say it is two, but we can all see, that it is four". The child separated the fingers, and then held three fingers together on both hands before separating them. "And, here we see that two times three gives six. So, multiplication can be trusted, but not addition. Therefore, please take away your number line addition with its flat-earth-math. Instead, please help us with the numbers we bring to school, multiplication bundle-numbers as 2 3s and 4 5s that we can see on this ten-by-ten peg board, a Bundle-Bundle board, or a BBBoard. And that we would like to add next-to as eights, or on-top as 3s or 5s. If we add them next-to, we add areas, which my uncle calls integral calculus, and if we add them on-top the units must be changed to the same unit, which my uncle calls proportionality. He says it is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school instead of having them colonized by your line numbers without units.

We know that you prefer to bundle in tens, and in ten tens, and in ten ten tens, but we like to bundle also in 2s, in 3s, in 4s, in half tens, etc. And we know that you have not been taught this and that the textbook doesn't teach it, but don't worry, we will teach you. Or better, instead of you colonizing our ways let us find out together what math may grow from our bundle-numbers with units. My uncle is a philosopher, and he calls it existentialism when you treat existence before essence.

So, let us begin with the fingers on a hand. You only see the essence, five, but we see all the ways the fingers may exist as five. That is, as 5 1s, as 1 5s or 1 Bundle 0 5s, as 1B 1 4s, as 1B 2 3s or 2B less 1 3s, as 1B 3 or 2B1 or 3B less 1 2s, or as 1BB 0B 1 2s."

What happened next is seen in this workshop↓ & this folder→



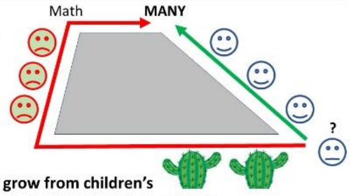


Workshop: Flexible Bundle Numbers
Develop the Childs Innate Mastery of Many
https://youtu.be/z_FM3Mm5RmE

Mathematics is easy if Many is mastered first with ManyMath
http://mathecademy.net/bundle-bundle-numbers-with-units/

The Goal of Math Education, is that well Defined? What comes first: to master Math - or to master Many?

All say:
The goal is to master Math,
to later master Many.
But Math is hard!
Why not first master Many,
to later master Math?



So, we may ask:

What Mathematics may grow from children's innate mastery of Many, as developed before school?

What to master first, Many or Mathematics? Many, because that makes math just so easy

- Many-math respects that MANY is described by the child's own bundle-numbers with units. Instead of having a false identity imposed with line-numbers without units that becomes 'Mathema-tism' by claiming that 2 + 1 = 3 always, despite the fact that 2days + 1week = 9days.
- Many-math is seen by asking a 3-year-old "How many years next?" Typically, the answer is 4, with 4 fingers shown. But held together 2 and 2, the child says "No, that is not 4, that is 2 2s." Thus, the child sees what exists in space and time, bundles of 2s in space, and 2 of them in time when counted. So, what exists are totals that can be counted and united in time and space, like T = 2 2s.
- Many-math is based on the philosophy called existentialism that holds that existence precedes essence.
- That is, what exists outside precedes what is said about it inside.

Aha, with existence before essence counting comes before adding

"No, that is not four, that is two twos". This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time.

The number 'two' thus exists both in space and in time.

In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Either horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or to 2B-1 3s with B for bundle.

Here 1 plus 1 does not add up to 2 since the units are not the same, whereas one 2-bundle + one 2-bundle may both be two 2-bundles or one 4-bundle, but not 2 4-bundles.

In time, 2 exists together with the unit that was counted, as 2 units, as a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit.

So, as space-numbers, 2+3 is 5, whereas as time-numbers 2+3 is undecided until their units are known.

The child thus opens our eyes for a different mathematics that, freed from its present essence-bounds, may return to its original identity as a natural science about Many in space and in time. This decolonization allows us to develop a natural number-language to communicate about outside existence instead of about inside essence only.

If existence comes before essence, then counting comes before adding, which is new since normally we just get numbers to add."

Kids use units (2 3s), Schools don't. So Math Differs INSIDE & OUTSIDE the 'NoUnitMath' GreenHouse

OUTSIDE	INSIDE
ManyMath	NoUnitMath GreenHouse
2+1 = ??	2+1 = 3 always
depends on the units	
2weeks + 1day = 15days	MatheMatism
2m + 1m = 3m	*True inside,*
2km + 1mm = 2km	*but seldom outside*
20$ + 10% = 22$	

*2+3 is 5 sometimes. But 2*3 is 6 always. Why? Because here 3 is the unit!*

Many-math uses bundle numbers with units

With units, there are only four kinds of numbers in the world: unit-numbers and per-numbers that can be unlike or like, and that may be reunited. The aim of mathematics is therefore not to 'math', because math is not an action word, but to act:

"ReUnite Unlike & Like Unit-Numbers & Per-Numbers"

- 3$ and 2$ are unlike unit-numbers where the calculation 3+2 = 5 predicts the result of uniting them.
- 3 times 2$ are like unit-numbers where the calculation 3*2 = 6 predicts the result of uniting them.
- 3 times 2% are like per-number where the calculation 102%^3 = 106.12% predicts the result of uniting them to 6% and 0.12% extra in 'compound interest'.
- Unlike per-numbers we meet in mixtures, 2kg at 3$/kg plus 4kg at 5$/kg. Here, unit-numbers 2 and 4 unite directly, while the per-numbers 3 and 5 must first be multiplied to unit-numbers before uniting as areas, called integration, where multiplication precedes plus: T = (2+4) kg at (2*3 + 4*5)$, or 6 kg at 26/6 $/kg.

2KG AT 3$/KG
4KG AT 5$/KG
6KG AT 8$/KG

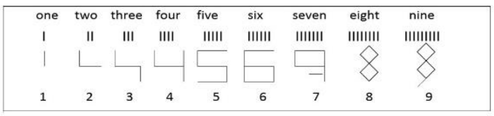
The algebra square shows the 4 types of numbers in the world, unlike & like unit- & per-numbers

- The Arabic word Algebra means to reunite, that is, to unite and split. Numbers are united in four ways: Addition unites unlike unit-numbers, multiplication unites like unit-numbers, integration unites unlike per-numbers, and power unites like per-number, as you add 5% by multiplying by 105%.
- The opposite is to split into: subtraction splits into unlike unit-numbers, division splits into like unit-numbers, differentiation splits into unlike per-numbers, and the factor-finding root and factor-counting logarithm splits into like per-numbers.

Unite *Split into*	Unlike	Like
Unit-numbers	T = a + b	T = a*b
(meter, second)	*T – b = a*	*T/b = a*
Per-numbers	T = ∫ f dx	T = a^b
(m/sec, m/100m = %)	*dT/dx = f*	*b√T = a logₐ(T) = b*

Digits are Icons: IIII → IIII → 4 →

Children love making number-icons of cars, dolls, spoons, sticks.
Changing four ones to one fours creates a 4-icon with four sticks.
An icon contains as many sticks as it represents, if written less sloppy.
Once created, icons become **UNITS** when counting in bundles, as kids do.



Totals are counted in Bundles with units

- 5 fingers are counted as '1 **Bundle** 2' 3s, short as '1**B**2' 3s, or simply '12' 3s.
- Ten fingers are counted as '3**B**1' 3s, or '**1BundleBundle** 0**Bundle** 1' 3s, or '1**BB** 0**B** 1 3s, or simply '101' 3s.
- A total, T, is counted in a unit, for example, T = 3 fours, or T = 3 times 4.
- This is a number-narrative, with a subject, 'T', a verb, 'equal to', and a predicate, '3 times 4'.
- T = 345 has omitted the units, T = 3**BB**4**B**5, where the bundle **B** = ten.
- Counting 5 in 2s is done in 3 ways: **normal**, or '**over-load**,' or '**under-load**'

This eases calculations:

45 + 27 = 4**B**5 + 2**B**7 = 6**B**12 = 7**B**2 = 72
7 * 56 = 7 * 5**B**6 = 35**B**42 = 39**B**2 = 392
392 /7 = 39**B**2 /7 = 35**B** 42 /7 = 5**B**6 = 56
Place values and carrying are unneeded

5 = 2**B**1 = 1**B**3 = 3**B**-1 2s
Likewise: T = 47 = 4**B**7 = 3**B**17 = 5**B**-3 tens

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Bundle-Bundles are squares

On a **BBB**oard we see, that all the bundle-bundles are squares, 2 2s, 3 3s, 4 4s, 5 5s, etc.

- 2 2s need 2 more 2s, and 1 corner, to be 3 3s.
  - So, 1 **BB** 3s is 1**BB** 2**B** 1 2s, or (2+2)**B** 1 2s or 4**B** 1 2s, or 0**B** 9 tens
- 4 4s need 2 more 4s, and 1 corner, to be 5 5s.
  - So, 1 **BB** 5s is 1**BB** 2**B** 1 4s, or (5+1)**B** 1 4s or 6**B** 1 4s, or 2**B** 5 tens

Going the other way, we see that 1 **BB** 4s comes if we from 1 **BB** 5s pull-away 2 bundles, and add the corner that we pull-away twice, which shows that minus times minus must be plus

- 1 **BB** 4s = 1**BB** -2**B** 1 5s = (5-2)**B** 1 5s = 3**B** 1 5s = 1**B** 6 tens
- 1 **BB** 2s = 1**BB** -2**B** 1 3s = (3-2)**B** 1 3s = 1**B** 1 3s = 0**B** 4 tens

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Divide & Multiply & Subtract & Add may be 'de-modeled' as Icons also

- From 9 **PUSH** away **4s** we write 9/4 iconizing a broom, called *division*.
- 2 times **LIFT** the **4s** to a stack we write 2x4 iconizing a lift called *multiplication*.
- "From 9 **PULL** away 2 **4s** to find un-bundled" we write 9 – 2x4 iconizing a rope, called *subtraction*.
- **UNITING** next-to or on-top we write **B**+**C** iconizing the two directions, called *addition*.

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Shifting Units Creates a Recount-Formula

8 = (8/2)x2
T = (T/B)xB

Shift unit from **1s** to **2s**:	8 = ? **2s**
Bundle-counting:	8 = 4 **2s** = 4x2
Predict by a calculation:	8/2 = 4
Recount result:	8 = (8/2)x2
Recount-Formula:	T = (T/B)xB "*T contains T/B Bs*"

ux2 = 8 = (8/2)x2 — Equations — Proportionality
u = 8/2

Move: OPPOSITE Side & Sign

Shifting unit	y = k * x
Linearity	Δy = (Δy/Δx) * Δx = m * Δx
Local linearity	dy = (dy/dx) * dx = y' * dx
Trigonometry	a = (a/b) * b = tanA * b
Trade	$ = ($/kg) * kg = price * kg
STEM	meter = (meter/sec) * sec = speed * sec

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Calculators Predict Counting Results Bundle-Counting a Total of 9 in 2s

9/2 → 4.some; 9 – 4x2 → 1

Acting	Predicting
9	From 9, 9/2 times, push away **2s** (9/2 → 4.5)
bundled in **2s** with 1 unbundled	From 9, pull away 4 **2s**, leaves 1. Calculator prediction: T = 9 = 4**B**1 **2s** (9-4x2 → 1)
stacked as 4x2 with 1 unbundled placed on-top: 4.1**B**, 4½**B** 2s, 5**B**-1	The unbundled are placed on-top • separated by a decimal point, 4.1 **2s** • counted in bundles as 1 = (1/2) x 2 giving 4½**B** **2s** • counting what is missing in a full bundle, 5**B**-1 **2s** So, inside decimals, fractions, and negatives are rooted in how we outside see the unbundled.

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Splitting

u + 2 = 8	u x 2 = 8	2^u = 8	u^8 = 2
u = 8 – 2	u = 8/2	u = log₂(8)	$u = \sqrt[8]{2}$

The reverse of uniting is splitting, which is predicted by reverse calculations called equations, where we use the letter *u* for the unknown number.

- In the reverse calculation (equation) '*u* + 2 = 5' we ask "What is it that united with 2 gives 5?". The answer, of course, is obtained by the reverse process, by now pulling-away the 2 from 5 by minus, *u* = 5 – 2. The unknown number is thus found by moving the known number to the opposite side with the opposite calculation sign, the "OPPOSITE SIDE & SIGN" method.
- In the equation *u**2 = 6, we ask "How many 2s are in 6?". The answer, of course, is obtained by counting 6 in 2s, 6 = (6/2)*2, so *u* = 6/2. So again, by the reverse process, by pushing 2s away. So, again 'opposite side & sign'.
- In the equation 2^*u* = 8, we ask "How many factors 2 are there in 8?". The answer is obtained by the factor-counting logarithm, *u* = log2(8). So, again 'opposite side & sign'.

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Equations are solved by moving the numbers to OPPOSITE side with OPPOSITE calculation sign

In the equation *u*^3 = 8 we ask: "What multiplication number is there 3 of in 8?". The answer is obtained by the factor-finding root, *u* = 3√8. So, again 'opposite side & sign'.

- In the equation 2*3 + *u**5 = 4*8, we ask "2 3s plus how many 5s give 4 8s?" The answer is obtained again by the reverse process, i.e., by pulling-away the 2 3s, and then counting the rest in 5s, also called differentiating where minus precedes division, i.e., the opposite of integrating.

u + 2 = 8	u x 2 = 8	u^8 = 2	2^u = 8
u = 8 – 2	u = 8/2	$u = \sqrt[8]{2}$	u = log₂(8)

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A hymn to equations

Equations are the best we know, they're solved by isolation. But first the brackets must be placed around multiplication. We change the sign, and take away, so only u itself will stay. We just keep on moving, we never give up. So feed us equations, we don't want to stop.	3**u* + 2 = 14 (3**u*) + 2 = 14 3**u* = 14 – 2 *u* = (14 – 2)/3 *u* = 4

Walking the equation 3*u + 2 = 14, in time:

u → (*3) **3*u** → (+2) **3*u + 2**
4 = (14-2)/3 ← (/3) 14-2 ← (-2) 14

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Recount between icons and 10s

- The question 'How many 8s in 32' is predicted by the equation *u**8 = 32, with the solution *u* = 32/8 since 32 recounted in 8s is 32 = (32/8)*8.
- The question 'How many tens in 6 7s' is predicted by placing them both as stacks with underload on a BundleBundle board, a BBBoard, to learn early algebra:
- 6*7 = (**B**-4) * (**B**-3) = 10**B** - 4**B** - 3**B** + 4*3, as the 4 3s are pulled away twice.

Act	Predict
4; 6 *B*-4; 7 *B*-3; 3	T = 6 **7s** = ? **tens** = 6 x 7 = (*B*-4) x (*B*-3) = ($\frac{B-4}{B-3}$) = *BB* - 3*B* - 4*B* - - 4 x 3 = 3*B* + 1*B*2 = 4*B*2 = 42, so 6 **7s** is 4*B*2 **tens** *- - is + since it is pulled away twice*

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Increasing the base will decrease the height. A Total may be STACK'ed, TEN'ed or SQUARE'd

STACK'ed	TEN'ed	SQUARE'd
4 x 5	2 x 10	4 x 4 (√20, t, 4)

The surplus 4 1s is shared by the two 4xt stacks:
8*t = 4, or t = 4/8 = 0.5
So √20 ≈ 4.5
Calculator: √20 = 4.472

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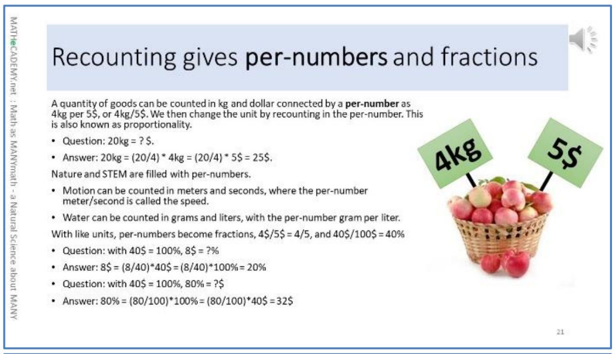
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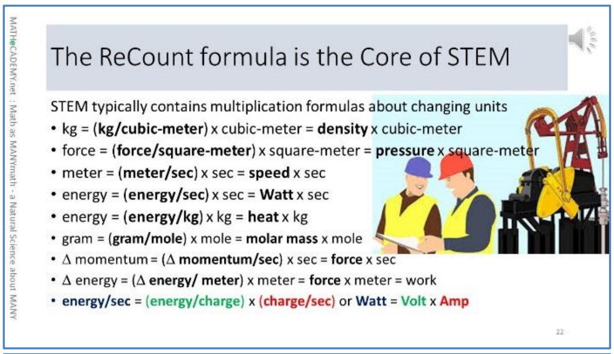
Recounting gives per-numbers and fractions
A quantity of goods can be counted in kg and dollar connected by a per-number as 4kg per 5$, or 4kg/5$. We then change the unit by recounting in the per-number. This is also known as proportionality.
• Question: 20kg = ? $.
• Answer: 20kg = (20/4) * 4kg = (20/4) * 5$ = 25$.
Nature and STEM are filled with per-numbers.
• Motion can be counted in meters and seconds, where the per-number meter/second is called the speed.
• Water can be counted in grams and liters, with the per-number gram per liter.
With like units, per-numbers become fractions, 4$/5$ = 4/5, and 40$/100$ = 40%
• Question: with 40$ = 100%, 8$ = ?%
• Answer: 8$ = (8/40)*40$ = (8/40)*100% = 20%
• Question: with 40$ = 100%, 80% = ?$
• Answer: 80% = (80/100)*100% = (80/100)*40$ = 32$
4kg
5$
21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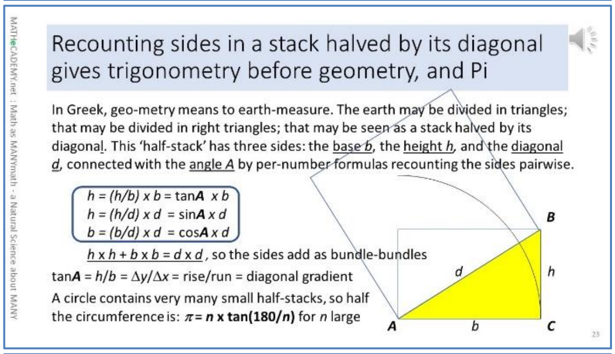
The ReCount formula is the Core of STEM
STEM typically contains multiplication formulas about changing units
• kg = (kg/cubic-meter) x cubic-meter = density x cubic-meter
• force = (force/square-meter) x square-meter = pressure x square-meter
• meter = (meter/sec) x sec = speed x sec
• energy = (energy/sec) x sec = Watt x sec
• energy = (energy/kg) x kg = heat x kg
• gram = (gram/mole) x mole = molar mass x mole
• Δ momentum = (Δ momentum/sec) x sec = force x sec
• Δ energy = (Δ energy/ meter) x meter = force x meter = work
• energy/sec = (energy/charge) x (charge/sec) or Watt = Volt x Amp
22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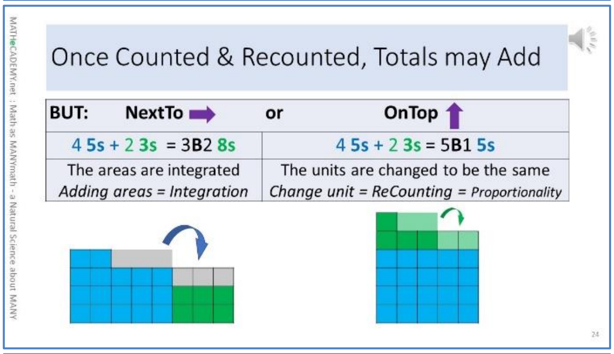
Recounting sides in a stack halved by its diagonal gives trigonometry before geometry, and Pi
In Greek, geo-metry means to earth-measure. The earth may be divided in triangles; that may be divided in right triangles; that may be seen as a stack halved by its diagonal. This 'half-stack' has three sides: the base b, the height h, and the diagonal d, connected with the angle A by per-number formulas recounting the sides pairwise.
h = (h/b) x b = tanA x b
h = (h/d) x d = sinA x d
b = (b/d) x d = cosA x d
h x h + b x b = d x d, so the sides add as bundle-bundles
tanA = h/b = Δy/Δx = rise/run = diagonal gradient
A circle contains very many small half-stacks, so half the circumference is: π = n x tan(180/n) for n large
A
B
C
b
h
d
23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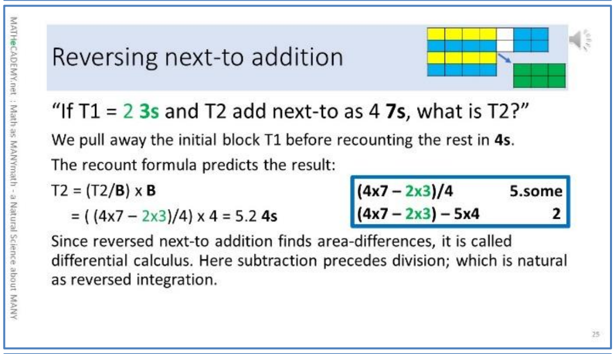
Once Counted & Recounted, Totals may Add
BUT: NextTo or OnTop
4 5s + 2 3s = 3B2 8s
4 5s + 2 3s = 5B1 5s
The areas are integrated
Adding areas = Integration
The units are changed to be the same
Change unit = ReCounting = Proportionality
24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Reversing next-to addition
"If T1 = 2 3s and T2 add next-to as 4 7s, what is T2?"
We pull away the initial block T1 before recounting the rest in 4s.
The recount formula predicts the result:
T2 = (T2/B) x B
= ((4x7 – 2x3)/4) x 4 = 5.2 4s
(4x7 – 2x3)/4 5.some
(4x7 – 2x3) – 5x4 2
Since reversed next-to addition finds area-differences, it is called differential calculus. Here subtraction precedes division; which is natural as reversed integration.
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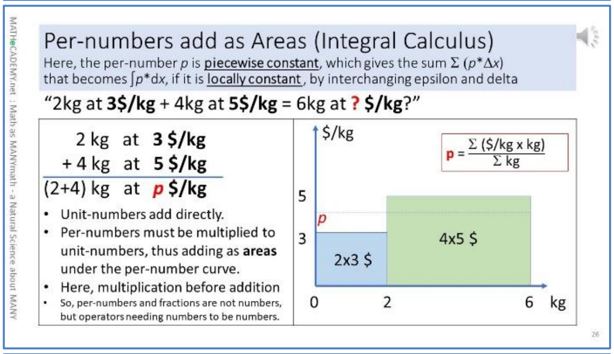
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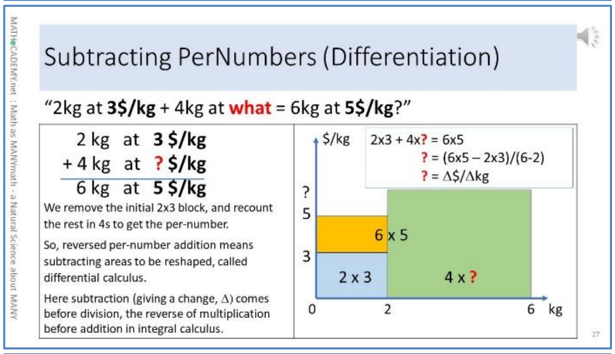
Per-numbers add as Areas (Integral Calculus)
Here, the per-number p is piecewise constant, which gives the sum Σ (p*Δx) that becomes ∫p*dx, if it is locally constant, by interchanging epsilon and delta
"2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg?"
2 kg at 3 $/kg
+ 4 kg at 5 $/kg
(2+4) kg at p $/kg
• Unit-numbers add directly.
• Per-numbers must be multiplied to unit-numbers, thus adding as areas under the per-number curve.
• Here, multiplication before addition
• So, per-numbers and fractions are not numbers, but operators needing numbers to be numbers.
$/kg
p = Σ ($/kg x kg) / Σ kg
5
p
3
2x3 $
4x5 $
0
2
6
kg
26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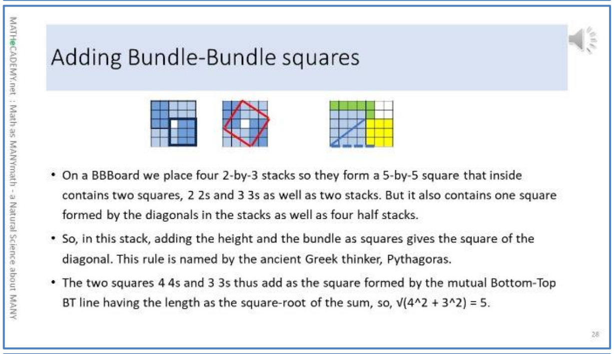
Subtracting PerNumbers (Differentiation)
"2kg at 3$/kg + 4kg at what = 6kg at 5$/kg?"
2 kg at 3 $/kg
+ 4 kg at ? $/kg
6 kg at 5 $/kg
We remove the initial 2x3 block, and recount the rest in 4s to get the per-number.
So, reversed per-number addition means subtracting areas to be reshaped, called differential calculus.
Here subtraction (giving a change, Δ) comes before division, the reverse of multiplication before addition in integral calculus.
$/kg
2x3 + 4x? = 6x5
? = (6x5 – 2x3)/(6-2)
? = Δ$/Δkg
?
5
3
6 x 5
2 x 3
4 x ?
0
2
6
kg
27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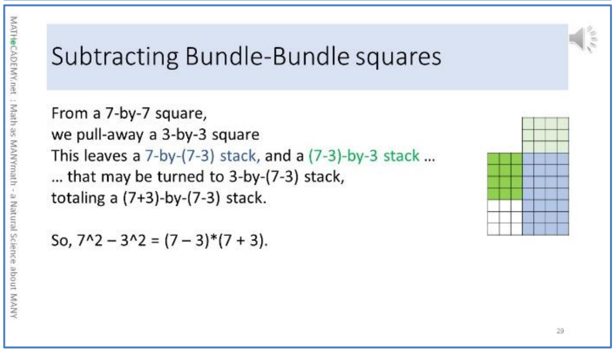
Adding Bundle-Bundle squares
• On a BBBoard we place four 2-by-3 stacks so they form a 5-by-5 square that inside contains two squares, 2 2s and 3 3s as well as two stacks. But it also contains one square formed by the diagonals in the stacks as well as four half stacks.
• So, in this stack, adding the height and the bundle as squares gives the square of the diagonal. This rule is named by the ancient Greek thinker, Pythagoras.
• The two squares 4 4s and 3 3s thus add as the square formed by the mutual Bottom-Top BT line having the length as the square-root of the sum, so, √(4^2 + 3^2) = 5.
28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Subtracting Bundle-Bundle squares
From a 7-by-7 square,
we pull-away a 3-by-3 square
This leaves a 7-by-(7-3) stack, and a (7-3)-by-3 stack ...
... that may be turned to 3-by-(7-3) stack,
totaling a (7+3)-by-(7-3) stack.
So, 7^2 – 3^2 = (7 – 3)*(7 + 3).
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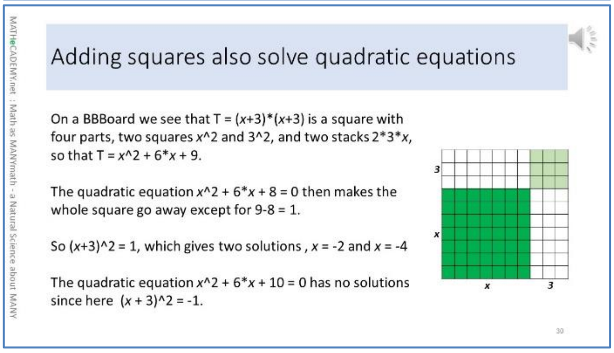
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Adding squares also solve quadratic equations
On a BBBoard we see that T = (x+3)*(x+3) is a square with four parts, two squares x^2 and 3^2, and two stacks 2*3*x, so that T = x^2 + 6*x + 9.
The quadratic equation x^2 + 6*x + 8 = 0 then makes the whole square go away except for 9-8 = 1.
So (x+3)^2 = 1, which gives two solutions , x = -2 and x = -4
The quadratic equation x^2 + 6*x + 10 = 0 has no solutions since here (x + 3)^2 = -1.
3
x
x
3
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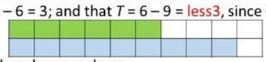
Subtracting Numbers with Like Units, 9-6

Inside the 'no-unit-math' paradigm, numbers subtract **serial** next-to on the number line. We find the result by counting backwards.

Outside, in the 'unit-math' paradigm, numbers subtract **parallel** on-top.
We see that $T = 9 – 6 = 3$; and that $T = 6 – 9$ = less3, since 6 = 9less3



Subtracted directly as less-numbers:

$T = 9 – 6 = B-1 – (B-4) = 0 -1 - -4 = -1 + 4 = 3$,

$T = 6 – 9 = B-4 – (B-1) = 0 -4 - -1 = -4 + 1 = -3$, both showing that - - is +

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Adding Numbers with Like Units, 7+9

Inside the 'no-unit-math' paradigm, numbers add **serial** next-to on the number line. We find the result by counting on from 7 or 9.

Outside, in the 'unit-math' paradigm, numbers add **parallel** on-top.

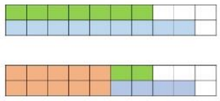
We see that $T = 7 + 9$ = 2B2 7s = 2B-2 9s = 2B-4 tens = 1B6 tens = 16

Added directly as less-numbers:

$T = 7 + 9 = B-3 + B-1 = 2B-4 = 16$

Added directly as half-bundles

T = 7 + 9 = ½B 2 + ½B 4 = 1B 6 = 16



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Decolonizing by demodeling & deconstruction

2 colonizations: BundleNumbers, first by MateMatism & LineNumbers, then by MetaMatism & sets

	Mathematism, ESSENCE	ManyMath, EXISTENCE
Digits	Symbols	Icons
345	The place value system tells it	T = 3BB 4B 5, BB = B^2, BBB = B^3
Operations	Functions, order + - x / ^	Icons, order ^ / x - +
3 + 4	3 + 4 = 7	Meaningless without units
3 * 4	3 * 4 = 12	3 * 4 = 3 4s
9 = ? 2s	Meaningless, only ten counting	9 = 3B3 = 5B-2 = 4B1 = 4½ 2s
8 = ? 2s	Meaningless, only ten counting	8 = (8/2)*2, T = (T/B)*B, proportionality
2*u = 8	(2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4	2*u = 8 = (8/2)*2, so u = 8/2
6*7 = ?	eh 44, eh 52, eh 42? OK	(B-4)*(B-3) = (10-4-3)*B+12 = 3B12 = 4B2 tens
4kg = 5\$, 6kg =?	1kg = 5/4\$, 6kg = 5/4*6\$	6kg = (6/4)*4kg = (6/4)*5\$
1/2 + 2/3 = ?	1/2 + 2/3 = 3/6 + 4/6 = 7/6	1/2*2 + 2/3*3 = 3/5*5
2*3 + 4*5	2*3+4*5 = 10*5, sorry, 6+20 = 26	2*3 + 4*5 = 2 3s + 4 5s = 3B2 8s, by integration
7+ 9 = ?	7 + 9 = 16	2B2 7s = 2B-2 9s = ½B2 + ½B4 = 1B6 tens = 16
Tangent = ?	Tan = sine/cosine	raise = (raise/run)*run, tan = raise/run

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Totals in time & space, growth & statistics I

In time, a total grows by being added or multiplied by a number, called addition-growth and multiplying-growth, or linear and exponential growth.

- Addition-growth: Final number = Initial number + growth-number * growth times, or shortly, $T = B + a*n$. The number a is also called the slope.
- Multiplying-growth: Final number = Initial number * growth-factor ^ growth times, or shortly, $T = B * a^n$, since 200\$ + 5% = (200*105%) \$, so here a is 1 + interest rate.
- Combined growth (savings in a bank): Here we have that $A/a = R/r$, where A is end-dollars, a is the period-dollars, R is the end-rate, r is the period-rate, and $1+R = (1+r)^n$, where n is the number of periods.

100% split in n parts will give the Euler number $e = (1+1/n)^n$ for n large.

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Totals in time & space, growth & statistics II

- Changing the growth-number constantly will give a quadratic growth with a parabola curving upwards or downwards if the change increases or decreases.
- Changing the curvature constantly will give cubic growth with a double parabola with curvature and counter-curvature.
- Decreasing the growth-factor constantly will give logistic saturation growth with a hill-curve in infections. Confusion between exponential and saturation growth can cause unnecessary damage.

In spaces, a total can be divided into several subtotals that could be as large as their average, but where the deviation then tells how far away from the average they are. However, averaging numbers only makes sense if they could be equal. Students in the 1st and 9th grades do not attend the 5th grade on average.

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Mathe-matism uses line numbers without units

- Many-math with units is based on the concrete existence 'Many', and uses bundle-numbers with units, and distinguishes between unit-numbers and per-numbers.
- Set-mathematics without units is based on the abstract essence 'set', and does not accept per-numbers, but is based upon line-numbers without units that become 'mathe-matism', always true inside but rarely outside class, by claiming that 2 + 1 = 3 even though 2 pairs + 1 = 5. And by claiming that digits and fractions are numbers when instead they are operators needing a number to become a number.

That sets lead to a self-reference paradox is neglected:

'The set of sets that do not belong to themselves', does it belong to itself or not?

Well, it does, if it does not. And it does not, if it does.

This is equivalent to asking:

"This sentence is false", is it true or false?

Well, it is true, if it is false. And it is false, if it is true.

In short, self-reference is meaningless.

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More sides of school mathematism

- Mathematics considers digits and operations as symbols just like letters. Multi-digit numbers are said to follow a place-value system, but ten is not called 'bundle', hundred is not called 'bundle-bundle', which would allow power as the first operation. Negative numbers are not allowed at any place.
- Reuniting occurs with the same operations; however, they are presented, not simultaneously, but in the opposite order: addition, subtraction, multiplication, division, power.
- 3+1 = 4 is presented as different number-names for the same total. And not as a tale about a total, T = 3+1 = 4. That is, both the subject and the verb are omitted. Only an equivalence between number-names is included. Over- and under-load are not accepted, carrying and borrowing are used instead.
- 2+3*4, is that 20 or14? This is determined by the definition known as the PEMDAS math hierarchy. Even though T = 2+3*4 = 2 **1s** + 3 **4s**, which can only be recounted as 1**B**4 **tens** or 14.

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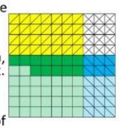
More sides of school mathematism

6*7 is presented as another number-name for 42, even though 6*7 is 6 7s, which may or may not be recounted to tens as 4**B**2 tens or 4.2*10 or 42 if leaving out the unit and the decimal point; and here increasing the width from 7 to 10 will decrease the height from 6 to 4.2.

8/2 is 8 divided into 2 4-bundles, instead of 8 counted in 4 2-bundles.

The tables are memorized, 6*7 = 42 instead of saying 6*7 = (**B**-4)*(**B**-3), or 6*7 = 6*(**B**-3) = 6**B** -18 = 6**B** – (2**B**-2) = 4**B**2 = 4.2*10, with less height.



Here we see that minus times minus gives plus.

Letter calculations such as 2ab + 3bc = (2a+3c)***b** are presented as applying the distributive law, where numbers can be moved in or out of parentheses.

And not by finding the common unity, **b**'s: Number of **b**'s is 2a + 3c, so

T = (2a + 3c) **b**'s = (2a+3c)***b**.

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More sides of school mathematism

- Division leads on to fractions, decimals, and percent. Fractions are presented without units: 1/2 + 2/3 = 7/6, even though 1/2 of 2apples + 2/3 of 3 apples is 3/5 of 5 apples, and, of course, not 7 apples of 6.
- Proportionality tasks are solved by going over the unit.
- Negative numbers are introduced as independent numbers, where minus times minus is defined to be plus.
- Splitting numbers is called solving equations with two numbers-names, whose equivalence is expressed in a statement that retains its truth value if the same operation is performed on both number-names. When transforming a number-name, three laws are used, a commutative, an associative and a distributive law. And two abstract concepts, 1 over 2 as the inverse element to 2, as well as the neutral element 1. Instead of simply moving to opposite side with opposite sign.

$$\begin{aligned} 2*x &= 8 \\ (2*x)*½ &= 8*½ \\ (x*2)*½ &= 4 \\ x*(2*½) &= 4 \\ x*1 &= 4 \\ x &= 4 \end{aligned}$$

$$\begin{aligned} 2*x &= 8 \\ x &= 8/2 \\ x &= 4 \end{aligned}$$

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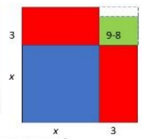
More sides of school mathematism



- **Quadratics** omit drawing $x^2 + 6x + 8 = 0$ as the square $(x+3)^2$ with 4 parts, x^2, and $3x$, and $3x$, and 3^2, that disappear except for $3^2 - 8 = 1$.
- If the school presented inner essence as stemming from external existence, then functions could be presented as number-language sentences that connect an outer subject with an inner predicate, as in the word-language.
- Instead, the school says: A function is a rule that assigns to each element in one set one and only one element in another set.
- And in teacher training: A function is a subset of a set-product in which first-component identity implies second-component identity.

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More sides of school mathematism

- Where *x* represents an unspecified number, *f*(*x*) represents an unspecified formula with *x* as a variable. The term *f*(2) is therefore meaningless, since 2 is not a variable but a constant number.
- Linear and exponential functions are then defined as examples of homomorphisms:
- *f*(*x*) = *a***x*, and *f*(*x*) = *a*^*x*, i.e., without initial number *b*.
- In geometry, plane geometry and coordinate geometry are presented before trigonometry.
- Calculus is presented last with differentiation before integration, although mixture calculations mean adding piecewise constant per-numbers, which later become locally constant per-numbers that are rewritten as increments, *p**dx = d*y*, which add as one difference between end-*y* and start-*y*, as all intermediaries disappear.
- In addition, mathematism introduces eight so-called 'mathematics competences', where many-math have only two: **C**ount & **A**dd in **T**ime & **S**pace, the **CATS** approach.

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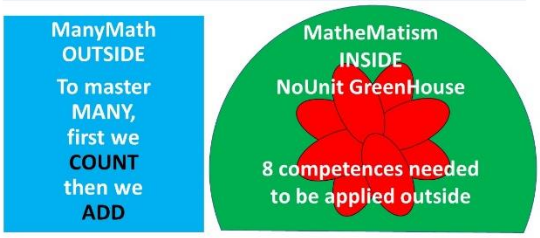
8 Competences in MatheMatism, only 2 in ManyMath: Count & Add in Time & Space, the CATS approach



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Fact & fiction & fake, the 3 genres of both the word-language and the number-language

- Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling.
- As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models.
- Fact models talk about the past and present, and they only need to have the units checked.
- Fiction models talk about the future, and they need to be supplemented with alternative models built upon alternative assumptions.
- And fake models typically add without units, e.g., when claiming that '2+3 = 5' always despite 2weeks + 3days = 17days, thus transforming mathematics to 'mathematism'.

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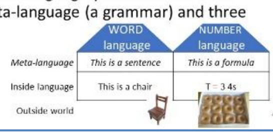
The 'Unit-Math' Paradigm allows Finally to have a Communicative Turn in Mathematics Education

- Modeling real world problems is difficult for mathematism that also fails to distinguish between the three genres fact, fiction, and fake ('Since-then/If-then/What-then, or 'room/rate/risk' models). All models are said to be approximations.
- By using formulas from the start, Many-math avoids modeling problems, as it sees itself as a number-language parallel to the word-language, both of which have a meta-language (a grammar) and three genres: fact, fiction, and fake.

Fake models are, e.g., mathematism adding numbers without units, as well as averages of numbers that could never be equal.



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A Final Question

SUSTAINABLE DEVELOPMENT GOALS

4 QUALITY EDUCATION

Should Ethical Quality Education force children inside a 'no-unit-math' greenhouse that slowly strangles their innate number-language by using line-numbers to learn no-unit addition that folds outside?

Where children's innate mastery of Many just waits to be developed by flexible bundle-numbers available at their fingertips.

Sociology explains mathematism

- Sociology sees people as individuals working towards goals. Both individually, and jointly through institutions with employees who should work for the common goal to be achieved, but who are tempted to make a so-called goal-displacement by instead working to ensure that the goal is not achieved, as this will ensure continued employment, and more resources for extra hours, and for more employees.
- The school's goal is to make children and young people more self-reliant than they already are. This is also true in terms of their mastery of Many as it appears in time and space.
- To achieve this goal, the school has hired mathematism, which unfortunately has fallen into the temptation to make a goal-displacement so that it has become the goal itself, and which therefore constantly wants more resources for extra hours, and for more employees to teach and research the 'difficult' mathematism.
- By instead firing mathematism, the school can avoid this goal-displacement and allow mastery of many to precede mastery of mathematics, which is then achieved automatically anyhow.

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Theoretical Background

Tarp, A. (2018). Mastering Many by counting and recounting before adding on-top and next-to. *Journal of Math Education, March 2018, 11*(1), 103-117.
Tarp, A. (2020). De-modeling numbers, operations and equations: from inside-inside to outside-inside understanding. *Ho Chi Minh City Univ. of Education Journal of Science 17*(3), 453-466.

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References to MATHeCADEMY.net material

- MATHeCADEMY.net, e.g.,
  http://mathecademy.net/math-with-playing-cards/
  http://mathecademy.net/calculus-adds-pernumbers/
  http://mathecademy.net/refugee-camp-math/
  http://mathecademy.net/trigonometry-before-geometry/
  http://mathecademy.net/dk/math-modeling-models/
  http://mathecademy.net/dk/two-competences-or-eight/

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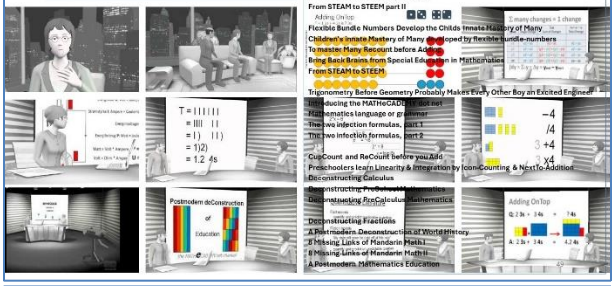
Some of MrAlTarp's 25 YouTube videoes



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MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

ManyMath solving a facebook puzzle

Question Without units	Answer With units
1 + 4 = 5	1 **1s** + 4 **1s** = 5
2 + 5 = 12	2 **1s** + 5 **2s** = 12
3 + 6 = 21	3 **1s** + 6 **3s** = 21
8 + 11 = ?	8 **1s** + 11 **4s** = **52**

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MATHeCADEMY.net : Mathematics as MANY-math, a natural science of Many

Conclusion

We began by observing the difference between 'mathematism' adding without units, which is true inside but seldom outside, and 'Many-math' instead using bundle-numbers with units inspired by how the uneducated child sees the outside existing fact Many. We then explored the consequences of letting existence come before essence by letting counting and recounting come before adding. We saw how Many-math may be learned by working with things and actions on a 2D Bundle-Bundle-Board.

But, will this allow the learner to learn mathematics or even to be numerate as wished by the UN Sustainable Development Goals?

Apparently, different definitions of 'numerate' exist where existence and essence have different order. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

In their common history, England once colonized America. So, the difference in the definitions is interesting. The former uses the passive term 'being' where the latter uses the active term 'having'. The former connects the definition to the inside essence of mathematics while the latter connects it directly to the outside existence of numbers.

The choice thus is: shall existence precede essence as philosophical Existentialism holds, or shall essence be allowed to colonize existence with a 'no-unit regime' to use a Foucault-phrase? Maybe it is time to see if children stay numerate if their own 2D bundle-numbers with units are not colonized by 1D line-numbers without units.

Maybe it is finally time for a Kuhnian paradigm shift in number-language education.

Therefore, think things. Or, in the Viking version: "Derfor, tænk ting".

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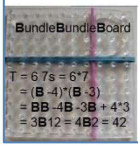
Many before Math!

Mathematics DeColonized by the Child's own 2D BundleBundle Numbers



Make Math Easy with Many-Math where Existence precedes Essence

Allan.Tarp@gmail.com
MATHeCADEMY.net
Spring 2024

06. Math Dislike Cured with BundleBundle Math

Math Dislike
CURED

with Flexible BundleNumbers

My Many Math Tears will not Stay – if I Bundle the Stray Away

BundleCOUNT before you ADD

T = 5 = ǁ I I I = 1B3 2s

T = 5 = ǁ ǁ I = 2B1 2s

T = 5 = ǁ ǁ ǁ = 3B-1 2s

3 ways to BundleCount: Overload, Normal, Underload

ReCount 47 in **tens**: T = 47 = 4B7 = 3B17 = 5B-3 **tens**

NO, 4x7 is not 28, it is 4 **7s** = 2B8 = 1B18 = 3B-2 **tens**
NO, 30/6 is not 30 split by 6, but **3 tens** recounted in **6s**

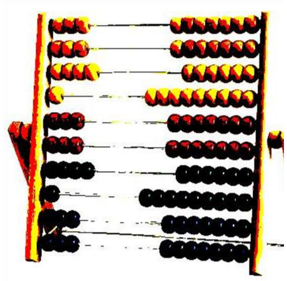
BundleWriting tells **InSide Bundles** from **OutSide Singles**

• **65 + 27**	=	6B5 + 2B7 = 8B12 = 9B2 =	**92**
• **65 – 27**	=	6B5 – 2B7 = 4B-2 = 3B8 =	**38**
• **7x 48**	=	7x 4B8 = 28B56 = 33B6 =	**336**
• **336 /7**	=	33B6 /7 = 28B56 /7 = 4B8 =	**48**
• **336 /7**	=	33B6 /7 = 35B-14 /7 = 5B-2 =	**48**

MatheMatics as ManyMath - a Natural Science about Many

Makes Math Potentials Blossom in Children, Adults & Migrants

Allan.Tarp@MATHeCADEMY.net

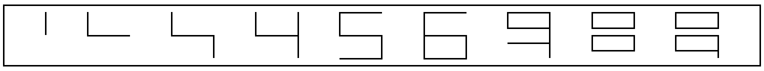


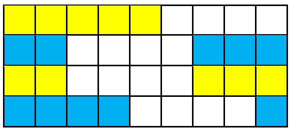
Count before you Add

MatheMatics as ManyMath

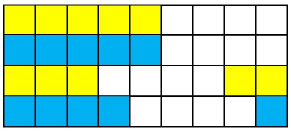
a Natural Science about MANY

MATHeCADEMY.net **Cure Math Dislike** with Kid's own 2D **BlockNumbers** with **Units:** 2 **3s**

Count in ***Icons*** in ***Bundles***	 T = I I I I = 4 = 4 T = 7 = IIII I I I I = IIII IIII I = IIII IIII IIII = 1B4 3s or 2B1 3s or 3B-2 3s
ReCount in *the same* ***Unit*** in *a new* ***Unit***	**ReBundle** to create Overload or Underload T = 7 = I I I I I I I = 1B4 3s = 2B1 3s = 3B-2 3s T = 7 = 2B1 3s = 1B3 4s = 1B2 5s = 3B1 2s = 1BB1B1 2s = 11B1 2s
ReCount in ***Tens*** from ***Tens***	3 7s = ? tens Answer: 3x7 = 21 = 2B1 tens ? 7s = 3 tens Answer: (30/7)x7 = 4B2 7s
DoubleCount in *PerNumbers* in *PerFive, 3/5* in *PerHundred, %*	With 4$ per 5kg or 4/5 $/kg, T = 20kg = (20/5) x 5kg = (20/5) x 4$ = 16$ 3/5 = 3$/5$ of 200$ = ?$. 200$ = (200/5) x 5$ gives (200/5) x 3$ = 120$ 70% = 70$/100$ of 300$ = ?$. 300$ = (300/100) x 100$ = gives (300/100) x 70$ = 210$
Calculations Predict with a ***RecountFormula***	T = 2 4s = ? 5s = 1B3 5s since 2x4/5 1.some; 2x4 – 1x5 3 T = (T/B) x B = T/B Bs
Add *NextTo* *OnTop*	T = 2 3s + 4 5s = 3B2 8s *Integration* T = 2 3s + 4 5s = 1B1 5s + 4 5s = 5B1 5s *Proportionality*
Multiply, Divide Use *BundleWriting*	7 x 63 = 7 x 6B3 = 42B21 = 44B1 = 441 245 /7 = 24B5 /7 = 21B35 /7 = 3B5 = 35



Geometry-mode



Algebra-mode

MrAlTarp
YouTube Videos
Allan.Tarp@MATHeCADEMY.net

MATHeCADEMY.net	*Teaching Teachers to Teach* *MatheMatics as ManyMath* PYRAMIDeDUCATION **CATS**: **C**ount & **A**dd in **T**ime & **S**pace

Piaget: Grasping with Fingers leads to Grasping Mentally

Four as an icon built by four cars, four rhinos, four sticks, a ruler folded in four parts, four beads on an abacus, LEGO blocks, pearls on a pearl board, etc.

Seven sticks bundle-counted as 1**B**2 5s, or as 2**B**1 3s or as 3**B**1 2s

The MATHeCADEMY.net stand at the MatematikBiennale in Sweden, 2014

Contents

Preface

"How old will you be next time?" I asked the child. "Four", he answered and showed me four fingers. "Four, you said?" I asked and showed him four fingers held together two by two. "No, that is not four, that is two twos!" the child replied. That opened my eyes. Children come to school with two-dimensional block numbers where all numbers have units. However, the school does not allow the children to count the numbers before being added. Instead, the school teaches cardinality as a one-dimensional number line where numbers have different names; thus disregarding the fact that numbers are two dimensional blocks where all numbers have a unit as shown when writing out fully

T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1.

This booklet allows schools and parents to choose an education that accepts and develops the 2D number blocks that the children bring to school instead of forcing a 1D number line upon them. Also, the booklet allows the children to practice 'counting before adding' and to include bundle-counting and re-counting to different units. The booklet thus is an answer to the question 'How to Save and Develop a Child's Math Potential?'

To master Many we ask 'how many?' To answer, we count by bundling and stacking to get a total T. Once counted, first a total can be recounted in the same unit to create overload or underload, or to create a different unit; next totals can be added NextTo, or OnTop if the units are the same.

Counting a total T of 7 ones in 3s we get the result T = 7 = 2 3s & 1 = 2**B**1 3s.

We separate the *inside* bundles from the *outside* unbundled singles by a *bundle* becoming a bracket when reporting the result with *bundle-writing*: T = III III I = II **B** I = 2**B**1 **3s**

Once counted, a total can be *recounted* to create *overload* or *underload*, deficit. To create an overload, we move a stick from the inside to the outside: T = II **B** I = I **B** III I = 1**B**4 **3s**.

To create an underload, we borrow foreign sticks to move a bundle from the outside to the inside

T = II **B** I = II **B** I II II = III **B** II = 3**B**-2 **3s**.

Thus a given total can be *recounted* in three ways: normal, with overload and with underload.

T = 7 = 2**B**1 **3s** = 1**B**4 **3s** = 3**B**-2 **3s**.

A total of 68 can be recounted in four different ways as T = 68 = 6**B**8 **tens** = 5**B**18 **tens** = 7**B**-2 **tens**.

Recounting and bundle-writing come in handy when we add, subtract, multiply or divide numbers:

Using bundle-writing to add 65 and 27 we get an overload outside the bundle allowing us to move 10 **1s** from the outside to the inside as 1 **tens**

T = 65 + 27 = 6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92

Using bundle-writing to subtract 27 from 65 we get an underload outside the bundle allowing us to move a bundle of 1 **tens** from the inside to the outside as 10 **1s** to remove the underload.

T = 65 – 27 = 6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38

Alternatively, before subtracting we can create an overload outside by moving 1 **tens** from the inside to the outside as 10 **1s**

T = 65 – 27 = 6**B**5 – 2**B**7 = 5**B**15 – 2**B**7 = 3**B**8 = 38

Using bundle-writing to multiply 48 with 7 we get an overload outside the bundle allowing us to move 50 **1s** from the outside to the inside as 5 **tens**

T = 7* 48 = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336

Alternatively, before multiplying we can create an underload outside by borrowing 2 **1s**. Later the underload can be removed by moving 2 **tens** outside as 20 **1s**

T = 7* 48 = 7* 4**B**8 = 7* 5**B**-2 = 35**B**-14 = 33**B**6 = 336

Using bundle-writing to divide 336 with 7 we prefer to have 28 instead of 33 inside the bundle, so we create an overload outside by moving 5 bundles outside as 50 **1s**

T = 336 = 33**B**6 = 28**B**56; so T /7 = 4**B**8 = 48

Alternatively, we can create an underload outside before dividing

T = 336 = 33**B**6 = 35**B**-14; so T /7 = 5**B**-2 = 4**B**8 = 48

To divide 338 by 7 we get 2 single leftovers that counted in 7s becomes a fraction 2/7

T = 338 = 33**B**8 = 28**B**58 = 28**B**56 + 2; so T /7 = 4**B**8 + 2/7 = 48 2/7

Allan.Tarp@MATHeCADEMY.net, September 2019

Introduction to the Chapters

Chapter 01, From Sticks to Icons, shows how rearranging four sticks creates a 4-icon with as many sticks as it represents; likewise with the other icons until ten having a name but no icon.

Chapter 02, Counting-sequences in Icons, shows that when counting by bundling, the bundle-icon is not used. Hence, when counting in tens, ten does not need an icon. A natural counting sequence will report both the bundles and the unbundled: 01, 02, …, 10, 11; or 0.1, 0.2, …, 1.0, 1.1 always including the bundle-name as the unit. Each bundle-size has its own counting sequence, but the standard is ten-counting in a sloppy version leaving out the unit and misplacing the decimal point by saying 23 instead of 2.3 tens.

Chapter 03, BundleCount in Icons, shows how a total T can be recounted in icon-bundles. Thus a total of nine things, represented by a line of sticks or beads on an abacus, can be counted in fours by a counting sequence. Also, they can be represented by a stack of bundles placed with one stick per bundle in a bundle that can written as a bracket (bundle-writing) and reported as a decimal number with a unit where the decimal point separates the bundles from the unbundled singles, T = 9 = 2**B**1 4s = 2.1 4s. Alternatively, a calculator can be asked to predict the counting result. Entering '9/4', we ask 'from 9, taking away 4s can be done how many times?' The calculator answers '2.some' so by entering '9 – 2x4' we ask 'from 9, once taking away 2 4s leaves what?' The answer '1' gives the calculator prediction T = 9 = 2.1 4s. Thus also operations are icons: /4 shows the broom wiping away 4 many times, – 4 shows the trace left when dragging away 4 only once, 2x shows the lifting needed to create a stack of 2 bundles, and +3 shows the juxtaposition of 3 singles left next to a stack of bundles. Moving 1 stick outside the bundle creates an overload T = 1**B**5 4s; and moving an extra stick in gives an underload, a deficit, T = 3**B**-3 4s. Thus by recounting, a total T of nine can be recounted in 4 different ways: T = nine = 9 1s = 2**B**1 4s = 1**B**5 4s = 3**B**-3 4s. This comes in handy when totals are added, subtracted, multiplied or divided. A good calculator says 2+3*4 = 14; a bad says 2+3*4 = 20.

Chapter 04, BundleCount with dices, shows how a total T can be recounted in icon-bundles where the total is shown on two similar dices and the icon-number is shown on a third dice.

Chapter 05, ReCount in the same Unit, shows how to recount a total T in the same unit by unbundling a bundle to singles thus creating an overload, or by borrowing extra singles that then has be counted for as a deficit. Thus a total of 2.1 5s can be written with overload as T = 1**B**6 5s or as T = 1.6 5s, or with borrowing as T = 3**B**-4 5s or as T = 3.-4 5s

Chapter 06, ReCount in a new Unit, shows how once counted in one unit, a total T can be recounted in another unit. Thus a total of 2 9s can be recounted in 6s as in chapter 3, again by lining, counting, bundling, stacking, bundle-writing and answering; and again checked by a calculator prediction using two formulas. The ReCount formula **T = (T/B)*B** saying that 'from T, T/B times Bs can be taken away'; and the ReStack formula **T = (T–B)+B** saying that 'From T, T–B is left when B is placed next to'. To change a unit is also called **proportionality**.

Chapter 07, ReCount in BundleBundles, shows how an overload in a bundle can be removed by an extra bundle for bundles-of-bundles. Thus counting a total T of 4 8s in 5s gives T = 6**B**2 5s. However, with 5 as the bundle-size, 5 bundles can be recounted as 1 bundle-of-bundles of 5s so that
T = 6**B**2 5s = B1**B**2 5s = 1**B**1**B**2 5s or T = 6.2 5s = 11.2 5s.

Chapter 08, ReCount in Tens on Squared Paper or an Abacus, shows how easy totals counted in icon-bundles can be recounted in tens since the calculator is programmed to give the answer directly in its sloppy version. Thus to recount 3 8s in tens we enter 3*8 and get the answer 24, so T = 3 8s = 2.4 tens. Recounting icon-numbers in tens systematically will provide the multiplication tables showing individual patterns in a ten by ten square or on an abacus.

Chapter 09, ReCount from Tens, shows, as in chapter 3, that we can get the answer through a calculator prediction or through lining, rebundling, and bundle-writing. Only this time we shorten the line by using Roman numbers as icons. Recounting large numbers from tens, we save time by using a multiplication table. Thus to recount a total T of 253 in 7s we use bundle-writing to create overloads guide by the table:
T = 253 = 25**B**3 = 21**B**43 = 21**B** 42+1 = 3**B**6 *7 +1, so T = 253 = 36 7s + 1.

Chapter 10, ReCount Large Numbers in Tens, show how bundle-writing may be used to create overloads later to be removed to get the final answer. Thus to recount 7 43s in tens gives a total
T = 7 43s = 7*43 = 7*4**B**3 = 28**B**21 = 30**B**1 = 301 as confirmed by a calculator.

Chapter 11, DoubleCount with PerNumbers, shows that counting a quantity in two different physical units will provide a per-number to be used as a bridge connecting the two units. Thus counting a quantity as 4$ and as 5 kg gives the per-number 4$/5kg or 4/5 $/kg. Asking '8$ = ? kg', the answer comes from recounting the 8s in 4s to be able to use the per-number as a bridge between the two units:
T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg. Likewise when asking e.g.'?$ = 12kg'

Chapter 12, DoubleCount with Fractions and Percentages, shows that fractions and percentages can be treated as per-numbers. Thus asking '3/5 of 200$' is the same as asking '3 per 5 of 200$ gives ?'. So we recount the 200 in 5s to get the answer: T = 200$ = (200/5)*5$ giving (200/5)*3$ = 120$. And asking '3% of 250$' is the same as asking '3 per 100 of 250$'. So we recount the 250 in 100s to get the answer:
T = 250$ = (250/100)*100$ gives (250/100)*3$ = 7.5$ as confirmed by writing '3/100*250' on a calculator.

Chapter13, ReCount PerNumbers, Fractions, shows how changing unit transforms per-numbers.

Chapter 14, Adding OnTop, shows that to add two totals T1 and T2 OnTop the units must be the same so recounting may be needed to change a unit. Thus adding 2 3s and 4 5s as 3s, the 4 5s must be recounted as 3s to give a total of 8.2 3s as confirmed by a calculator.

Chapter 15, Reversed Adding OnTop, shows that to reverse OnTop addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 5 3s, we take away the 2 3s from the 5 3s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 1.4 5s as confirmed by a calculator. Subtraction followed by division is called differentiation.

Chapter 16, Adding NextTo, shows that adding two totals T1 and T2 NextTo means adding their areas, also called integration. Thus adding 2 3s and 4 5s NextTo each other as 8s on a ten by ten square or on an abacus gives 3.2 8s as confirmed by a calculator.

Chapter 17, Reversed Adding NextTo, shows that to reverse NextTo addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 3 8s, we take away the 2 3s from the 3 8s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 3.3 5s as confirmed by a calculator. Together, integration and differentiation is called **calculus**.

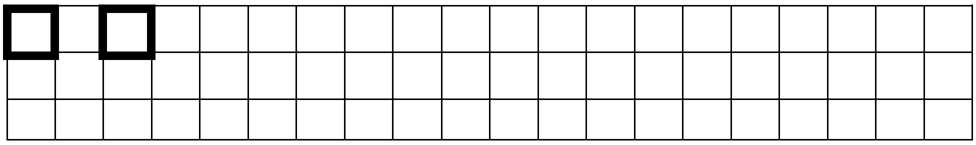
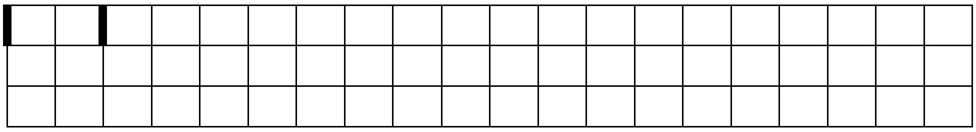
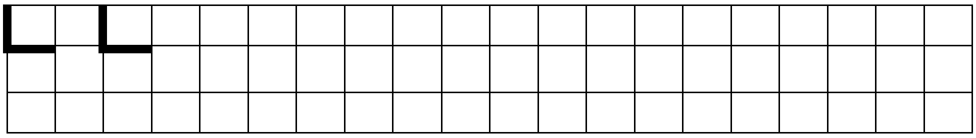
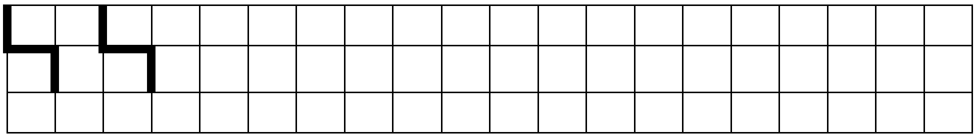
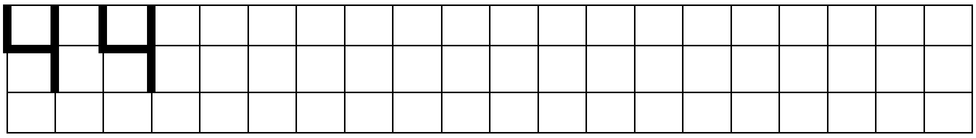
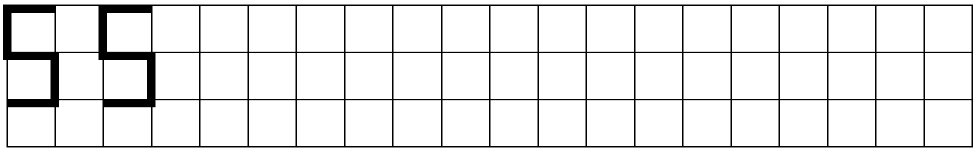
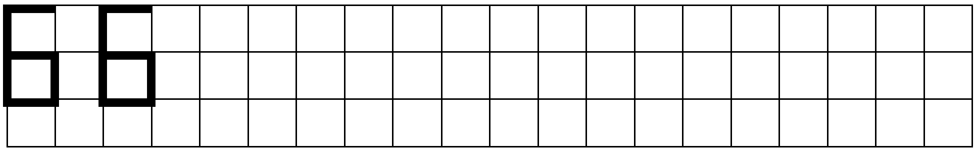
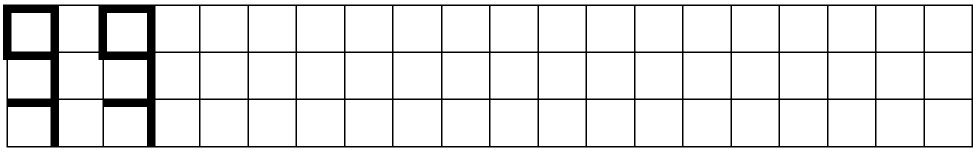
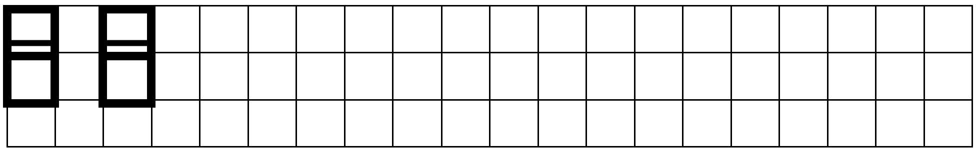
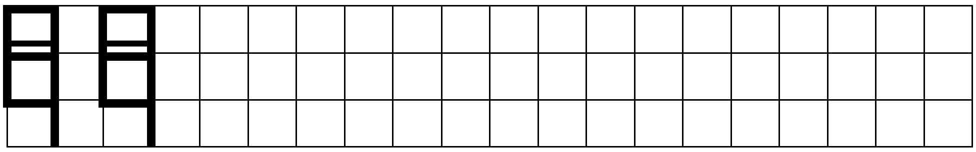
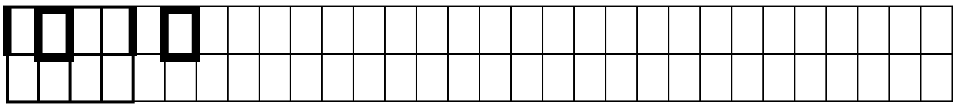
Chapter 18, Adding Tens, shows that when adding tens, bundle-writing can be used to create and remove overloads. Thus adding totals as 27 and 85 creates an overload that can be removed by bundle-writing,
T = 27 + 85 = 2**B**7 + 8**B**5 = 10**B**12 = 11**B**2 = 112 as confirmed by a calculator.

Chapter 19, Reversed Adding Tens, the number added must be taken away which might result in a deficit calling for a unbundling a bundle, unless this is done first resulting in an overload that allows taking the number away without creating a deficit. Thus asking '? + 27 = 85' or '85 – 27', bundle-writing is used to remove the deficit, 85 – 27 = 8**B**5 – 2**B**7 = 6**B**-2 = 5**B**8 = 58; or used to create an overload,
85 – 27 = 8**B**5 – 2**B**7 = 7**B**15 – 2**B**7 = 5**B**8 = 58, both confirmed by a calculator.

Chapter 20, Recounting Solves Equations, shows that equations expressing a reversed calculation can be solved by recounting and restacking. Thus to solve the equation u*2 = 8, 8 is recounted in 2s as 8 = (8/2)*2 = 4*2, so that u = 4, checked by a calculator by entering 4*2. With u*2 = 8 solved by u = 8/2 we get a shortcut for solving equations: *Move to the opposite side with the opposite sign.*

u*2 = 8 = (8/2)*2 = 4*2	Here we recount 8 in 2s as 8 = (8/2)*2 = 4*2	**u = 4**
u+2 = 9 = (9-2)+2 = 7+2	Here we restack 9 to 9-2+2 = 7+2	**u = 7**
u/3 = 2	Here we recount 2 in 3s as 2 = (2/3)*3 = 2*3/3 = 6/3	**u = 6**
u-2 = 6	Here we restack 6 to 6-2+2 = 6+2-2 = 8-2	**u = 8**
2*u+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and 2*u = 12 = 12/2*2 = 6*2	**u = 6**
2*u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2*u = 18 = 18/2*2 = 9*2	**u = 9**
u/2+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and u/2 = 12 = 12/2*2 = 12*2/2 = 24/2	**u = 24**
2/u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2/u = 18 = 18/2*2 = 18*2/2 = 36/2	**u = 36**

01. From Sticks to Icons

Job	Do
	
I	
II	
III	
IIII	
IIIII	
IIIII I	
IIIII II	
IIIII III	
IIIII IIII	
IIIII IIIII	

Count & Color Squares, Odd & Even

1		2			3			4			5			6			7			8			9		

9					8			7			6			5			4			3		2			1

2			4			6			8			1		3			5			7			9		

1		2			4			8			3			5			9				6			7	

3			6			9				8			5			4			7			1		2	

1		3			5			7			9				8			6			4			2	

5			3			1		2			4			7			9				8			6	

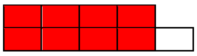
02. Counting-sequences in Icons

	I	I	I	I	I	I	I	I	I	I	I	I	I	I
ten	1	2	3	4	5	6	7	8	9	10	11	12	13	14
ten	01	02	03	04	05	06	07	08	09	1B	1B1	1B2	1B3	1B4
ten	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.	1.1	1.2	1.3	1.4
9	01	02	03	04	05	06	07	08	1B	1B1	1B2	1B3	1B4	1B5
9	.1	.2	.3	.4	.5	.6	.7	.8	1.	1.1	1.2	1.3	1.4	1.5
8														
8														
7														
7														
6														
6														
5														
5														
4														
4														
3	01	02	1B	1B1	1B2	2B	2B1	2B2	BB	1BB1	1BB2	1BB1B	1BB1B1	1BB1B2
3	.1	.2	1.	1.1	1.2	2.	2.1	2.2	10.	10.1	10.2	11.	11.1	11.2
2														
2														
11	1	2	3	4	5	6	7	8	9	X	1B	1B1	1B2	1B3
11	01													
11	.1													

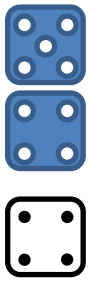
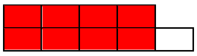
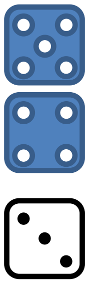
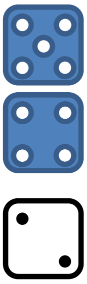
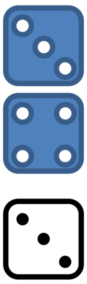
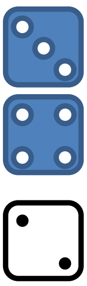
	II	II	II	II	II	II	II	II	II	II	II	II	II	II
ten	02				1B							2B2		
ten														
9														
9														
8														
8														
7														
7														
6														
6														
5														
5														
4														
3														
3														

	III	III	III	III	III	III	III	III	III	III	III	III	III	III
ten														
9														
8														
7														
6														
5														
4														
3														
2														

03. BundleCount in Icons

Job		Do	Calculator
9 in 5s	Line Count Bundle Stack B-write Answer	T = I I I I I I I I I 1, 2, 3, 4, B, 1B1, 1B2, 1B3, <u>**1B4**</u> T = ~~IIIII~~ I I I I  T = 1**B**4 5s = 0**B**9 5s = 2**B**-1 5s <u>T = 9 = 1.4 5s</u>	9/5 1.some 9 – 1*5 4 *9 – 0*5 9* *9 – 2*5 -1*
9 in 4s	Line Count Bundle B-write Stack Answer	T = I I I I I I I I I 1, 2, 3, B, 1B1, 1B2, 1B3, 2B, <u>**2B1**</u> T = ~~IIII~~ ~~IIII~~ I T = 2**B**1 4s = 1**B**5 4s = 3**B**-3 4s  <u>T = 9 = 2.1 4s</u>	9/4 2.some 9 – 2*4 1 *9 – 1*4 5* *9 – 3*4 -3*
9 in 3s	Line Count Bundle B-write Stack Answer		9/ 9 –
8 in 4s	Line Count Bundle B-write Stack Answer		8 8
8 in 3s	Line Count Bundle B-write Stack Answer		8 8

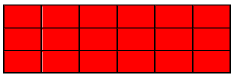
04. BundleCount with Dices

Job		Do	Calculator
 9 i 4s	Line Count Bundle B-write Stack Answer	T = I I I I I I I I I 1, 2, 3, B, 1B1, 1B2, 1B3, 2B, <u>2B1</u> T = ~~IIII~~ ~~IIII~~ I T = 2**B**1 4s = 1**B**5 4s = 3**B**-3 4s  <u>T = 9 = 2.1 4s</u>	9/4 2.some 9 – 2*4 1 *9 – 1*4 5* *9 – 3*4 -3*
	Line Count Bundle B-write Stack Answer		9/ 9 –
	Line Count Bundle B-write Stack Answer		9 9
	Line Count Bundle B-write Stack Answer		7 7
	Line Count Bundle B-write Stack Answer		7 7

05. ReCount in the Same Unit

Job		Do	Bundle	Answer
2.1 5s in 5s	Line UnBundle Embezzle	T = IIIII IIIII I T = IIIII I I I I I I T = IIIII IIIII IIIII	2B1 1B6 3B-4	T = 2.1 **5s** T = 1.6 **5s** T = 3.-4 **5s**
2.1 4s in 4s	Line UnBundle Embezzle			
2.1 3s in 3s	Line UnBundle Embezzle			
2.1 6s in 6s	Line UnBundle Embezzle			
2.1 7s in 7s	Line UnBundle Embezzle			
3.2 7s in 7s	Line UnBundle Embezzle			
3.2 6s in 6s	Line UnBundle Embezzle			
3.2 5s in 5s	Line UnBundle Embezzle			
3.2 4s in 4s	Line UnBundle Embezzle			
3.2 3s in 3s	Line UnBundle Embezzle			

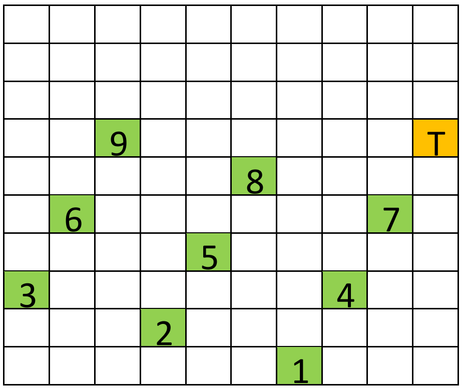
06. ReCount in a New Unit

Job		Do	Calculator
2 9s in 6s	Line	T = I I I I I I I I I I I I I I I I I I	
	Count	1, 2, 3, 4, B, 1B1, 1B2, 1B3, 1B4, …, **3B**	
	Bundle	T = ~~IIIIII~~ ~~IIIIII~~ ~~IIIIII~~	2*9/6 3
	Stack		2*9 – 3*6 0
	B-write	T = 3**B**	
	Answer	T = 2 9s = 3 6s	
2 9s in 5s	Line		
	Count		
	Bundle		2*9/
	Stack		2*9 –
	B-write		
	Answer		
2 8s in 6s	Line		
	Count		
	Bundle		2*8
	Stack		2*8
	B-write		
	Answer		
2 8s in 5s	Line		
	Count		
	Bundle		2*8
	Stack		2*8
	B-write		
	Answer		
2 7s in 6s	Line		
	Count		
	Bundle		2*7
	Stack		2*7
	B-write		
	Answer		

07. Recount in BundleBundles

Job		Do	Calculator
4 8s in 5s	B-write Answer	T = 4 **8s** = 6**B**2 = B1**B**2 **5s** = 1**B**1**B**2 T = 4 **8s** = 6.2 **5s** = 11.2 **5s** = 12.-3 **5s**	4*8/5 6.some 4*8 – 6*5 2
5 8s in 6s	B-write Answer		
6 9s in 7s	B-write Answer		
9 9s in 8s	B-write Answer		
3 9s in 4s	B-write Answer		
4 5s in 3s	B-write Answer		
6 8s in 5s	B-write Answer		
6 8s in 4s	B-write Answer		
7 8s in 5s	B-write Answer		
7 8s in 4s	B-write Answer		
8 8s in 5s	B-write Answer		
8 8s in 4s	B-write Answer		

08. ReCount in Tens on Squared Paper or an Abacus

Job		Do	Calculator
7s **in tens**			10 * 7 = 70 9 * 7 = 63 8 * 7 = 56 7 * 7 = 49 6 * 7 = 42 5 * 7 = 35 4 * 7 = 28 3 * 7 = 21 2 * 7 = 14 1 * 7 = 7
8s **in tens**			10 * 8 = 9 * 8 = 8 * 8 = 7 * 8 = 6 * 8 = 5 * 8 = 4 * 8 = 3 * 8 = 2 * 8 = 1 * 8 =
9s **in tens**			10 * 9 = 9 * 9 = 8 * 9 = 7 * 9 = 6 * 9 = 5 * 9 = 4 * 9 = 3 * 9 = 2 * 9 = 1 * 9 =
6s **in tens**			10 * 6 = 9 * 6 = 8 * 6 = 7 * 6 = 6 * 6 = 5 * 6 = 4 * 6 = 3 * 6 = 2 * 6 = 1 * 6 =

09. Recount From Tens

Job		Do	Calculator
37 in 9s	Line ReBundle B-write Answer	X X X V II 9I 9I 9I V II -> 9 9 9 X -> 9 9 9 9 1 3**B** 7 = **B**37 = **B**36 + 1 = **B**4*9 + 1 T = 37 = 4*9 + 1 = 4.1 **9s** = 4 1/9 **9s**	 37/9 4.some 37 – 4*9 1
37 in 7s	Line ReBundle B-write Answer		
37 in 5s	Line ReBundle B-write Answer		
42 in 7s	Line ReBundle B-write Answer		
42 in 5s	Line ReBundle B-write Answer		
26 in 7s	Line ReBundle B-write Answer		
26 in 5s	Line ReBundle B-write Answer		

253 in 7s	B-write Answer	T = 2**B**5**B**3 = 25**B**3 = 21**B**43 = 21**B**42 + 1 T = 3**B**6 * 7 + 1 = 36 * 7 + 1 = <u>36 1/7 **7s**</u>	253/7 36.some 253 – 36*7 1
253 in 9s	B-write Answer		
253 in 5s	B-write Answer		
253 in 3s	B-write Answer		
842 in 7s	B-write Answer		
842 in 5s	B-write Answer		
842 in 4s	B-write Answer		
842 in 2s	B-write Answer		
904 in 8s	B-write Answer		
904 in 7s	B-write Answer		
904 in 5s	B-write Answer		
904 in 3s	B-write Answer		
789 in 8s	B-write Answer		
789 in 7s	B-write Answer		
789 in 5s	B-write Answer		
789 in 4s	B-write Answer		

10. Recount Large Numbers in Tens

Job		Do	Calculator
7 43s	B-write Answer	T = 7 * 4**B**3 = 28**B**21 = 30**B**1 = 301 T = 7 **43s** = 30.1 **tens** = 301	7*43 301
8 43s	B-write Answer		
9 43s	B-write Answer		
6 43s	B-write Answer		
5 62s	B-write Answer		
4 62s	B-write Answer		
3 62s	B-write Answer		
2 62s	B-write Answer		
27 436s	B-write Answer		
3 436s	B-write Answer		
4 436s	B-write Answer		
5 436s	B-write Answer		
6 436s	B-write Answer		
7 436s	B-write Answer		
8 436s	B-write Answer		

17 43s	B-write Answer	T = 17 * 4**B**3 = 68**B**51 = 73**B**1 = 731 T = 17 **43s** = 73.1 **tens** = 731	17*43 731
27 43s	B-write Answer		
37 43s	B-write Answer		
47 43s	B-write Answer		
57 43s	B-write Answer		
67 43s	B-write Answer		
77 43s	B-write Answer		
87 43s	B-write Answer		
32 243s	B-write Answer	T = 32 * 2**B**4**B**3 = 64**B**128**B**96 = 64**B**137**B**6 = 77**B**7**B**6 = 777.6 **tens** = 7776	32*243 7776
35 413s	B-write Answer		
43 343s	B-write Answer		
56 453s	B-write Answer		
62 637s	B-write Answer		
74 843s	B-write Answer		
87 543s	B-write Answer		
92 493s	B-write Answer		

11. DoubleCount with PerNumbers

Job	Do	Formula
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**	8$ = (8/4)*4$ = (8/4)*5kg = 10kg 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$	Kg = (kg/$)*$ Kg = (5/4)*8 = 10 $ = ($/kg)*kg $ = (4/5)*12 = 9.6
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 7 kg **8$ = ?kg** **?$ = 12 kg**		

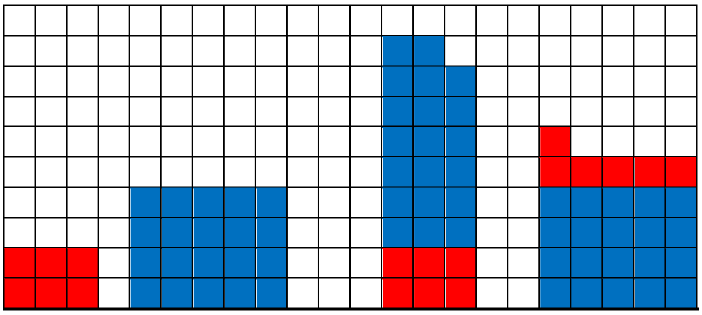
12. DoubleCount with Fractions and Percentages

Job	Do	Calculator
3 per 5 of 200$	200$ = (200/5)*5$ Giving (200/5)*3$ = 120$	3/5*200 120
3 per 5 of 400$		
2 per 5 of 200$		
1 per 5 of 200$		
3 per 6 of 240$		
2 per 6 of 240$		
5 per 6 of 300$		
3 per 100 of 250$ or 3% of 250$	250$ = (250/100)*100$ Giving (250/100)*3$ = 7.5$	3/100*250 7.5
8 per 100 of 200$ or 8% of 200$		
20 per 100 of 200$ or 20% of 200$		
3 per 100 of 560$ or 3% of 560$		
8 per 100 of 560$ or 8% of 560$		
12 per 100 of 560$ or 12% of 560$		
20 per 100 of 560$ or 20% of 560$		
60 per 100 of 560$ or 60% of 560$		

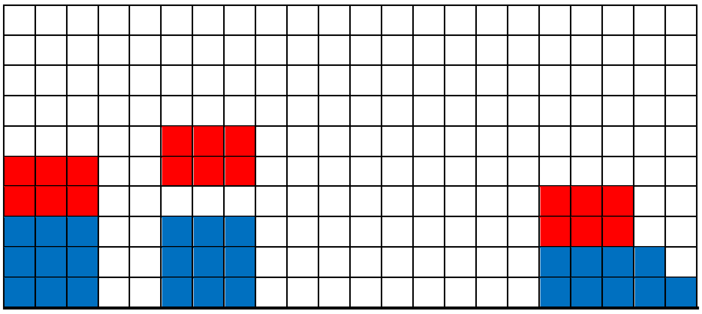
13. ReCount PerNumbers, Fractions

Job	Do	Do	Calculator	Calculator
2/3 = ?	2/3 = 2 2s / 3 2s = 4/6 2/3 = 2 3s / 3 3s = 6/9	2/3 = 2 4s / 3 4s = 8/12 2/3 = 2 5s / 3 5s = 10/15	2/3 = 0.66.. 4/6 = 0.66..	8/12 = 0.66.. 10/15 = 0.66..
1/3 = ?				
1/5 = ?				
2/5 = ?				
3/5 = ?				
4/5 = ?				
4/6 2/6 6/8 2/8	4/6 = 2 2s / 3 2s = 2/3 2/6 = 1 2s / 3 2s = 1/3	6/8 = 3 2s / 4 2s = 3/4 2/8 = 1 2s / 4 2s = 1/4	4/6 = 0.66.. 2/3 = 0.66.. 2/6 = 0.33.. 1/3 = 0.33..	6/8 = 0.75 3/4 = 0.75 2/8 = 0.25 1/4 = 0.25
2/10 4/10 6/10 8/10				
2/12 4/12 6/12 8/12 10/12				
2/14 4/14 6/14 8/14 10/14 12/14				
2/16 4/16 6/16 8/16 10/16 12/16 14/16				

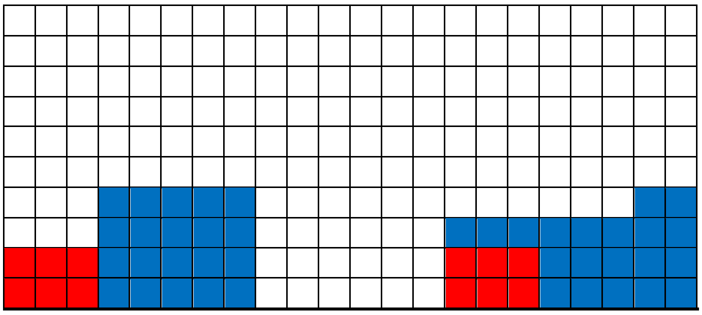
14. Add OnTop

Job	Do	Calculator
2 3s + 4 5s = ? 3s ? 5s		(2*3+4*5)/3 8.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 8.2 3s</u> (2*3+4*5)/5 5.some (2*3+4*5) - 5*5 1 <u>2 3s + 4 5s = 5.1 5s</u>
2 4s + 3 5s = ? 4s ? 5s		
3 2s + 4 6s = ? 2s ? 6s		
2 5s + 4 3s = ? 5s ? 3s		
5 2s + 3 4s = ? 2s ? 4s		

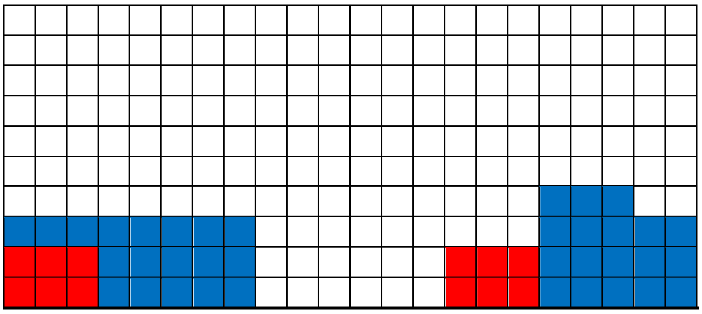
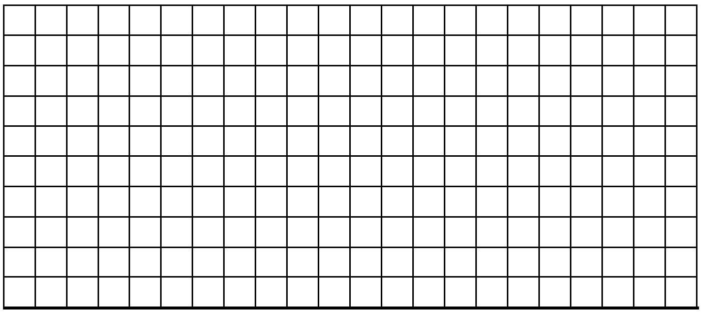
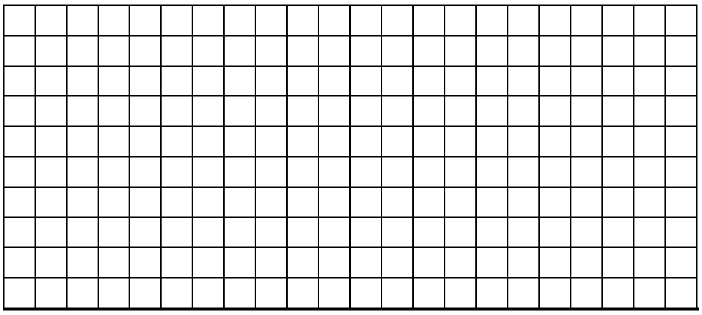
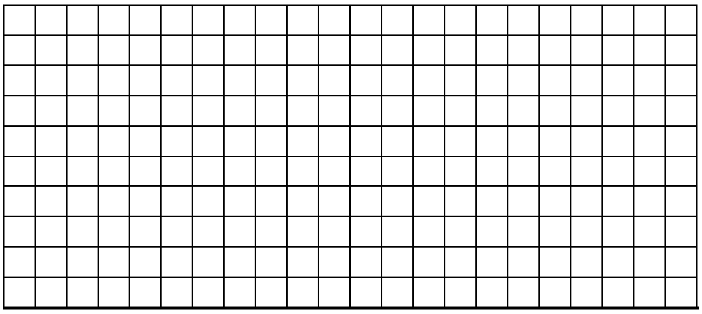
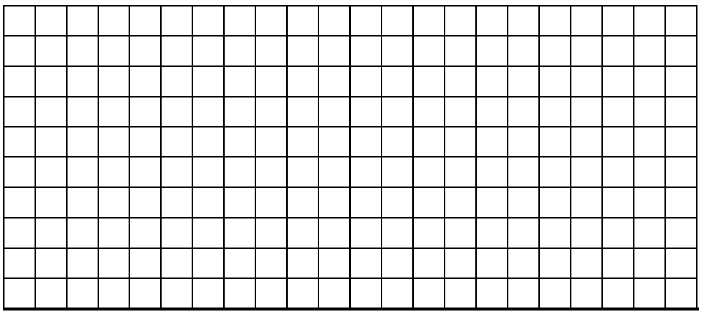
15. Reversed Adding OnTop

2 3s **+** **? 5s** **=** **5 3s**		(5*3-2*3)/5 1.some (5*3-2*3) - 1*5 4 2 3s + **1.4 5s** = 5 3s
2 4s **+** **? 5s** **=** **5 4s**		
2 6s **+** **? 5s** **=** **4 6s**		
2 7s **+** **? 5s** **=** **6 5s**		
2 6s **+** **? 5s** **=** **4 5s**		

16. Add NextTo

Job	Do	Calculator
2 3s **+** **4 5s** **=** **? 8s**		(2*3+4*5)/8 3.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 3.2 8s</u>
3 2s **+** **4 5s** **=** **? 7s**		
2 3s **+** **4 6s** **=** **? 9s**		
2 4s **+** **4 5s** **=** **? 9s**		
4 3s **+** **2 4s** **=** **? 6s**		

17. Reversed Adding NextTo

2 3s **+** **? 5s** **=** **3 8s**		(3*8-2*3)/5 3.some (3*8-2*3) - 3*5 3 <u>2 3s + **3.3 5s** = 3 8s</u>
2 4s **+** **? 5s** **=** **3 9s**		
2 3s **+** **? 4s** **=** **3 7s**		
4 3s **+** **? 5s** **=** **3 8s**		
5 2s **+** **? 5s** **=** **3 7s**		

18. Add Tens

Job		Do	Calculator
27 + 85	B-write Answer	T = 2**B**7 + 8**B**5 = 10**B**12 = 11**B**2 = 112 T = 27 + 85 = 11.2 tens = 112	27+85 112
27 + 85	B-write Answer		
33 + 78	B-write Answer		
39 + 71	B-write Answer		
45 + 67	B-write Answer		
58 + 57	B-write Answer		
57 + 49	B-write Answer		
27 + 205	B-write Answer		
33 + 198	B-write Answer		
39 + 191	B-write Answer		
45 + 187	B-write Answer		
58 + 177	B-write Answer		
57 + 169	B-write Answer		
127 + 385	B-write Answer		
433 + 578	B-write Answer		

19. Reversed Adding Tens

Job		Do	Calculator
27 + ? = 85 **85 – 27**	B-write Answer	D = 8**B**5 – 2**B**7 = 6**B**-2 = 5**B**8 = 58 D = 8**B**5 – 2**B**7 = 7**B**15 – 2**B**7 = 5**B**8 = 58 T = 85 – 27 = 5.8 **tens** = 58	85 – 27 58
63 – 17	B-write Answer		
55 – 36	B-write Answer		
35 – 17	B-write Answer		
185 – 27	B-write Answer		
235 – 128	B-write Answer		
242 – 128	B-write Answer		
245 – 167	B-write Answer		
312 – 159	B-write Answer		
421 – 268	B-write Answer		

20. ReCounting solves Equations

Do	Equation	Calculator
ReCount Answer	**u*2 = 30** = (30/2)*2 = 15*2 u = 15	15*2 30
ReCount Answer	**u*3 = 15**	
ReCount Answer	**u*4 = 32**	
ReCount Answer	**u*5 = 40**	
ReCount Answer	**u/3 = 12** = (12/3)*3 = 12*3/3 = 36/3 u = 36	36/3 12
ReCount Answer	**u/3 = 10**	
ReCount Answer	**u/4 = 8**	
ReCount Answer	**u/5 = 6**	
ReCount Answer	**u+2 = 30** = (30-2)+2 = 28 + 2 u = 28	28+2 30
ReCount Answer	**u+3 = 24**	
ReCount Answer	**u+4 = 20**	
ReCount Answer	**u+5 = 12**	
ReCount Answer	**u-2 = 30** = (30-2)+2 = 30+2-2 = 32- 2 u = 32	32-2 30
ReCount Answer	**u-3 = 20**	
ReCount Answer	**u-5 = 10**	

ReCount ReCount Answer	**2*u+3 = 15** = (15-3)+3 = 12 + 3 2*u = 12 = (12/2)*2 = 6*2 u = 6	2*6+3	15
ReCount ReCount Answer	**3*u+4 = 19**		
ReCount ReCount Answer	**4*u+6 = 38**		
ReCount ReCount Answer	**2*u-3 = 15** = (15-3)+3 = 15+3-3 = 18 - 3 2*u = 18 = (18/2)*2 = 9*2 u = 9	2*9-3	15
ReCount ReCount Answer	**3*u-4 = 8**		
ReCount ReCount Answer	**4*u-5 = 23**		
ReCount ReCount Answer	**u/2+3 = 15** = (15-3)+3 = 12 + 3 u/2 = 12 = (12/2)*2 = (12*2)/2 = 24/2 u = 24	24/2+3	15
ReCount ReCount Answer	**u/3+4= 12**		
ReCount ReCount Answer	**u/2-3 = 15** = (15-3)+3 = (15+3)-3 = 18 - 3 u/2 = 18 = (18/2)*2 = (18*2)/2 = 36*2 u = 36	36/2-3	15
ReCount ReCount Answer	**u/4-7 = 5**		

07. Bundle-counting and Next-to Addition Roots Linearity and Integration

Allan Tarp, the MATHeCADEMY.net, October 2019

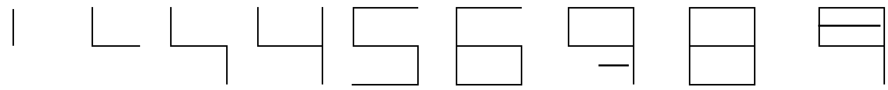
”How old next time?” “Four” the child said and showed 4 fingers. “Four?” I asked and showed 4 fingers held together 2 by 2. “No, that is not four, that is two twos!” the child replied thus insisting upon what exists, bundles of twos, and two of them. From this observation we can ask:

What kind of mathematical learning can take place when children count in bundles less than ten?

The methodology mixes French skepticism and American pragmatism: Traditions are deconstructed and grounded theory creates categories freely when working with the physical fact Many. Thus adding 2 5s and 4 3s on-top means changing units by recounting 4 3s in 5s; and 2 5s and 4 3s are added next-to in 8s by their area. So bundle-counting and next-to addition allows preschoolers to learn proportionality and integration, to be tested by designing appropriate preschool micro-curricula.

In the first micro-curriculum M1 children learn to use sticks to build the number icons up to nine, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second, children learn to count a given total in bundles manually, using an abacus and by using a calculator. In the third, children learn to recount a total in the same unit. In the fourth, children learn to recount a total in a different unit. In the fifth, children learn to add two bundle-numbers on top of each other. In the sixth, children learn to add two bundle-numbers next to each other. In the seventh, children learn to reverse on-top addition. And in the eights, children learn to reverse next-to addition.

In most cases a calculator (to the right) predicts the result of the counting and re-counting jobs.

M1	I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII  1 2 3 4 5 6 7 8 9	
M2	7 1s is how many 3s? I I I I I I I → III III I → B B I → 2B1 3s → 2.1 3s	**7/3 2.some** **7 – 2x3 1**
M3	‘2.7 5s is also how many 5s?’ 2.7 5s = 3.2 5s = 4.-3 5s IIIII IIIII IIIIIII = IIIII IIIII IIIII II = V V V II III = V V V V III	
M4	2 5s is also how many 4s?’ IIIII IIIII = IIII I IIII I = IIII IIII I I so 2 5s = 2.2 4s	**2x5 / 4 2.some** **2x5 – 2 x 4 2**
M5	‘2 5s and 4 3s total how many 5s?’ IIIII IIIII III III III III = V V V V I I so 2 5s + 4 3s = 4.2 5s	**(2x5+4x3) /5 4.some** **2x5+4x3 – 4x5 2**
M6	‘2 5s and 4 3s total how many 8s?’ IIIII IIIII III III III III = IIIIIIII IIIIIIII III III so 2 5s + 4 3s = 2.6 8s	**(2x5+4x3) /8 2.some** **2x5+4x3 – 2x8 6**
M7	‘2 5s and ? 3s total 4 5s?’ IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	**(4x5 – 2x5)/3 3.some** **4x5 – 2x5 – 3x5 1**
M8	‘2 5s and ? 3s total 2.1 8s?’ IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	**(4x5 – 2x5)/3 3.some** **4x5 – 2x5 – 3x5 1**

An abacus can show how next-to addition of 2.2 5s and 1.2 3s gives 2.1 8s

08. Research Project in Bundle-counting and Next-to Addition

Allan.Tarp@MATHeCADEMY.net, October 2019

”How old will you be next time?” I asked the child. “Four”, he answered and showed me four fingers. “Four, you said?” I asked and showed him four fingers held together two by two. “No, that is not four, that is two twos!” the child replied thus insisting upon what exists, bundles of twos, and two of them. Likewise, preschool children have no difficulties counting in other units than ten, even if they only learn how to count in tens. This observation motivates the following question:

What kind of mathematical learning takes places when children count in bundles less than ten?

The methodology comes from the two Enlightenment republics by mixing French skepticism and American pragmatism. First postmodern contingency research will deconstruct the ruling traditions by uncovering hidden differences that might make a difference when tested by designing different micro-curricula, in this case for the last year in preschool mathematics.

When testing the design in a group of students, grounded theory resonating with Piaget assimilation/ accommodation is used to gather observations and create categories. Finally, the categories are validated or refined by tests in other student groups.

The following micro-curricula use activities with concrete material to obtain its learning goals in accordance with Piaget’s principle ‘greifen vor begrifen’ (grasp to grasp). In the first, children learn to use sticks to build the number icons up to nine, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second, children learn to count a given total in bundles manually, using an abacus and by using a calculator. In the third, children learn to recount a total in the same unit. In the fourth, children learn to recount a total in a different unit. In the fifth, children learn to add two bundle-numbers on top of each other. In the sixth, children learn to add two bundle-numbers next to each other. In the seventh, children learn to reverse on-top addition. And in the eights, children learn to reverse next-to addition.

As to concrete material, anything goes in the first micro-curriculum. The others will use sticks and strokes, beads on an abacus, LEGO-like blocks and squares, and a calculator respecting the priority of the operations. Fingers, pegs on a pegboard and other concrete material might also be used.

Some of the curricula can be tested using silent education where the teacher is allowed to demonstrate and guide through actions, but not through words; or by using words form a foreign language not understood by the child.

Curricula:

Micro-curriculum 1. Creating Icon-numbers

A. Deconstructing the tradition

The tradition sees digits and letters as socially constructed symbols receiving meaning through their use. Skepticism would ask if this is indeed the case. Both describe something in the world, sounds and many-ness. However, there is no universal a-sound, b-sound etc. whereas there are many specific examples of four-ness, five-ness etc. and the digit 4 can be seen as four sticks rearranged into a four-icon thus directly showing the degree of many it represents.

So, letters are symbols that do not label something universal but receive their meaning through local use, thus being socially constructed, both as to the sound they represent and as to the symbols used. Digits, on the other hand, are icons containing as many sticks or strokes as they label if written in a special way, that might be socially constructed, but that is globally recognized and that label something that is universal and not socially constructed, different degrees of Many.

To emphasize this fundamental difference between digits and letters, the traditional digits are deconstructed and replaced by digits that more clearly show the degree of many they label; and with a similarity, which does not disturb the meaning of traditional digits. Instead, these can be seen as a quick and slightly sloppy way to write the 'natural' number digits, clearly showing their degree of Many.

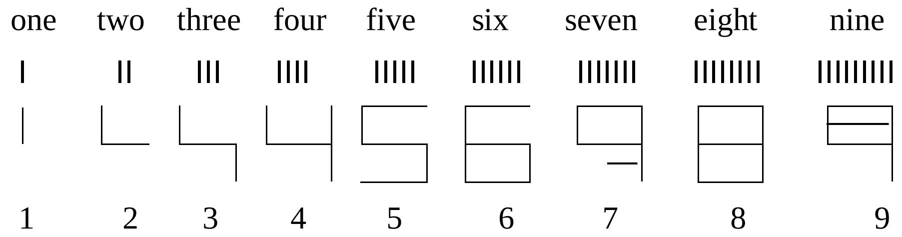


Figure 101. Creating a 5-icon out of five sticks, etc.

Also the educational tradition presenting digits as arbitrary socially constructed symbols of the same kind as letters can be deconstructed by allowing preschool children to experience themselves the construction of digits from 0 to 9 that contains as many things as they represent.

This deconstruction thus raises the following question:

What happens to children when allowed to create number-bundles themselves?

B. Designing a Micro-curriculum

On the floor the children place six hula hoop rings next to each other as six different lands: empty-land, 1-land, 2-land, 3-land, 4-land and 5-land shown by the corresponding number of chopsticks on a piece of paper outside the ring. Each child is asked to find a thing to place in 1-land, and to explain why. Then they are asked to turn their thing so it has the same direction as the chopstick. Finally the group walks around the room and points out examples of 'one thing' always including the unit, e.g. 1 chair, 1 ball, etc.

In the same way each child finds a thing to place in 2-land. The instructor shows how to rearrange two chopsticks to form one 2-icon. The children are asked to pick up two sticks and do the same; and to draw many examples of the 2-icon on a paper discussing with the instructor why the 2-icon on the wall is different from the ones they draw. Now the children are asked to rearrange their 2s in 2-land so they have the same form as the 2-icon. Again the group walks around the room and points out examples of 'two things' that is also called 'one pair of things'. This is now repeated with 3-land where three things are called one triplet.

Before going on to 4-land the instructor asks the children to do the same with empty-land. Since the empty-icon cannot be made by chopsticks the instructor ask for proposals for an empty-icon hoping that one or more will suggest the form of the ring, i.e. a circle. Again the group walks around the room to try to locate examples of ‘no things’ or zero things.

Now, the activity is repeated with 4-land where the children are asked to suggest an icon for four made by four sticks. When summing up the teacher explains that the adults have rejected the square since it reminds too much of a zero, so the top stick is turned and placed below the square to the right. Here the children are asked to rearrange their 4s in 4-land so they have the same form as a square, and as the 4-icon. And again, the group walks around the room and points out examples of ‘four things’ that is also called ‘a double pair’.

Now the activity is repeated with 5-land. Here the instructor asks the children to suggest an icon for five made by 5 sticks. When summing up the teacher explains that the adults have decided to place the five stick in an s-form. When walking around the room to point out examples a discussion is initiated if ‘five things’ is the same as a pair plus a triplet, and as a double pair plus one. This activity can carry on to design icons for the numbers from six to nine realizing that the existing icons can be recycled if bundling in tens.

Finally, the instructor invites the children to experience that when counting in 5s we do not use the word five or the icon five since the number sequence becomes 1, 2, 3, 4, Bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1 etc. So when counting in tens we do not need a ten-icon.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these. In this case, the different ways to construct icons allow for additional observations as to preferences and performances.

Examples of observations

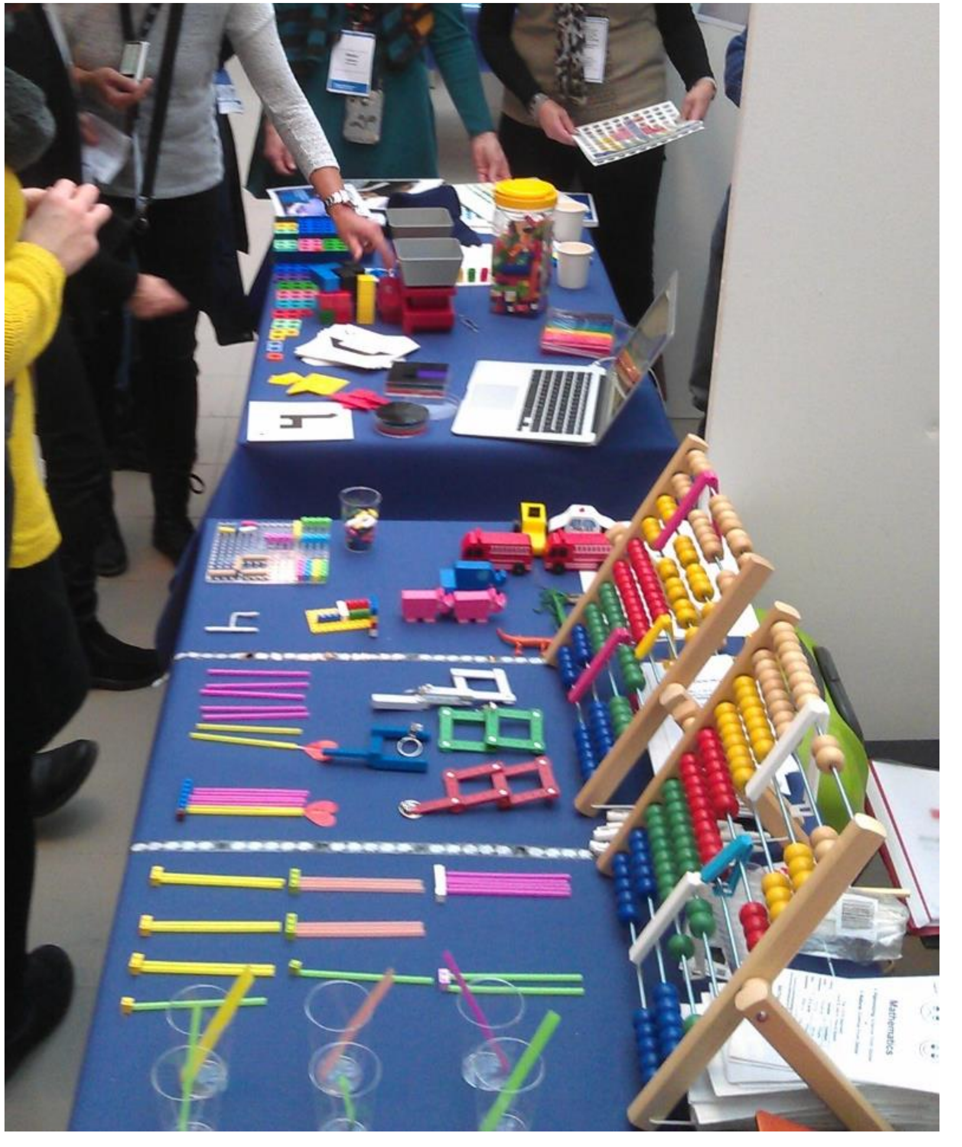
When trying out the micro-curriculum, the preschool teachers are asked to look for examples of re-cognition and new cognition. One teacher noticed the confusion created by asking the children to bring things to empty-land. It disappeared when one child was asked what he had just put into the ring and answered no elephant. Now all of the children were eager to put no cars, no planes etc. into the ring. Later the teacher witnessed children discussing why the 3-icon was not a triangle, and later used the word four-angle for the square. Also this teacher noticed that some children began to use their fingers instead of the chopsticks. Under the walk around the room a fierce discussion about cheating broke out when a child suggested that clapping his hand three times was also an example of three things. Its not, another child responded. It is. No its not! Why not? Because you cannot bring it to 3-land! Let’s ask the teacher! After telling about space and time, children produced other examples as three knocks, three steps, three rounds around a table, three notes. Other children began to look at examples of threes at their own body soon finding three fingers, three parts on a finger, and three hands twice when three children stood side by side and the middle one lent out his two hands to his neighbors.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Photos from the MATHeCADEMY.net stand at the Mathematics Biennale in Sweden 2014

A difference between its Piaget based approach using concrete real-world materials and the neighbor stand using a Vygotsky approach with symbols.



Micro-curriculum 2. Counting in Bundles

A. Deconstructing the Tradition

Traditionally, counting takes place by stacking bundles of tens where e.g. 345 means a total of 3 ten-tens and 4 tens and 5 ones. Using ten as bundle-size is based on the biological fact that we have ten fingers.

Skepticism could point out that only accepting ten as a bundle-size will give totals the same unit. This might hide learning possibilities coming from changing units when counting in different icon-bundles less than ten; and that might be problematic since changing units, also called proportionality or linearity, is a core part of mathematics. Furthermore, a calculator cannot predict the counting result since ten does not have its own icon on a calculator.

This deconstruction raises the question: '*What learning possibilities occur if allowing preschool children in to count in bundles less than ten?*

B. Designing a Micro-curriculum

It is a natural fact that we live in space and in time. To include both when counting, we can introduce two different ways of counting: counting in space, geometry-counting, and counting in time, algebra-counting.

Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, and with squares.

Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, and with strokes.

<u>Counting task 1:</u> '6 1s is how many 2s?'

Using geometry-counting, 6 blocks are placed on a table and counted in 2s by taking away a 2-bundle 3 times to be stacked on top of each other, and reported orally as '6 1s can be counted as 3 2s'. Then the stack of 3 2s is written down as T = 3 x 2 = 3 2s using the multiplication symbol as an icon showing the three times lifting up of the 2s. To use an abacus in geometry mode, again 6 beads are moved to the right on the first from below separated from the above line by a rubber band. Now we move 1 2s to the left and 1 2s to the right on the second line. Again we move 1 2s to the left and 1 2s to the right on the third line. And, again we move 1 2s to the left and 1 2s to the right on the fourth line. Again, the result is reported as '6 1s can be counted as 3 2s'. On a squared paper we draw the total as 3x2 squares.

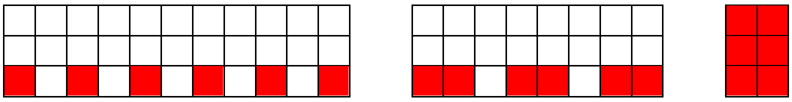


Figure 201. Counting 6 1s as 3 2s by bundling and stacking squares

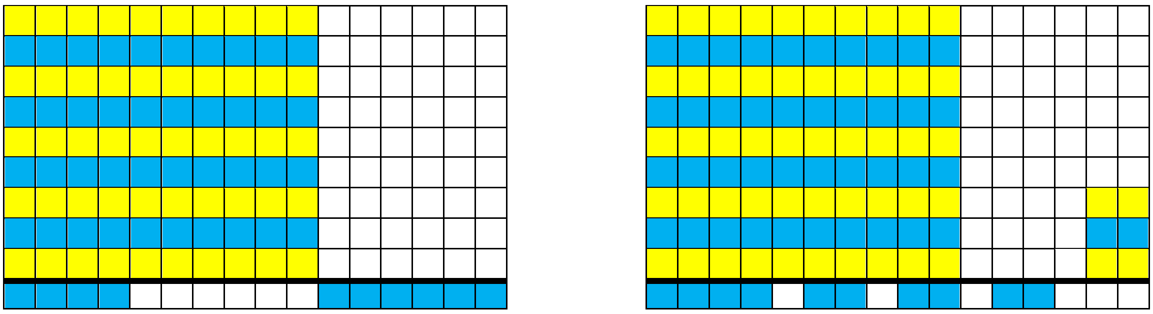


Figure 202. Counting 6 1s as 3 2s on a western ten by ten abacus in geometry mode

Using algebra-counting, 6 sticks are placed on a table and counted in 2s by taking away a 2-bundle 3 times. Orally we report this as '6 1s can be counted as 3 2s'. Then the result is written down using ‘bundle-writing’ and ‘decimal-writing’ with a dot to separate the bundles from the unbundled, i.e. as ‘T = 3B0 2s = 3.0 2s.

On an abacus in algebra-mode, 6 beads are moved to the right on the bottom line. For each 2-bundle moved to the left one bead is moved to the right on the above line, again showing that 6 1s can be counted as 3 2s.

I I I I I I → II II II → 3B 2s → 3.0 2s

Figure 203. Counting 6 1s as 3 2s by sticks, bundle-writing and decimal-writing

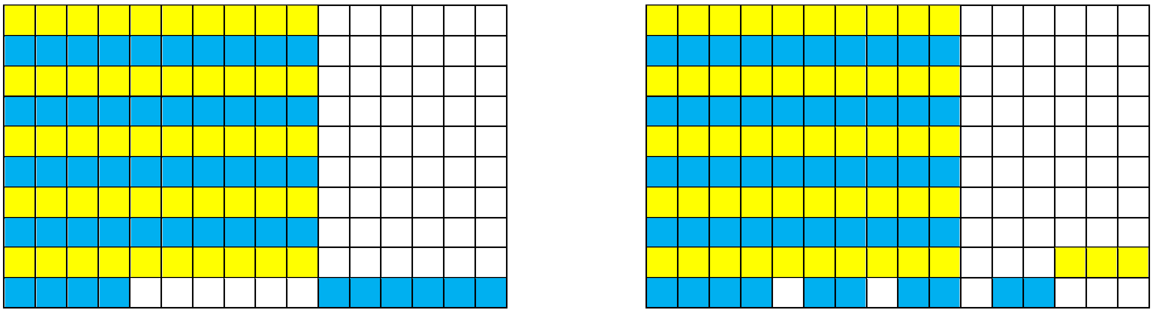


Figure 204. Counting 6 1s as 3 2s on a western ten by ten abacus in algebra mode

Finally, we use the calculator to predict the result. The calculator has two icons for taking away, subtraction showing the trace left when taking away just once, and division showing the broom wiping away several times. So entering ‘6/2’ means asking the calculator ‘from 6 we take away 2s how many times?’ The calculator gives the result ‘3’. To test the result we enter ‘6 – 3x2’ to ask the calculator ‘from 6 we take away 3 2s leaving what?’ As expected, the answer is 0.

6 / 2	3
6 – 2 x 3	0

Figure 205. Using a calculator to predict the result of counting 6 1s in 2s, and to check for singles

<u>Counting task 2:</u> ‘6 1s is how many 3s?’, is performed in a similar way, but first we ask the calculator to predict the result.

<u>Counting task 3:</u> ‘8 1s is how many 2s?’ is performed in a similar way, but first we ask the calculator to predict the result.

<u>Counting task 4:</u> ‘8 1s is how many 4s?’ is performed in a similar way, but first we ask the calculator to predict the result.

<u>Counting task 5:</u> ‘7 1s is how many 2s?’

Using geometry-counting, 7 blocks are placed on a table and counted in 2s by taking away a 2-bundle 3 times to be stacked on top of each other leaving 1 unbundled to the right, and reported orally as ‘7 1s can be counted as 3 2s and 1’. Then the stack of 3 2s is written down as T = 3.1 2s to use an abacus in geometry mode, again 7 beads are moved to the right of the bottom line. Now we move 1 2s to the left and 1 2s to the right on the first line above. Again we move 1 2s to the left and 1 2s to the right on the second line above. And, again we move 1 2s to the left and 1 2s to the right on the third line above leaving 1 unbundled on the bottom line. Again, the result is reported as ‘7 1s can be counted as 3 2s and 1’. On a squared paper, we draw the total as 3x2 squares with an additional square to the right in a different colour.

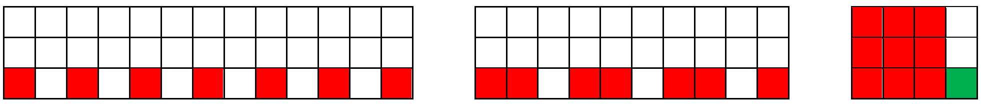


Figure 206. Counting 7 1s as 3.1 2s by bundling and stacking squares

Using algebra-counting, 7 sticks are placed on a table and counted in 2s by taking away a 2-bundle 3 times leaving 1 stick unbundled. Orally we report this as ‘7 1s can be counted as 3 2s and 1’.

Then the result is written down using ‘bundle-writing’ and ‘decimal-writing’ with a dot to separate the bundles from the unbundled, i.e. as ‘T = 3B1 2s = 3.1 2s.

On an abacus, 7 beads are moved to the right of the bottom line. For each 2-bundle moved to the left one bead is moved to the right on the above line, again showing that 7 1s can be counted as 3 2s and 1, i.e. as 3.1 2s.

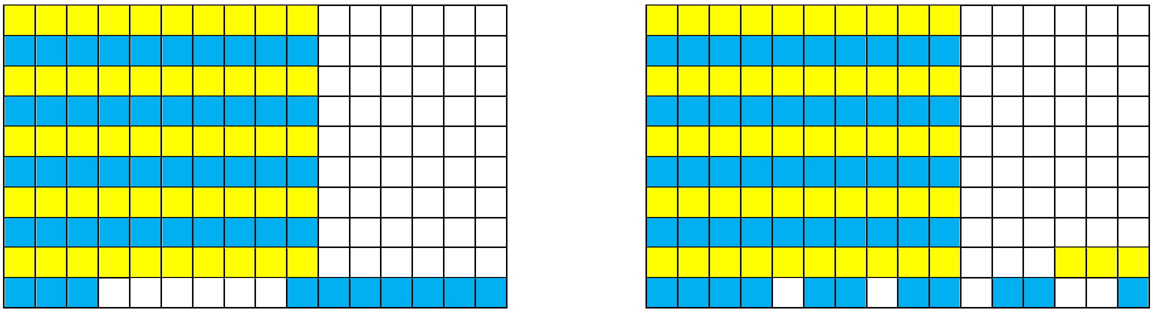


Figure 207. Counting 7 1s as 3.1 2s on a western ten by ten abacus in algebra mode

Finally, we use the calculator to predict the result. Entering ‘7/2’ means asking the calculator ‘from 7 we take away 2s how many times?’ The calculator gives the result ‘3.some’. To see how much is left unbundled we enter ‘7 – 3x2’ to ask the calculator ‘from 7 we take away 3 2s leaving what?’. As expected, the answer is 1. A display showing that 7 - 3x2 = 1 indirectly predicts that 7 can be counted as 3 2s and 1.

7 / 2	3.some
7 – 3 x 2	1

Figure 208. Using a calculator to predict the result of counting 7 1s in 2s, and to check for singles

Counting task 6: ‘7 1s is how many 3s?’ is performed in a similar way, but first we ask the calculator to predict the result.

Counting task 7: ‘7 1s is how many 5s?’ is performed in a similar way, but first we ask the calculator to predict the result.

Rolling dices can inspire additional counting tasks, e.g. by adding 1 or 2 to the numbers, so that rolling a 2 and a 5 creates the counting tasks ‘5 is how many 2s?’, ‘6 is how many 3s?’, ‘7 is how many 4s?’, etc.

B2. Designing an Additional Micro-curriculum

Most languages use ten as the bundle size calling ‘bundle and one’ for ‘ten-one’ and ‘bundle and two’ for ‘ten-two’, etc. English instead uses the words ‘eleven’ and ‘twelve’ meaning ‘one-left’ and ‘two-left’ before passing on to thirteen, ten-three. Inspired by this, nine and eight could be called ‘less 1’ and ‘less 2’ to be symbolized as ‘-1’ and ‘-2’ thus creating the following counting sequence to be practiced counting fingers and counting beads on an abacus:

1, 2, 3, 4, 5, les 4, less 3, less 2, less 1, ten.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to perform counting in bundles, algebra-counting in time, geometry-counting in space and calculator prediction, allow for additional observations as to preferences and performances.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Micro-curriculum 3. Re-counting Bundle-numbers in the Same Bundle

A. Deconstructing the Tradition

Traditionally, recounting does not takes place when using ten as the only allowed bundle-size.

Skepticism could point out that this will provide one and only one way to count a total, which will hide the learning possibilities coming from recounting a total in the same unit to deal with overloads, created when adding and needed when subtracting totals; and that might be problematic since adding and subtracting are core part of mathematics.

This deconstruction raises the question:

What learning possibilities occur if allowing preschool children to recount bundle-numbers in the same bundle?

B. Designing a Micro-curriculum

<u>Counting task 1:</u> 'Recount 3 4s in 4s?'

Using geometry-counting, a stack of 3 4-bundles is placed on a table. One is split into 1s and placed vertically to the right of the stack. Orally we report this as '3 4s can be recounted as 2 4s and 4'. On an abacus, 4 beads are moved to the right on the lines 2 to 4 from below reserving the bottom line for the singles. Moving the top 4-bundle to the left allows moving 4 singles to the right on the bottom single-line, again showing that 3 4s can be recounted as 2.4 2s. On a squared paper we draw the total as a stack of 2x4 squares and 4 1s to the right of the stack. Likewise, we can show that 3 4s can be recounted to 1.8 4s.

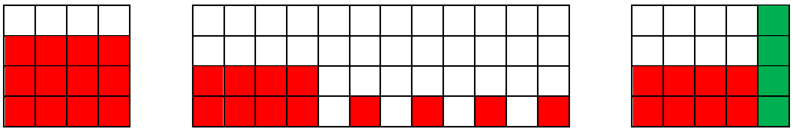


Figure 301. Recounting 3 4s as 2.4 4s by bundling and stacking squares

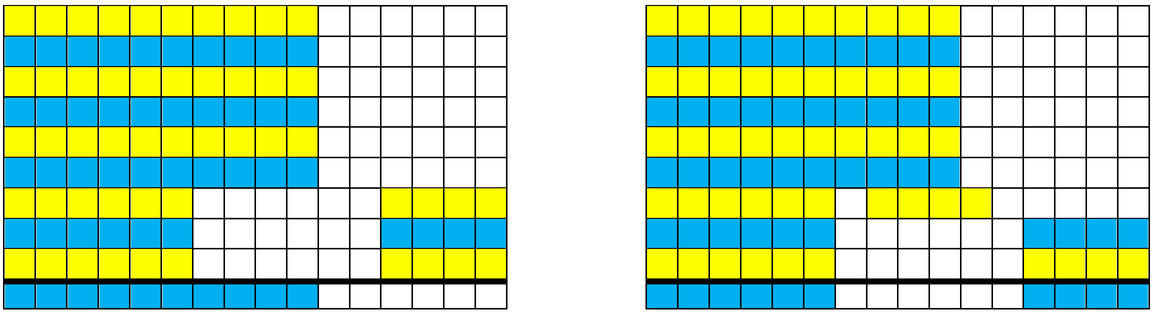


Figure 302. Recounting 3 4s as 2.4 4s on a western abacus in geometry mode

Using algebra-counting, a total of 3 4s are placed on a table. One if the 4-bundles is split into 1s. Orally we report this as '3 4s can be recounted as 2 4s and 4'.

Then the result is written down using 'bundle-writing' and 'decimal-writing' with a dot to separate the bundles from the unbundled, i.e. as 'T = 3 4s = 2B4 4s = 2.4 4s.

On an abacus, 3 bundle-beads are moved to the right on the bundle-line. Moving one bundle-bead to the left allows moving 4 beads to the right on the single line, again showing that 3 4s can be recounted as 2.4 4s. Likewise, we can show that 3 4s can be recounted to 1.8 4s.

IIII IIII IIII → IIII IIII I I I I → 2B 4 4s → 2.4 4s

Figure 303. Recounting 3 4s as 2.4 4s by sticks, bundle-writing and decimal-writing

IIII IIII I I I I → 3B-4 4s → 3.-4 4s

Figure 304. Recounting 2 4s as 3 less 4 4s by sticks, bundle-writing and decimal-writing

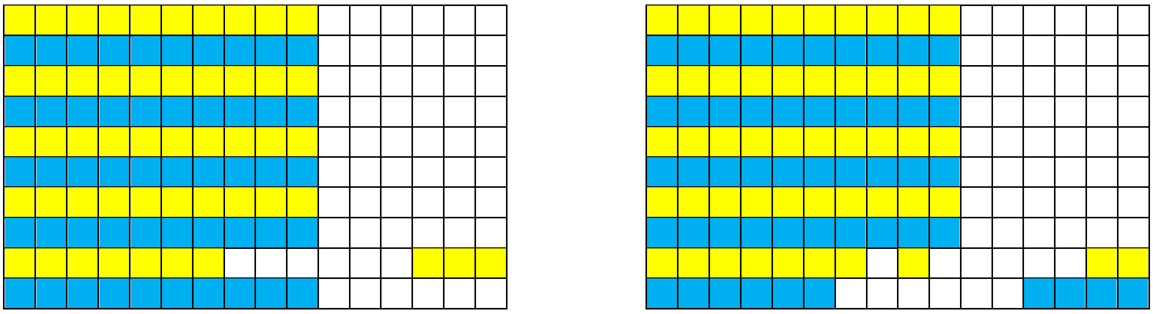


Figure 305. Recounting 3 4s as 2.4 4s on a western abacus in algebra mode

When recounting in the same unit, a calculator cannot be used to predict the result. Instead, bundles or bundle-writing can be used to recount in the same unit by showing directly how removing one bundle of 4s allows adding 4 1s.

3 4s:	***		**	****	*	********

3 4s: 3B0 4s = 3-1B0+4 4s = 2B4 4s = 2.4 4s = 2-1B4+4 4s = 1B8 4s = 1.8 4s

Figure 305. Recounting 3 4s as 2.4 4s and 1.8 4s using bundles and bundle-writing

Vice versa, removing 4 1s allows adding 1 bundle of 4s

1.8 4s:	*	********	**	****	***	

1.8 4s: 1B8 4s = 1+1B 8-4 4s = 2B4 4s = 2.4 4s = 2+1B4-4 4s = 3B0 4s = 3 4s

Figure 306. Recounting 1.8 4s as 2.4 4s and 3 4s using bundles and bundle-writing

Counting task 2: ‘Recount 2 3s in 3s?’ is performed in a similar way.

Rolling dices can inspire additional counting tasks, e.g. by adding 1 or 2 to the numbers, so that rolling a 2 and a 4 creates the counting tasks ‘Recount 4 2s in 2s?’, ‘Recount 5 3s in 3s?’etc.

Note that recounting in small units will avoid numbers above nine.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to perform counting in bundles, algebra-counting in time, geometry-counting in space allow for additional observations as to preferences and performances.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Micro-curriculum 4. Re-counting Bundle-numbers in a Different Bundle

A. Deconstructing the Tradition

Traditionally, recounting does not take place when using ten as the only allowed bundle-size.

Skepticism could point out that this will give all totals the same unit, which will hide the learning possibilities coming from changing units when counting in different icon-bundles less than ten; and that might be problematic since changing units, also called proportionality or linearity, is a core part of mathematics. Furthermore, where it cannot predict the counting result with no ten button, now a calculator gets a central role as a recount-predictor.

This deconstruction raises the question:

What learning possibilities occur if allowing preschool children to recount bundle-numbers in a different unit?

B. Designing a Micro-curriculum

Counting task 1: '3 4s is how many 5s?'

Using geometry-counting, a stack of 3 4s is placed on a table. The top bundle is changed to 1s in the single stack to the right and twice a stick is removed from there to enlarge the 4- bundles to a 5 bundle. This shows that '3 4s can be recounted as 2.2 5s.' To use an abacus in geometry mode, 3 4s are moved to the right leaving the bottom line for the singles. From the top, a 4-bundle is moved to the left to change the 4- bundles above to 5-bundles. Again, the result is reported as '3 4s can be recounted as 2.2 5s'. On a squared paper we draw the total as a 2x5 block with a 2x1 block to the right.

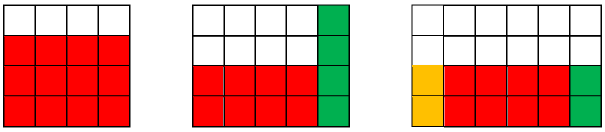


Figure 401. Recounting 3 4s as 2.2 5s by bundling and stacking squares

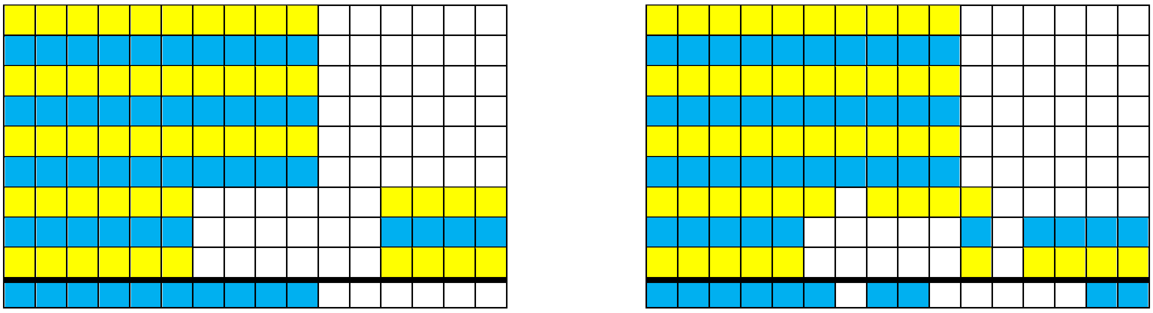


Figure 402. Recounting 3 4s as 2.2 5s on a western abacus in geometry mode

Using algebra-counting, a total of 3 4s is placed on a table and split into 1s. Now the total is counted in 5s by taking away a 5-bundle 2 times. Orally we report this as '3 4s can be counted as 2 5s and 2'.

Then the result is written down using 'bundle-writing' and 'decimal-writing' with a dot to separate the bundles from the unbundled, i.e. as 'T = 2B2 5s = 2.2 5s. On an abacus, 3 beads are moved to the right on the bundle-line. For each bundle moved to the left, 4 beads are moved to the right on the single line. Moving 2 beads to the left on the bundle-line allows moving 8 beads to the right on the single-line. This allows moving 5 beads to the left on the single line and to move 1bead to the right on the top-line counting the 5-bundles. Moving the last 3-bundle to the left gives 7 beads on the single line, from which 5 1s can be changed to 1 5s on the top line.

Again, we see that 3 4s can be recounted as 2.2 5s.

IIII IIII IIII → IIIII IIIII I I → 2B2 5s → 2.2 5s

Figure 403. Recounting 3 4s as 2.2 5s by sticks, bundle-writing and decimal-writing

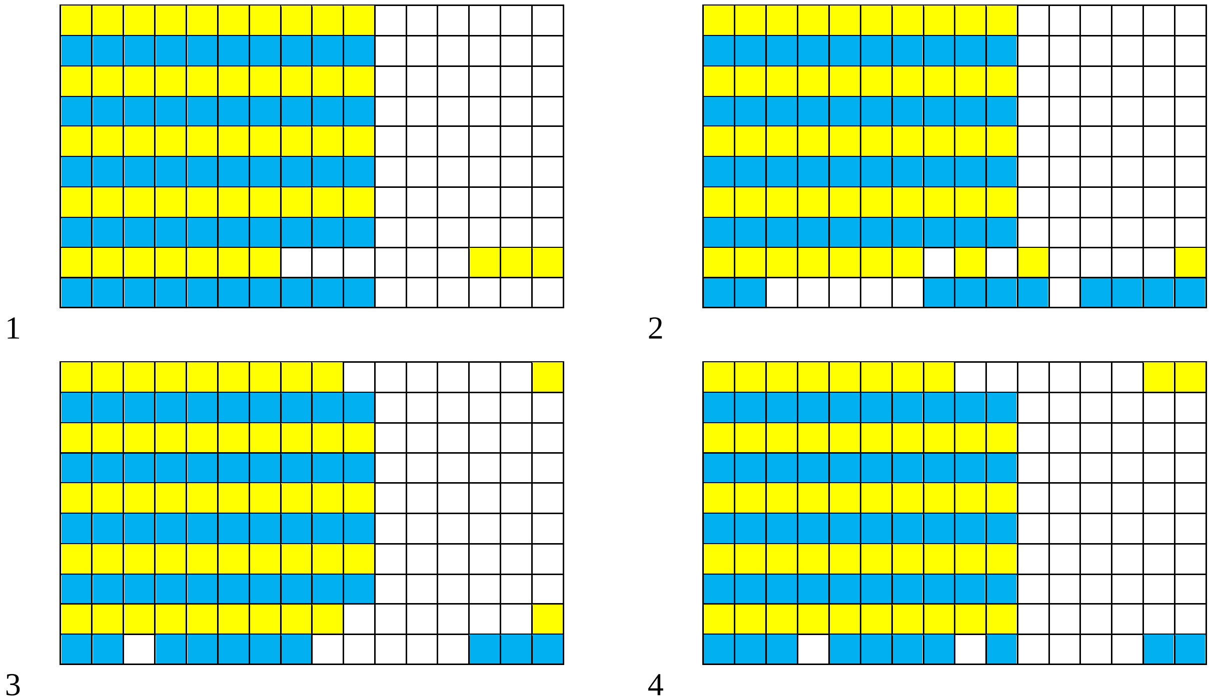


Figure 404. Recounting 3 4s as 2.2 5s on a western abacus in algebra mode

Finally, we use the calculator to predict the result. Entering ‘3x4/5’ means asking the calculator ‘from 3 4s we take away 5s how many times?’ The calculator gives the answer ‘2.some’. To see what is left we enter ‘3x4 – 2x5’ to ask the calculator ‘from 3 4s we take away 2 5s leaving what?’ As expected, the answer is 2. A display showing that 3x4 – 2x5 = 2 indirectly predicts that 3 4s can be recounted as 2 5s and 2.

3 x 4 / 5	2.some
3 x 4 – 2 x 5	2

Figure 405. Using a calculator to predict the result of recounting 3 4s in 5s, and to check for singles

<u>Counting task 2</u>, ‘4 5s is how many 6s?’, is performed in a similar way, but first we ask the calculator to predict the result.

<u>Counting task 3</u>: ‘3.2 4s is how many 5s?’ is reduced to ‘3 4s is how many 5s?’, adding 2 singles in the end: ‘Since 3 4s can be recounted to 2.2 5s, 3 4s and 2 can be recounted to 2.4 5s.’

<u>Counting task 4</u>, ‘4.3 5s is how many 6s?’ is performed in a similar way, but first we ask the calculator to predict the result.

Rolling dices can inspire additional counting tasks, e.g. by adding 1 or 2 to the numbers, so that rolling a 2 and a 4 and a 5 creates the counting tasks ‘Recount 2 4s in 5s?’, ‘Recount 3 5s in 6s?’, etc.

Note that recounting in a bigger unit will avoid numbers above nine.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these. In this case, the three ways to recount totals in a different bundle, algebra-counting in time, geometry-counting in space and calculator prediction allow for additional observations as to preferences and performances.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Micro-curriculum 5. Adding Bundle-numbers OnTop

A. Deconstructing the Tradition

Traditionally, we add totals on top of each other creating no problems when counted in tens except for dealing with overloads by carrying.

Skepticism could point out that this does not give a method to deal with nor experience with on-top addition of totals counted in different units, as e.g. adding 2 3s and 4 5s on top as 3s or as 5s. And that might be problematic since changing units, also called proportionality or linearity, is a core part of mathematics. This deconstruction raises the question: *What learning possibilities occur if allowing preschool children to perform on-top addition of bundle-numbers.*

B. Designing a Micro-curriculum

Adding task 1: '4 5s and 2 3s total how many 5s?'

To be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 4 5s gives a grand total of 5.1 5s.

On an abacus in geometry mode, leaving the bottom line empty, a stack of 4 5s is moved to the right and a stack of 2 3s is moved to the middle. Now, the 2 3s is changed to 6 1s on the bottom line allowing one additional 5s to be moved to the top of the stack of 5s to show the grand total is 5.1 5s.

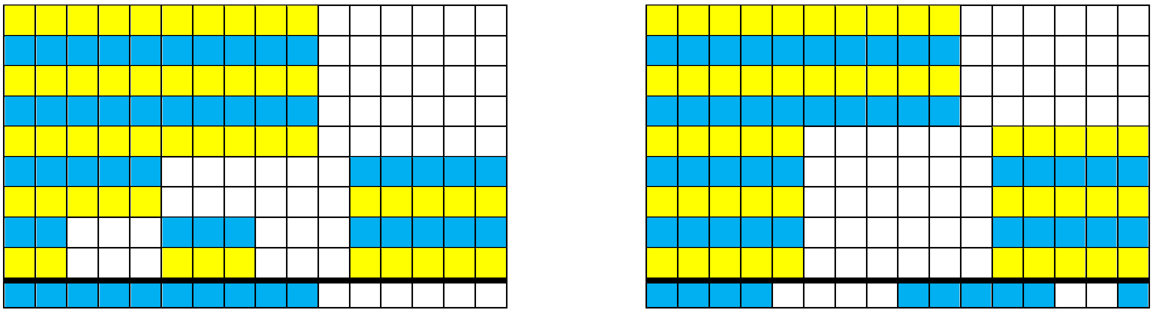


Figure 501. Adding 2 3s to 4 5s as 5.1 5s on a western abacus in geometry mode

On an abacus in algebra mode and split in the middle by a horizontal rubber band, the 4 5s are moved to the right on the bundle line below the band, and the 2 3s on the bundle line above the band. Now the 2 3s above the band is changed to 6 1s below, where moving 5 1s to the left allows moving 1 5s to the right, showing the grand total to be 5.1 5s.

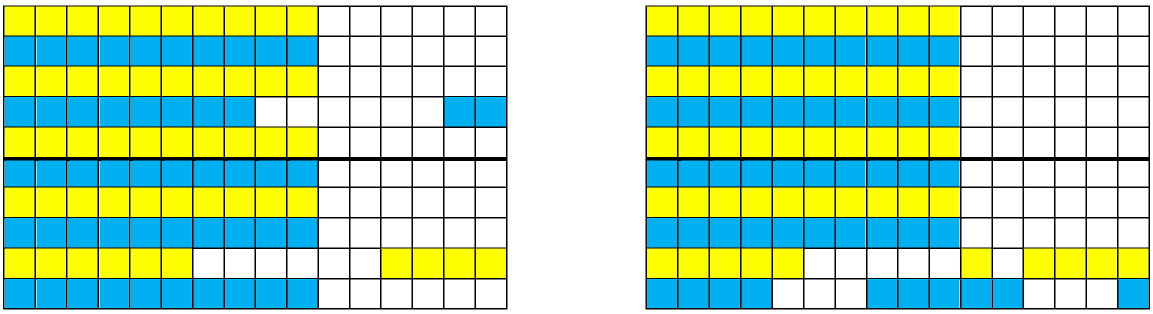


Figure 502. Adding 2 3s to 4 5s as 5.1 5s on a western abacus in algebra mode

When using a calculator to predict the result, we include the two totals in a bracket before counting in 5s: Asking '(4x5 + 2x3)/5' gives the answer 5.some. Then asking '(4x5 + 2x3) – 5x5 gives the answer 1. So the grand total is 5.1 5s. A display showing that (4x5 + 2x3) – 5x5 = 1 indirectly predicts that 2 3s can be added to 4 5s as 5.1 5s.

(2 x 3 + 4 x 5)/ 5	5.some
(2 x 3 + 4 x 5) – 5 x 5	1

Figure 503. Using a calculator to predict the result of adding 2 3s to 4 5s as 5.1 5s

<u>Adding task 2</u>: ‘4.4 5s and 2.2 3s total how many 5s?’

To be added on-top, the units must be the same, so 2.2 3s must be recounted in 5s giving 1.3 5s that added to the 4.4 5s gives a grand total of 5.7 5s that can be recounted to 6.2 5s to remove the overload.

On an abacus in geometry mode, a stack of 4.4 5s is moved to the right and a stack of 2.2 3s is moved to the middle. Now, the 2.2 3s is changed to 1s on the bottom line allowing 2 additional 5s to be moved to the top of the stack of 5s to show the grand total is 6.2 5s.

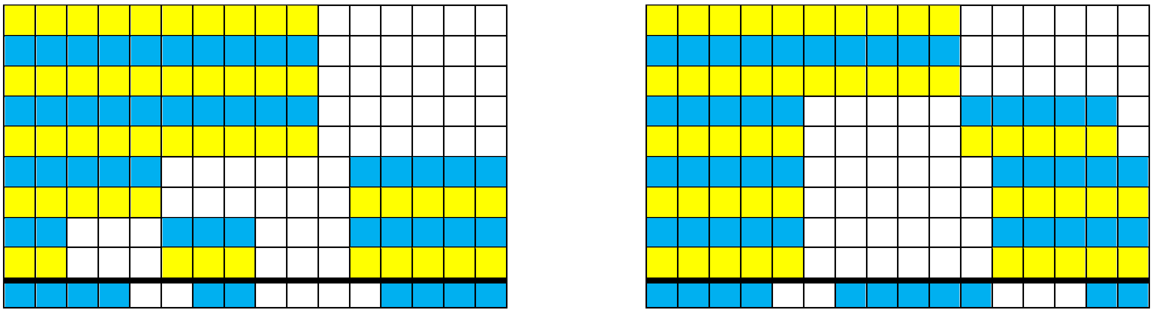


Figure 504. Adding 2.2 3s to 4.4 5s as 6.2 5s on a western abacus in geometry mode

On an abacus in algebra mode and split in the middle by a horizontal rubber band, the 4.4 5s are moved to the right below the band, and the 2.2 3s above the band. Now the 2.2 3s is changed to 1s added to the 4 1s on the single line below the band and bundled in 5s when possible to give a grand total of 6.2 5s.

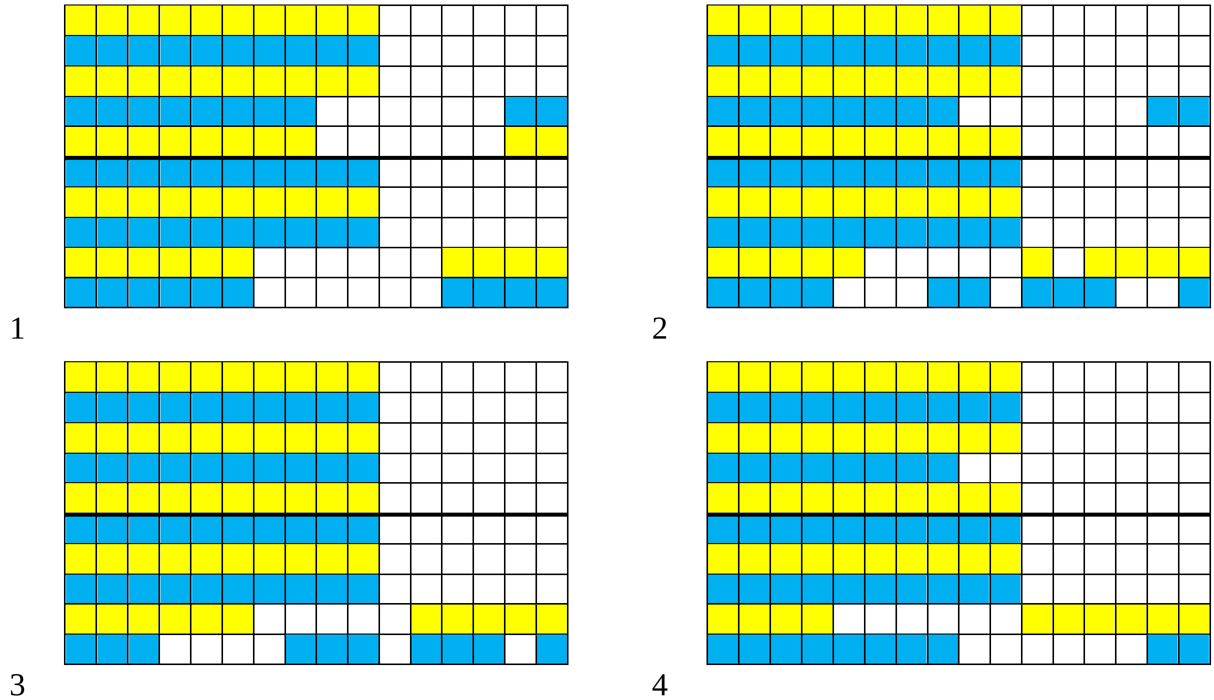


Figure 505. Adding 2.2 3s to 4.4 5s as 6.2 5s on a western abacus in algebra mode

When using a calculator to predict the result, we include the two totals in a bracket before counting in 5s: Asking ‘(4x5+4 + 2x3+2)/5’ gives the answer 6.some. Then asking ‘(4x5+4 + 2x3+2)– 6x5 gives the answer 2. So the grand total is 6.2 5s.

(4 x 5 + 4 + 2 x 3 + 2)/ 5	6.some
(4 x 5 + 4 + 2 x 3 + 2) – 6 x 5	2

Figure 506. Using a calculator to predict the result of adding 2.2 3s to 4.4 5s as 6.2 5s

Rolling dices can inspire additional adding tasks, e.g. by adding 1 or 2 to the numbers, so that rolling a 2 and a 3 and a 4 and a 5 creates the adding tasks ‘Add 2 4s to 3 5s?’, ‘Add 3 5s to 4 6 s?’, etc.

Note that adding in the bigger unit will avoid numbers above nine.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to add totals using geometry or algebra or a calculator for prediction allow for additional observations as to preferences and performances.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Micro-curriculum 6. Adding Bundle-numbers NextTo

A. Deconstructing the Tradition

Traditionally, we add totals on top of each other creating no problems when counted in tens except for dealing with overloads by carrying. Adding next-to is called integration and introduced as the inverse operation to differentiation. Both are seen as examples of the abstract limit concept, hence postponed to gifted students at the end of high school.

Skepticism could point out that not allowing counting in bundle-numbers forces the unit ten upon all numbers, which makes it irrelevant to change units or to add next-to, as e.g. adding 2 3s and 4 5s as 8s. That might be problematic since adding next-to, also called integration, is a core part of mathematics, and accepting counting in bundle-numbers makes integration a natural way to add in preschool.

This deconstruction raises the question:

What learning possibilities occur if allowing preschool children to perform next-to addition of bundle-numbers.

B. Designing a Micro-curriculum

<u>Adding task 1</u>: '4 5s and 2 3s total how many 8s?'

Adding the two totals next-to each other as 8s is possible if recounting both in 8s to be added on top, showing that 4 5s = 2.4 8s and 2 3s = 0.6 8s, thus resulting in a grand total of 2.4 8s + 0.6 8s which can be recounted to 3.2 8s.

On an abacus in geometry mode, leaving the bottom line empty, a stack of 4 5s is moved to the right and a stack of 2 3s is moved to the middle and then to the right. Moving 3 1s to the left on the top line allows filling up the 8 on the line below. The reaming 2 1s on the top line is moved to the bottom line showing the grand total to be 3.2 8s.

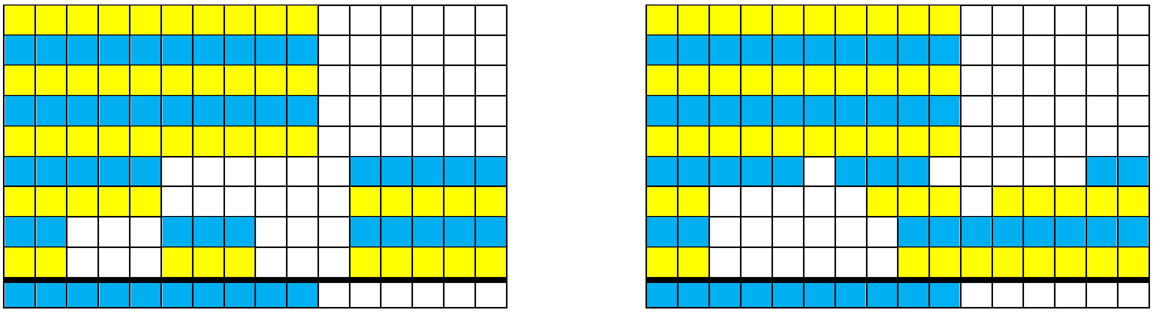


Figure 601. Adding 2 3s to 4 5s as 3.2 8s on a western abacus in geometry mode

An abacus in algebra mode can be used to recount the two totals in 8s to be added afterwards.

When using a calculator to predict the result, we include the two totals in a bracket before counting in 5s: Asking '(4x5 + 2x3)/8' gives the answer 3.some. Then asking '(4x5 + 2x3) – 3x8 gives the answer 2. So the grand total is 3.2 8s.

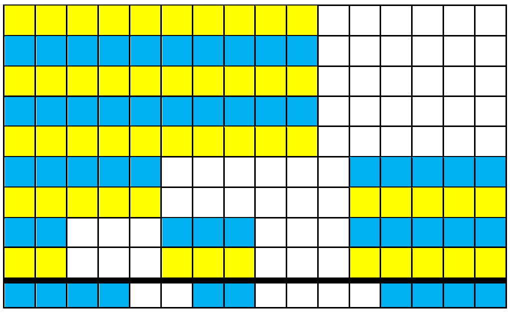
(4 x 5 + 2 x 3)/ 8	3.some
(4 x 5 + 3 x 2) – 3 x 8	2

Figure 602. Using a calculator to predict the result of adding 2 3s to 4 5s as 3.2 8s

<u>Adding task 2</u>: '4.4 5s and 2.2 3s total how many 8s?'

Adding the two totals next-to each other as 8s is possible if recounting both in 8s to be added on top, showing that 4.4 5s = 3 8s and 2.2 3s = 1 8s and, thus resulting in a grand total of 3 8s + 1 8s which is 4 8s.

On an abacus in geometry mode, 2 1s is moved to the right on the bottom line, and a stack of 2 3s is moved to the right above. Then 4 1s and a stack of 4 5s is moved to the middle and then to the right. Moving 3 1s to the left on the top line allows filling up the 8 on the line below. The reaming 2 1s on the top line is moved to the bottom line showing the grand total to be 4 8s.



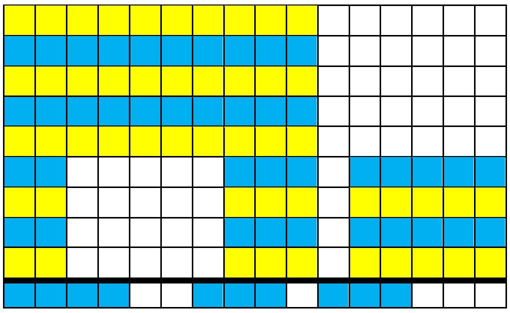


Figure 603. Adding 2.2 3s to 4.4 5s as 4 8s on a western abacus in geometry mode

An abacus in algebra mode can be used to recount the two totals in 8s and added afterwards.

When using a calculator to predict the result, we include the two totals in a bracket before counting in 5s: Asking '(4x5+4 + 2x3+2)/8' gives the answer 4. So the grand total is 4 8s.

(4 x 5 + 4 + 2 x 3 + 2)/ 8	4
(4 x 5 + 4 + 3 x 2 + 2) – 4 x 8	0

Figure 604. Using a calculator to predict the result of adding 2.2 3s to 4.4 5s as 4 8s

Rolling dices can inspire additional adding tasks, e.g. by adding 1 or 2 to the numbers, so that rolling a 2 and a 3 and a 4 and a 5 creates the adding tasks 'Add 2 4s to 3 5s?', 'Add 3 5s to 6 4 s?', etc.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to add totals using geometry or algebra or a calculator for prediction allow for additional observations as to preferences and performances. Furthermore, using bundle-numbers to perform integration in preschool is a new idea that has never been researched before.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Some of the micro-curricula can be tasted as 'silent-math' where you are not allowed to speak.

Micro-curriculum 7. Reversing Adding Bundle-numbers OnTop

A. Deconstructing the Tradition

Traditionally, reversed addition as '2 + ? = 5' is reformulated to '2 + u = 5', called an equation and defined as an example of an equivalence relation between the two 'number-names' u+2 and 5. Performing identical operation to both number-names will keep the equivalence intact. By applying the laws of associativity and community as well as the definitions of a neutral and an inverse element, it is possible to change one of the number-names to a single u called the solution of the equation. Testing the solution is not necessary because of the equivalence relation. Because of the degree of abstractness, equations are traditionally postponed to secondary school.

2 + u = 5	Addition has 0 as its neutral element, and 2 has -2 as its inverse element
(2 + u) + (-2) = 5 + (-2)	Adding 2's inverse element to both number-names
(u + 2) + (-2) = 3	Applying the commutative law to u + 2. 3 is the short number-name for 5+(-2)
u + (2 + (-2)) = 3	Applying the associative law
u + 0 = 3	Applying the definition of an inverse element
u = 3	Applying the definition of a neutral element

Figure 701. Solving the equation 2 + u = 5 the traditional algebraic way

Skepticism could point out that instead of seeing reversed equation as an example of an equation, the solution of which is seen as an example of applying abstract algebra, reversed addition could be treated as what it is, reversed addition. Furthermore equations with only one occurrence of the unknown number as is the case in most formulas in geometry, commence and physics could be treated as reversed calculations. Finally, a solution to a reversed addition should always be tested by the corresponding forward addition.

Because of the close relationship between forward and reversed addition, equations should be introduced in preschool along with forward addition. This deconstruction raises the question:

What learning possibilities occur if allowing preschool children to solve equations as reversed addition?

B. Designing a Micro-curriculum

Reversing task 1: '2.3 5s and ? 5s total 4.1 5s?'

On an abacus in geometry mode, a stack of 4.1 5s is moved to the right. Changing the top 5-bundle to 5 1s on the single line allows taking away the 2.3 5s to leave 1.3 5s

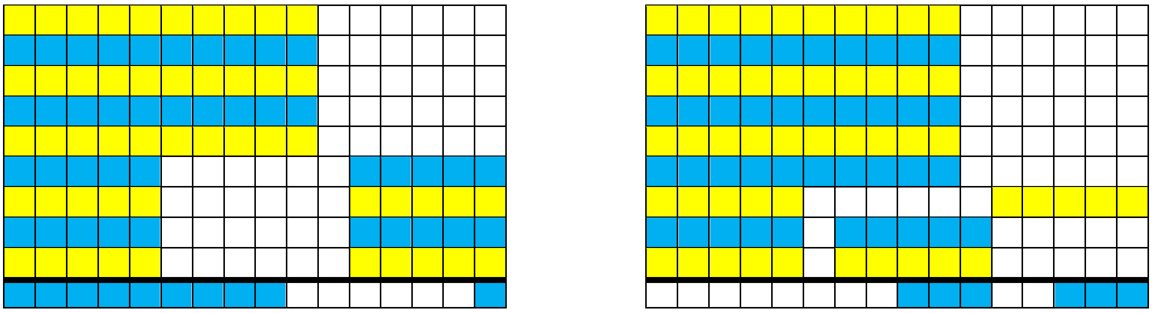


Figure 702. Solving the equation 2.3 5s + ? 5s total 4.1 5s on a western abacus in geometry mode

Using algebra-counting we recount the 4.1 5s to 3.6 5s to be able to take away what was added, the 2.3 5s, thus leaving 1.3 5s

4B1 5s = 4-1B1+5 5s = 3B6 5s = 3.6 5s = 2.3 5s + 1.3 5s

4B1 5s – 2B3 5s = 4-2B1-3 5s = 2B-2 5s = 2-1B-2+5 5s = 1B3 5s = 1.3 5s

Figure 703. Solving the equation 2.3 5s + ? 5s total 4.1 5s using bundle-writing and decimal-writing

On an abacus in algebra mode, the 4.1 5s are moved to the right below the band. Changing 1 5-bundle to 5 1s on the single line allows taking away the 2.3 5s to leave 1.3 5s

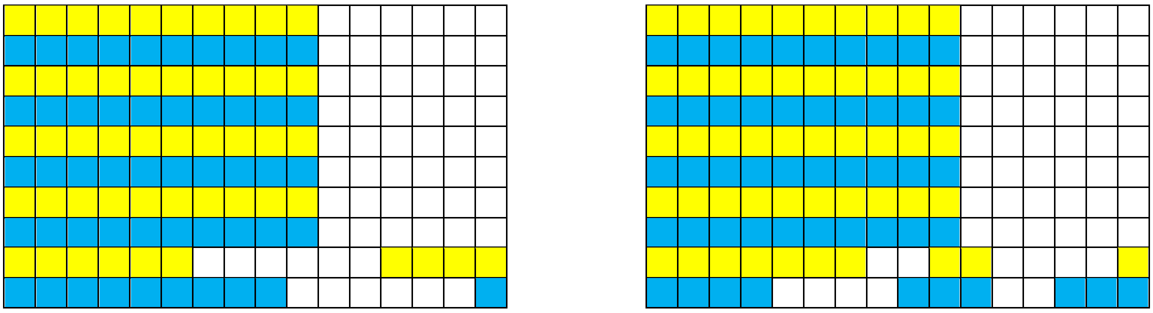


Figure 704. Solving the equation 2.3 5s + ? 5s total 4.1 5s on a western abacus in algebra mode

When using a calculator to predict the result, we include the two totals in a bracket before counting in 3s: Asking '(4x5+1 – 2x5 – 1)/5' gives the answer 1.some. Then asking '(4x5+1 – 2x5 – 1) – 1x5 gives the answer 1. So the answer is 1.3 5s.

(4 x 5 + 1 – 2 x 5 – 1)/ 5	1.some
(4 x 5 + 1 – 2 x 5 – 1) – 1 x 5	3

Figure 705. Using a calculator to solve the equation 2.3 5s + ? 5s total 4.1 5s

Reversing task 2: '4 5s and ? 3s total 6 5s?'

On an abacus in geometry mode, leaving the bottom line empty, a stack of 6 5s is moved to the right. To get back to the original 4 5s, 2 5s are moved to the middle and moved as 1s to the single line to be bundled if possible, thus giving the answer 3.1 3s. To test this result by performing the forward addition '4 5s and 3.1 3s total ? 5s?'

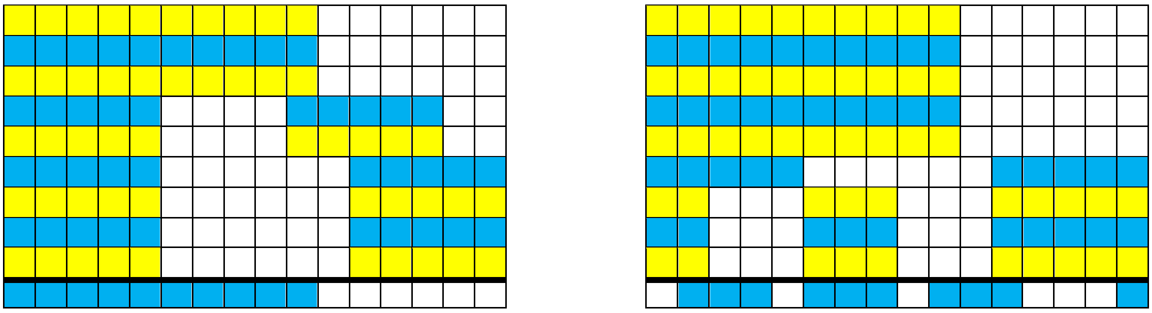


Figure 706. Solving the equation 4 5s + ? 3s total 6 5s on a western abacus in geometry mode

On an abacus in algebra mode and split in the middle by a horizontal rubber band, the 6 5s are moved to the right on the bundle line below the band. To get back to the original 4 5s, 2 5s are moved to the middle and moved as 1s to the single line above the band to be bundled if possible, thus giving the answer 3.1 3s. To test this result by performing the forward addition '4 5s and 3.1 3s total ? 5s?'

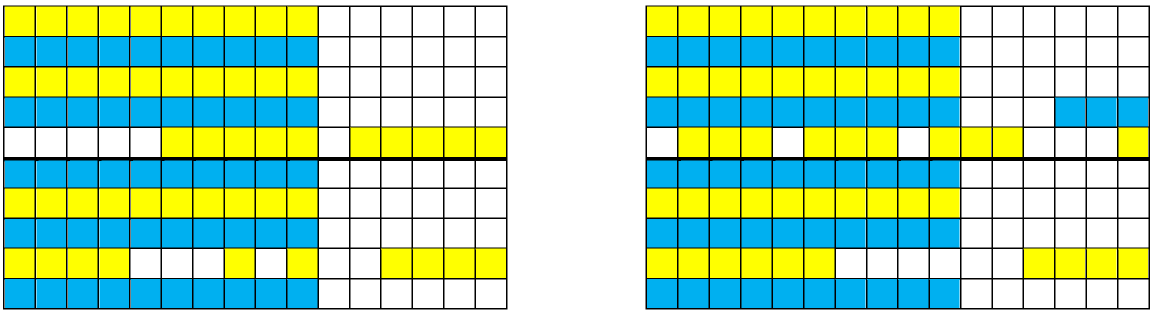


Figure 707. Solving the equation 4 5s + ? 3s total 6 5s on a western abacus in algebra mode

When using a calculator to predict the result, we include the two totals in a bracket before counting in 3s: Asking '(6x5 – 4x5)/3' gives the answer 3.some. Then asking '(6x5 – 4x5) – 3x3 gives the answer 1. So the answer is 3.1 3s. A display showing that (4x5 + 2x3) – 5x5 = 1 indirectly predicts that 2 3s can be added to 4 5s as 5.1 5s.

(6 x 5 – 4 x 5)/ 3	3.some
(6 x 5 – 4 x 5) – 3 x 3	1

Figure 708. Using a calculator to solve the equation 4 5s + ? 3s total 6 5s

Rolling dices can inspire additional adding tasks, e.g. by adding 1 or 2 to the numbers, so that rolling 4 dices creates the reversed addition tasks '2 4s and ? 3s total 6 4s?', '3 5s and ? 4s total 7 5s?', etc.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to solve equations using geometry or algebra or a calculator for prediction allow for additional observations as to preferences and performances. Furthermore, using bundle-numbers in equations in preschool is a new idea that has never been researched before.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

Micro-curriculum 8. Reversing Adding Bundle-numbers NextTo

A. Deconstructing the Tradition

Traditionally, reversed addition next-to as '2 3s + ? 5s = 4 8s' is reformulated to '2 x 3 + 5 x u = 4 x 8', called a linear equation and defined as an example of an equivalence relation between the two 'number-names' 2 x 3 + 5 x u and 4 x 8. Performing identical operation to both number-names will keep the equivalence intact. By applying the laws of associativity and community as well as the definitions of a neutral and an inverse element, it is possible to change one of the number-names to a single u called the solution of the equation. Testing the solution is not necessary because of the equivalence relation. Because of the degree of abstractness, equations are traditionally postponed to secondary school. Alternatively, the left hand side is seen as an example of a linear function, likewise postponed to secondary school.

2 x 3 + *u* x 5 = 4 x 8	Simplifying the number-names where possible
6 + 5 x *u* = 32	Addition has 0 as its neutral element, and 6 has -6 as its inverse element
(6 + 5 x *u*) + (-6) = 32 + (-6)	Adding 6's inverse element to both number-names
(5 x *u* + 6) + (-6) = 26	Applying the commutative law to 6 + 5 x *u*
5 x *u* + (6 + (-6)) = 26	Applying the associative law
5 x *u* + 0 = 26	Applying the definition of an inverse element
5 x *u* = 26	Applying the definition of a neutral element Multiplication has 1 as neutral element, and 5 has 1/5 as inverse element
(5 x *u*) x (1/5) = 26 x (1/5)	Multiplying 5's inverse element to both number-names
(*u* x 5) x (1/5) = 5.2	Applying the commutative law to 5 x *u*
u x (5 x (1/5)) = 5.2	Applying the associative law
u x 1 = 5.2	Applying the definition of an inverse element
u = 5.2	Applying the definition of a neutral element

Figure 801. Solving the equation 2 x 3 + u x 5 = 4 x 8 the traditional algebraic way

Skepticism could point out that instead of seeing reversed equation as an example of an equation, the solution of which is seen as an example of applying abstract algebra, reversed addition could be treated as what it is, reversed addition, asking '2 3s + ? 5s = 4 8s' and not asking '6 + 5 x u = 32' . Finally, a solution to a reversed addition should always be tested by the corresponding forward addition.

Because of the close relationship between forward addition next-to, also called integration, and reversed addition next-to, also called differentiation, differentiation should be introduced in preschool along with integration.

This deconstruction raises the question:

What learning possibilities occur if allowing preschool children to reverse integration?

B. Designing a Micro-curriculum

<u>Reversing task 1</u>: '2 3s and ? 5s total 4 8s?'

Using algebra-counting, 2 3s is recounted as 0.6 8s, that removed from the 4 8s leaves what must be counted in 5s. Since 4 8s can be recounted to 3.8 8s, removing 0.6 8s leaves 3.2 8s that can be recounted as 5.1 5s. So reversing integration to differentiation means first subtracting, then dividing, later called a difference quotient (T2 – T1)/n.

On an abacus in geometry mode, leaving the bottom line empty, a stack of 4 8s is moved to the right. First we remove what was added by moving 2 3s to the left. The surplus 2 3s are moved to the single line and recounted in 5s to give the result 5.1. 5s

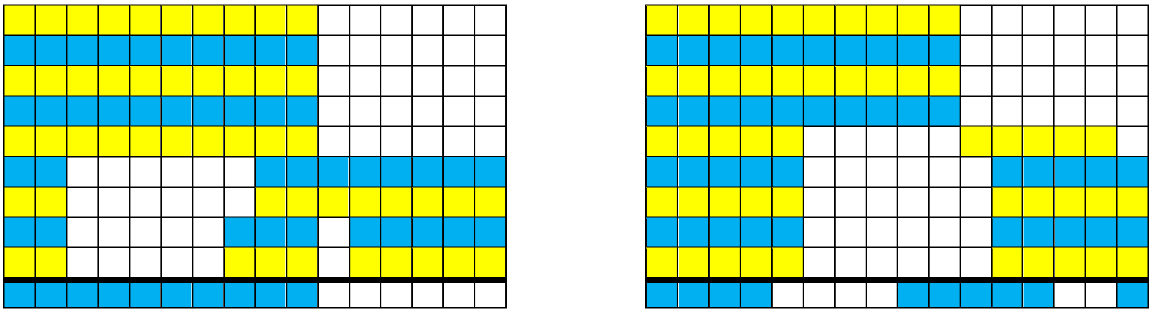


Figure 802. Solving the equation 2 3s + ? 5s total 4 8s on a western abacus in geometry mode

When using a calculator to predict the result, we include the two totals in a bracket before counting in 3s: Asking '(4x8 – 2x3)/5' gives the answer 5.some. Then asking '(4x8 – 2x3) – 5x5 gives the answer 1. So the answer is 5.1 5s. A display showing that (4x8 + 2x3) – 5x5 = 1 indirectly predicts that 2 3s can be added to 5.1 5s as 4 8s.

(4 x 8 – 2 x 3)/ 5	5.some
(4 x 8 – 2 x 3) – 5 x 5	1

Figure 803. Using a calculator to solve the equation 2 3s + ? 5s total 4 8s

Rolling dices can inspire additional adding tasks, e.g. by adding 1 or 2 to the numbers, so that rolling 4 dices creates the reversed addition tasks '4 3s and ? 5s total 2 8s?', '5 4s and ? 5s total 3 9s?', etc.

C. Researching the Micro-curriculum

The researcher(s) make observations of what the children do and say, if possible some sessions will be videotaped. Based upon the observations the researcher(s) will look for patterns in statements, in actions and in personal behavior allowing the construction of proto-categories. Bringing these back to the preschool, the researcher(s) will use further observation to refine the categories, typically by dividing them into sub-categories, and if possible look for relations between these.

In this case, the three ways to solve equations using geometry or algebra or a calculator for prediction allow for additional observations as to preferences and performances. Furthermore, using bundle-numbers to reverse integration in preschool is a new idea that has never been researched before.

D. Reporting the Findings

The findings are reported in the traditional way in journals and more interactively on a Facebook profile supported by YouTube videos.

M1 Creating Icons, Material for MicroCurriculum 1

Curricula

Micro-curriculum 1. Creating Icon-numbers

Micro-curriculum 2. Counting in Bundles

Micro-curriculum 3. Re-counting Bundle-numbers in the Same Bundle

Micro-curriculum 4. Re-counting Bundle-numbers in a Different Bundle

Micro-curriculum 5. Adding Bundle-numbers OnTop

Micro-curriculum 6. Adding Bundle-numbers NextTo

Micro-curriculum 7. Reversing Adding Bundle-numbers OnTop

Micro-curriculum 8. Reversing Adding Bundle-numbers NextTo

Preface

M1, Creating Icons, is the first of a series of 8 micro-curricula in mathematics for preschool and home schooling. They present mathematics as MANYmatics, a natural science about the physical fact Many. They all build upon the following observation:

”How old will you be next time?” I asked the child. “Four”, he answered and showed me four fingers. “Four, you said?” I asked and showed him four fingers held together two by two. “No, that is not four, that is two twos!” the child replied thus insisting upon what exists, bundles of twos, and two of them. Likewise, preschool children have no difficulties counting in other units than ten, even if they only learn how to count in tens.

The micro-curricula use activities with concrete material to obtain its learning goals in accordance with Piaget’s principle ‘greifen vor begrifen’ (grasp to grasp).

In the first, children learn to use sticks to build the number icons up to nine, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second, children learn to count a given total in icons manually, using an abacus and by using a calculator. In the third, children learn to recount a total in the same unit. In the fourth, children learn to recount a total in a different unit. In the fifth, children learn to add two bundle-numbers on top of each other. In the sixth, children learn to add two bundle-numbers next to each other. In the seventh, children learn to reverse on-top addition. And in the eights, children learn to reverse next-to addition

As to concrete material, anything goes in the first micro-curriculum. The others will use sticks and strokes, beads on an abacus, LEGO-like blocks and squares, and a calculator respecting the priority of the operations. Fingers, pegs on a pegboard and other concrete material might also be used.

Some of the curricula can use silent education where the instructor is allowed to demonstrate and guide through actions, but not through words; or by using words form a foreign language not understood by the child.

1

Say it

I	I	A	I	I	1
One	1 ones	Greek	Roman	Icon	Arabic

Write it

Stamp and Shade it

Form it

Repeat it

1	**2**	3	4	5	6	7	8	9

1

Round it up & Color it

✂	✂	✎	✆	✆	D	D	D	🔔	🔔
✂	✂	✎	✆	✆	D	D	D	🔔	🔔
✂	✂	✎	✆	✆	✈	✈	✈	✈	✈
✂	✂	✎	✎	✆	✈	✈	✈	✈	✈
✂	✂	✎	✎	✆	✉	✉	✉	✉	✉

Clap, Sing, Walk, Act & Letter it

✋	♩		☺	A

Reward: Stickers, each counting two

2

Say it

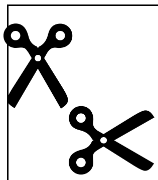
I I	II	B	II	└	2
Two 1s	1 twos	Greek	Roman	Icon	Arabic

Write it

Stamp and Shade it

Form it

Repeat it

1	**2**	3	4	5	6	7	8	9

2

Round it up & Color it

✂	✂	✐	ɔ	ɔ	D	D	D	🔔	🔔
✂	✂	✐	ɔ	ɔ	D	D	D	🔔	🔔
✂	✂	✐	ɔ	ɔ	✈	✈	✈	✈	✈
✂	✂	✐	✐	ɔ	✈	✈	✈	✈	✈
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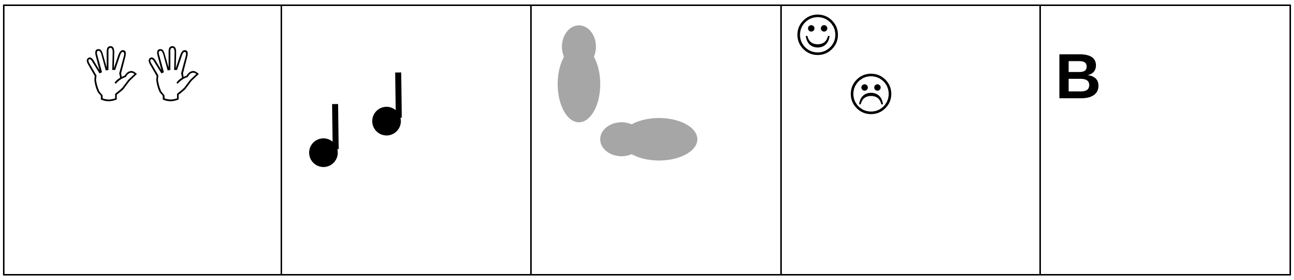
Clap, Sing, Walk, Act & Letter it



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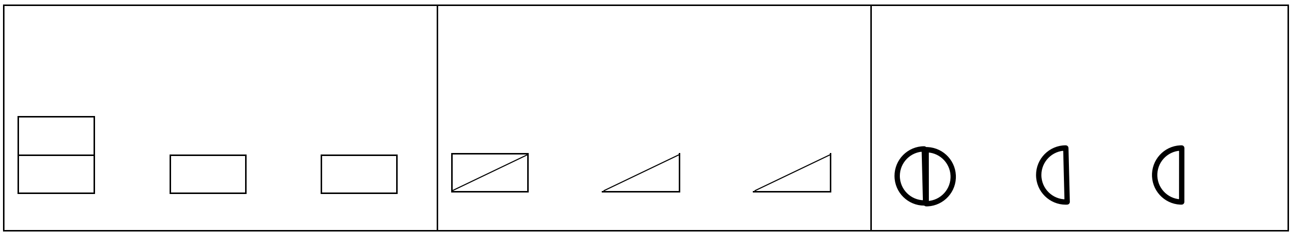

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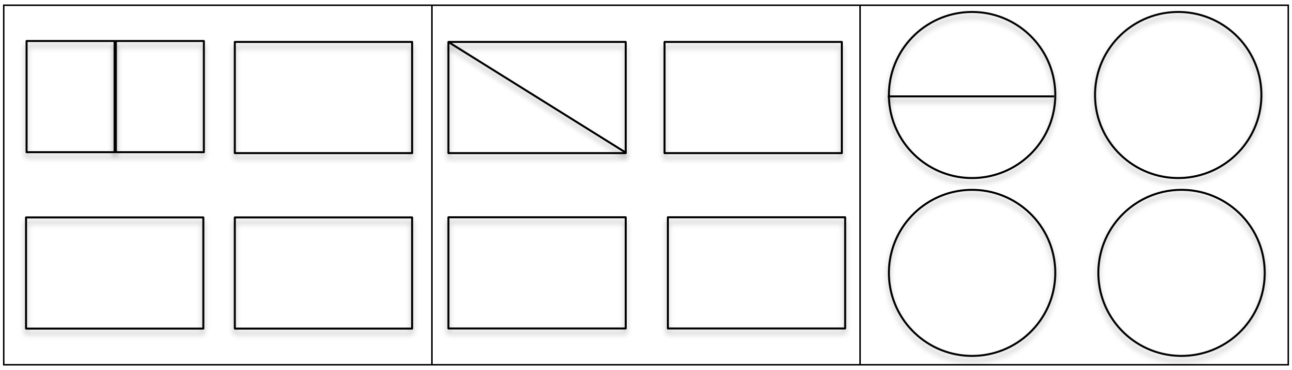


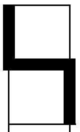
Reward: Stickers, each counting two 

3

Say it

I I I	III	C	III	[illegible]	3
Three 1s	1 threes	Greek	Roman	Icon	Arabic

Write it



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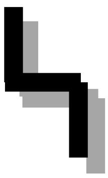

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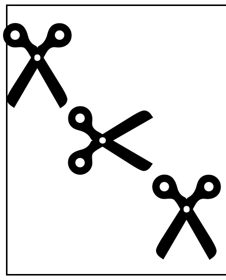


Repeat it

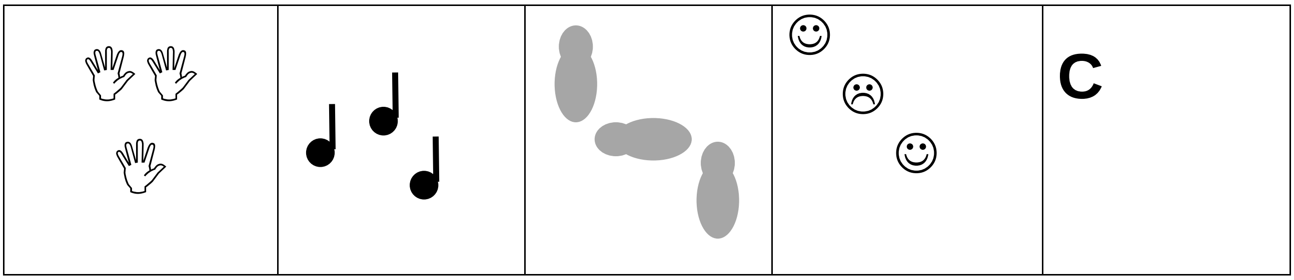
1	**2**	**3**	4	5	6	7	8	9

3

Round it up & Color it

✂	✂	✎	❳	❳	D	D	D	🔔	🔔
✂	✂	✎	❳	❳	D	D	D	🔔	🔔
✂	✂	✎	❳	❳	✈	✈	✈	✈	✈
✂	✂	✎	✎	❳	✈	✈	✈	✈	✈
✂	✂	✎	✎	❳	✉	✉	✉	✉	✉

Clap, Sing, Walk, Act & Letter it



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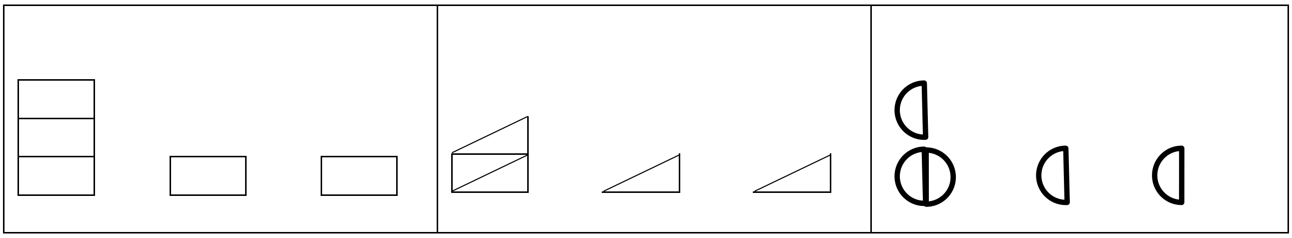

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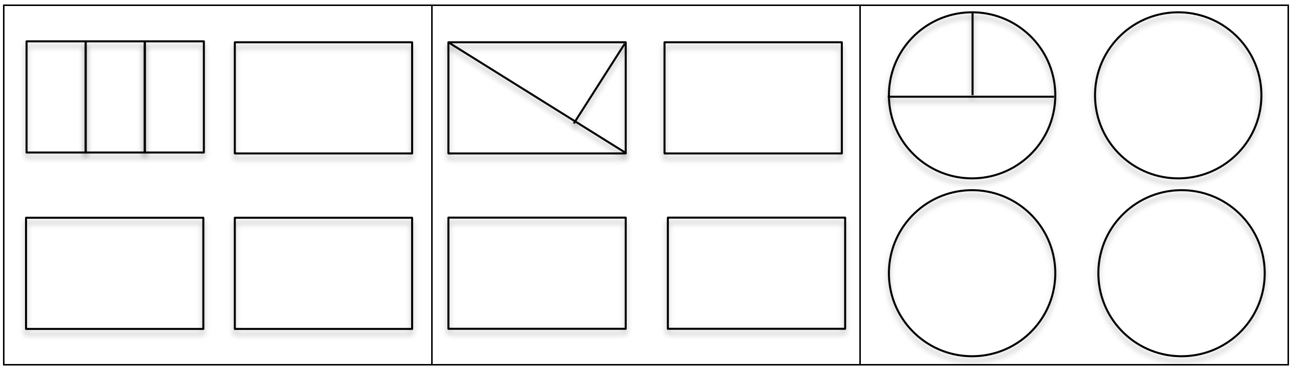
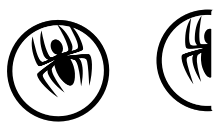


Reward: Stickers, each counting two



4

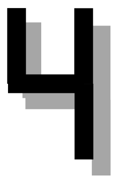
Say it

I I I I	IIII	D	IIII	५	4
Four 1s	1 fours	Greek	Roman	Icon	Arabic

Write it

4

Stamp and Shade it



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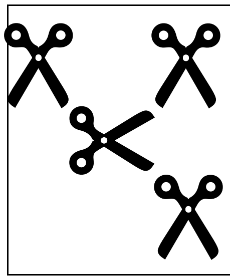
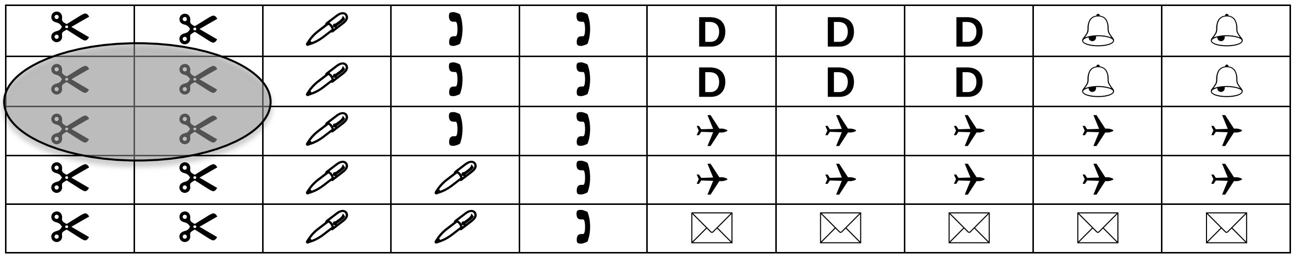


Repeat it

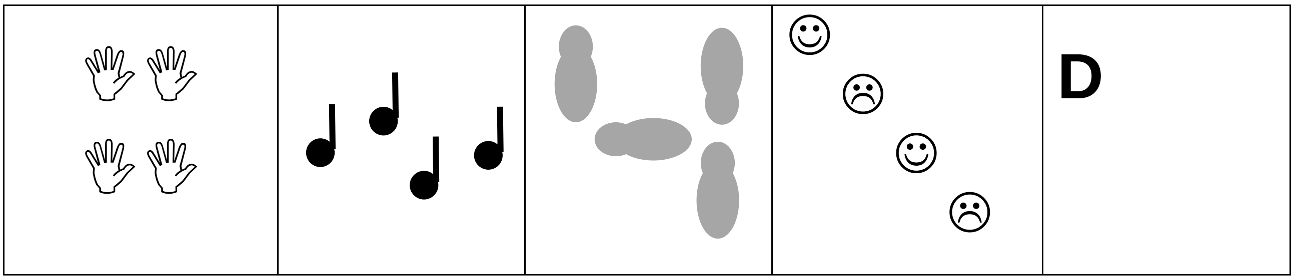
1	**2**	**3**	**4**	5	6	7	8	9

4

Round it up & Color it



Clap, Sing, Walk, Act & Letter it



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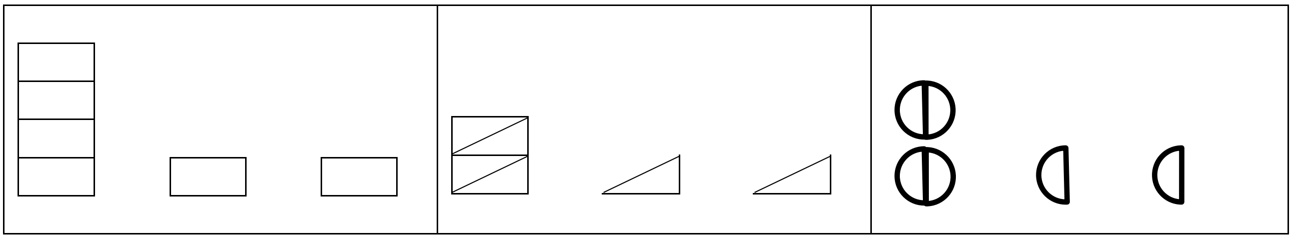

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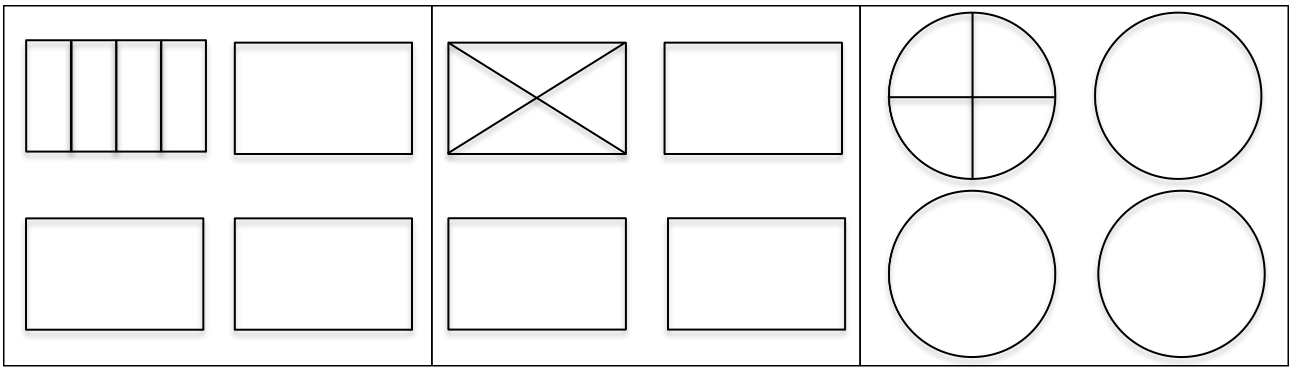
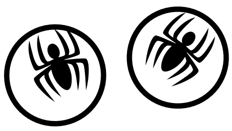


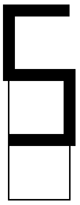
Reward: Stickers, each counting two 

5

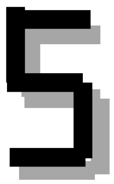
Say it

I I I I I	IIIII	E	V	5	5
Five 1s	1 fives	Greek	Roman	Icon	Arabic

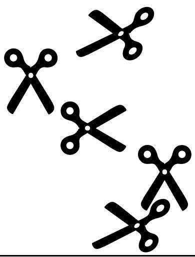
Write it



Stamp and Shade it



Form it

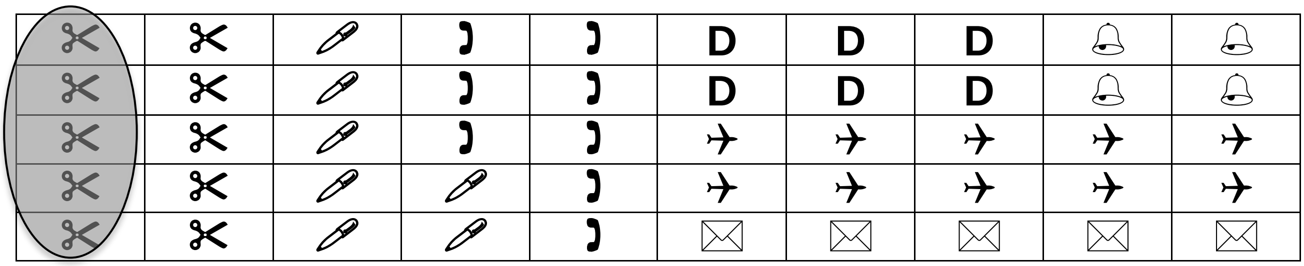


Repeat it

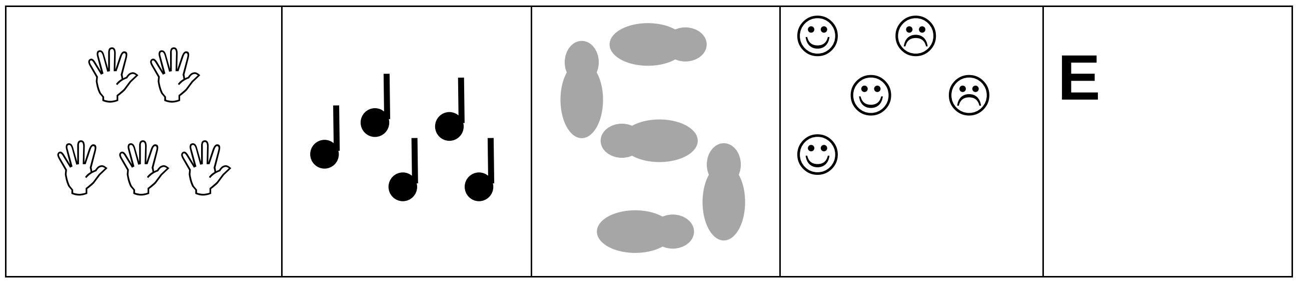
1	**2**	**3**	**4**	**5**	6	7	8	9

5

Round it up & Color it



Clap, Sing, Walk, Act & Letter it



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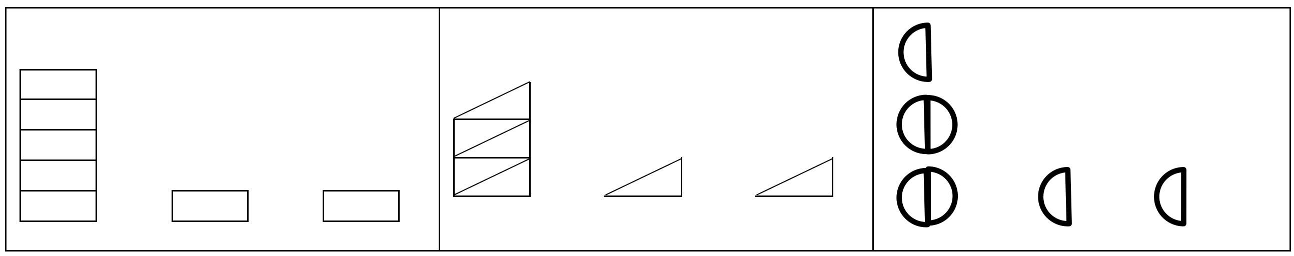
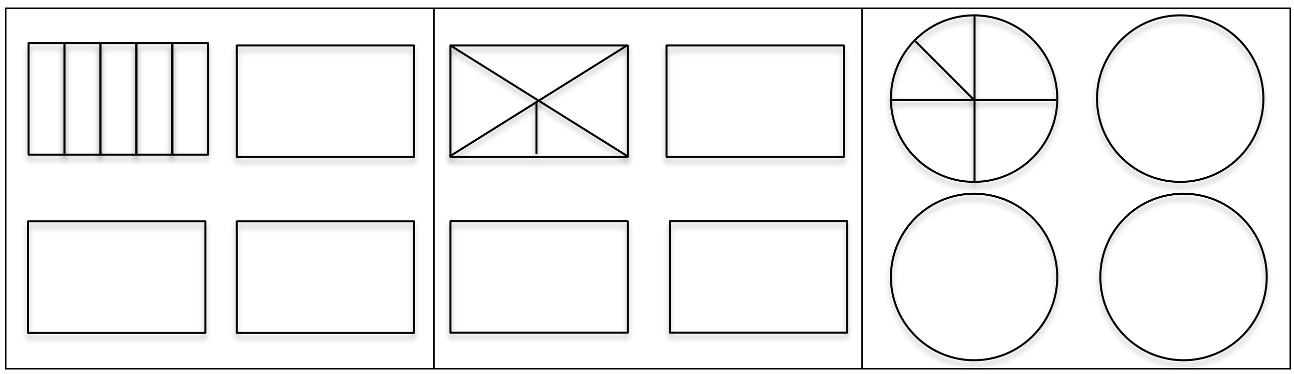
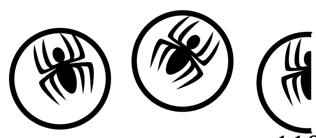


Split it



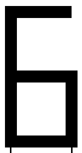
Reward: Stickers, each counting two 

6

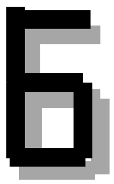
Say it

I I I I I I	IIIIII	F	VI		6
Six 1s	1 sixs	Greek	Roman	Icon	Arabic

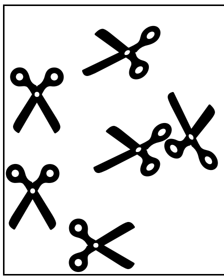
Write it



Stamp and Shade it



Form it

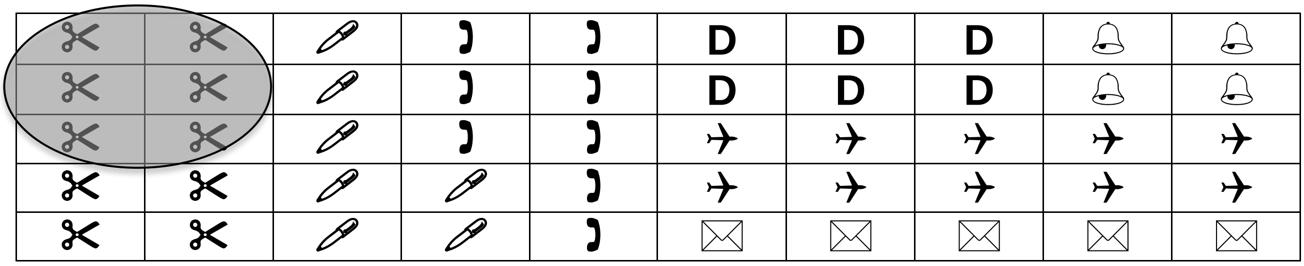


Repeat it

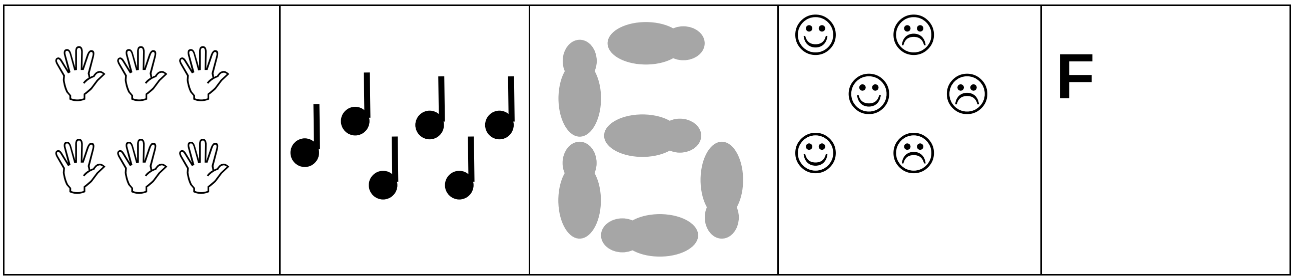
1	**2**	**3**	**4**	**5**	**6**	7	8	9

6

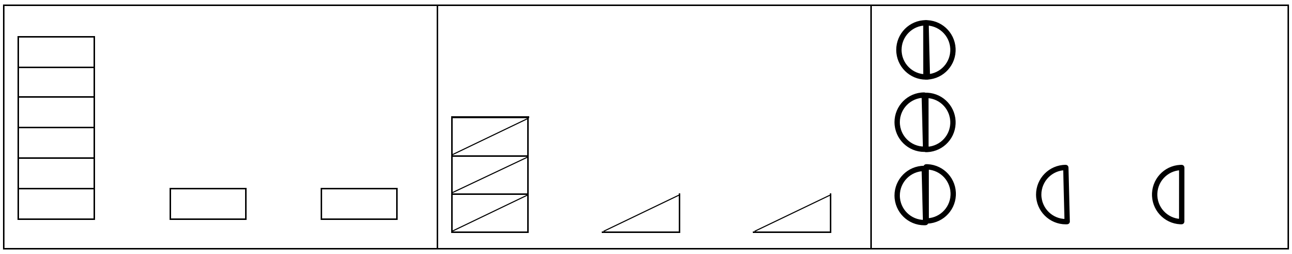
Round it up & Color it



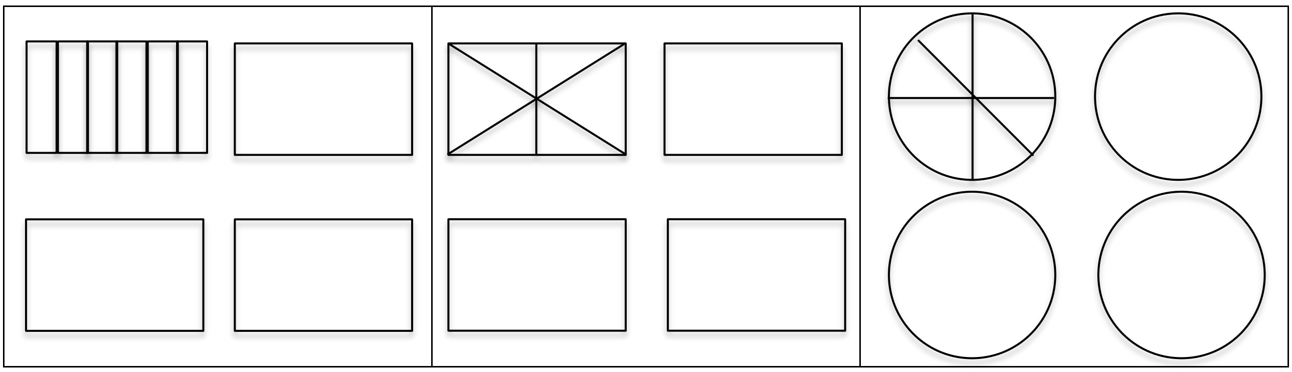
Clap, Sing, Walk, Act & Letter it



Unite it



Split it



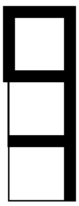
Reward: Stickers, each counting two 

7

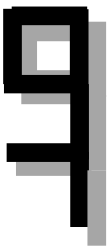
Say it

I I I I I I I	IIIIIII	G	VII		7
Seven 1s	1 sevens	Greek	Roman	Icon	Arabic

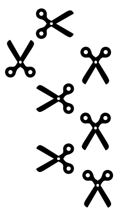
Write it



Stamp and Shade it



Form it

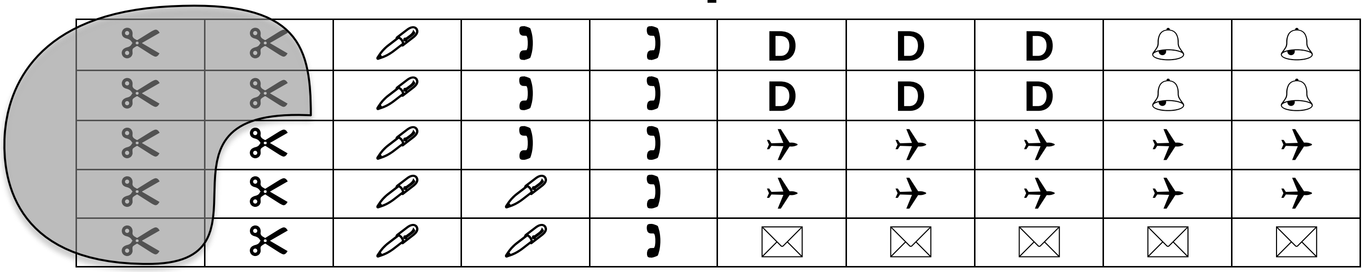


Repeat it

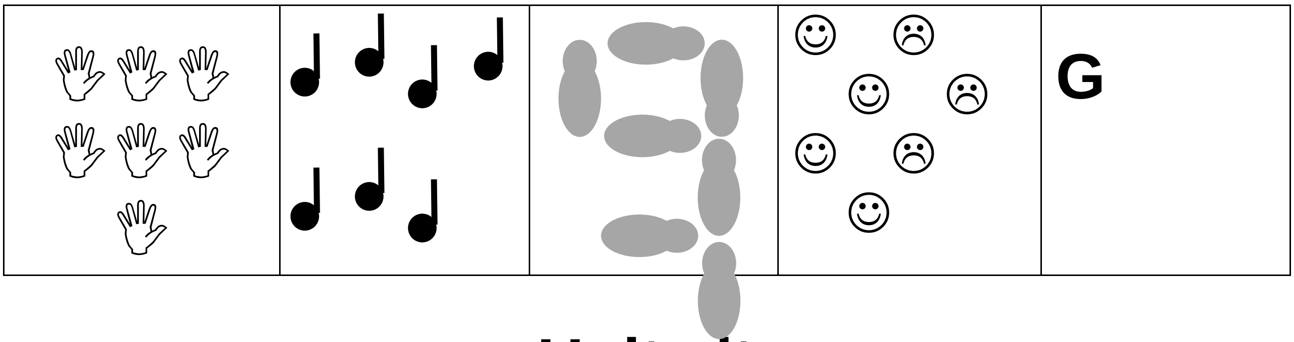
1	**2**	**3**	**4**	**5**	**6**	**7**	8	9

7

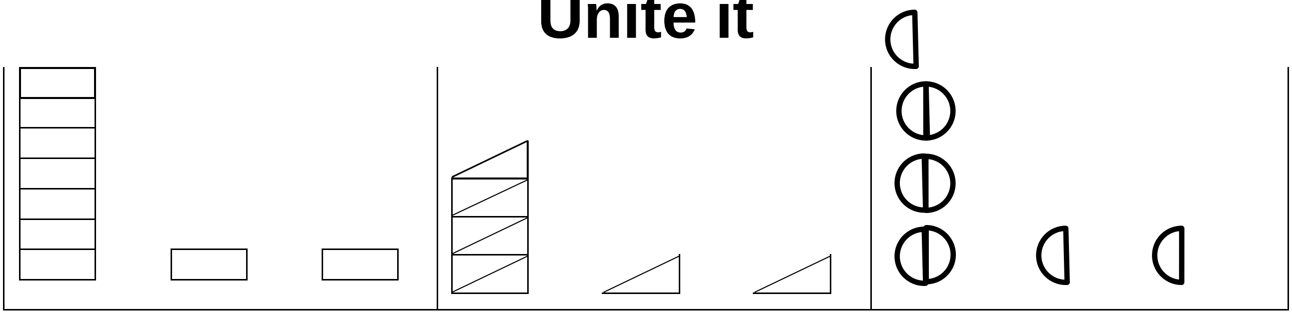
Round it up & Color it



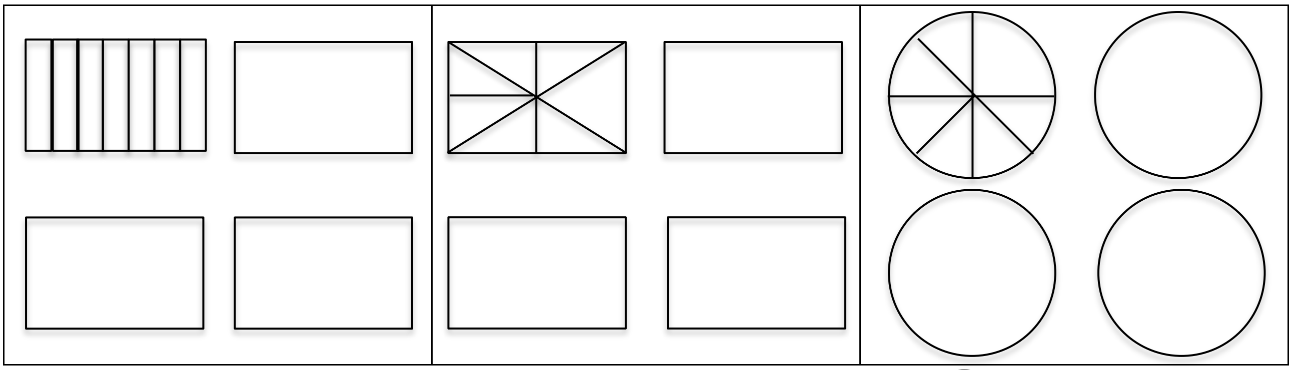
Clap, Sing, Walk, Act & Letter it



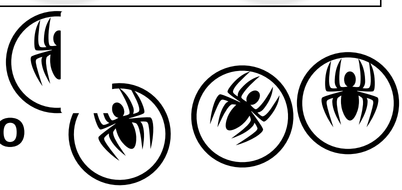
Unite it



Split it



Reward: Stickers, each counting two

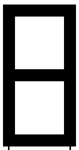


8

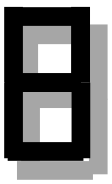
Say it

I I I I I I I I	IIIIIIII	H	VIII	🯸	8
Eight 1s	1 eights	Greek	Roman	Icon	Arabic

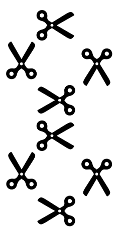
Write it



Stamp and Shade it



Form it

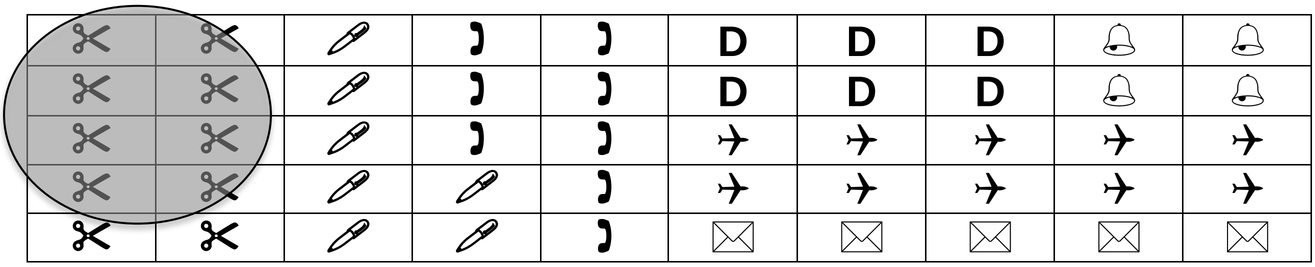


Repeat it

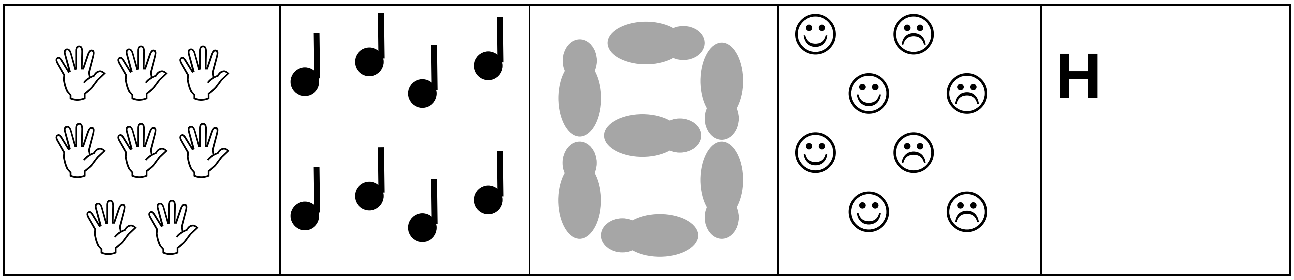
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	9

8

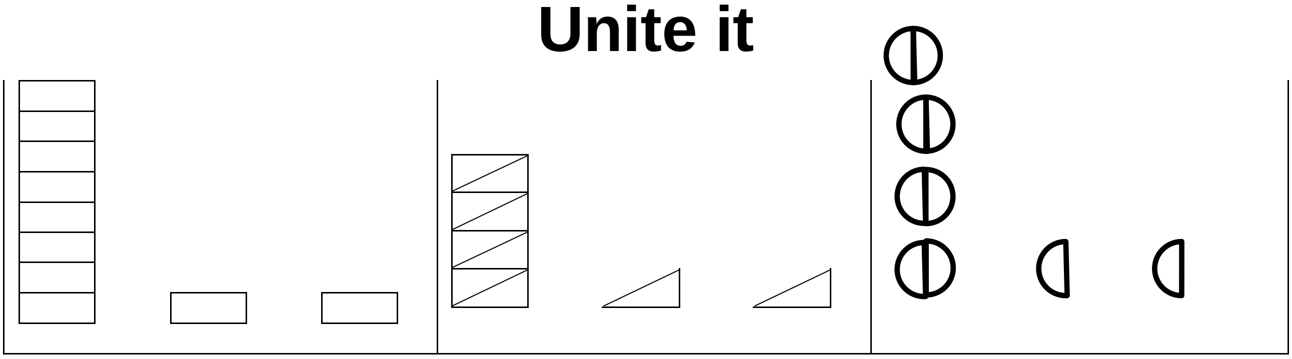
Round it up & Color it



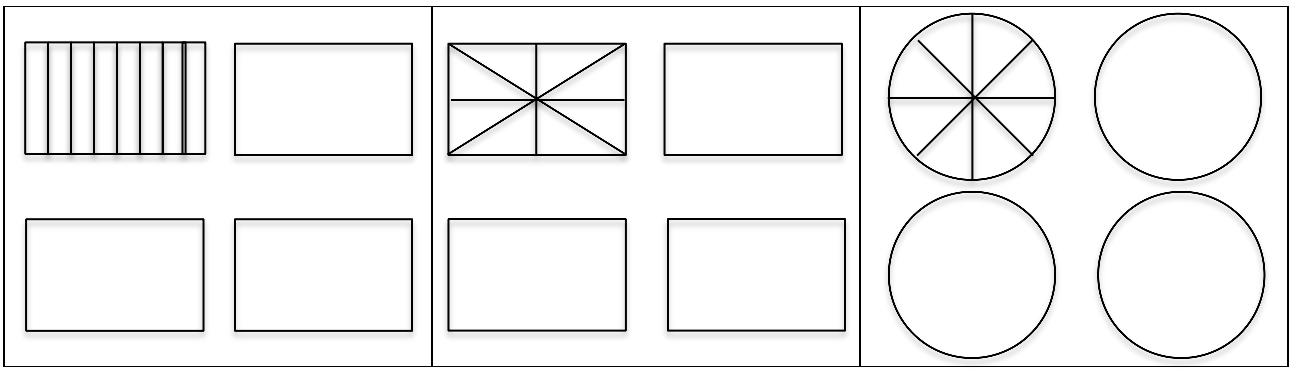
Clap, Sing, Walk, Act & Letter it



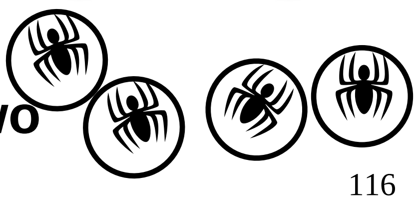
Unite it



Split it

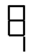


Reward: Stickers, each counting two

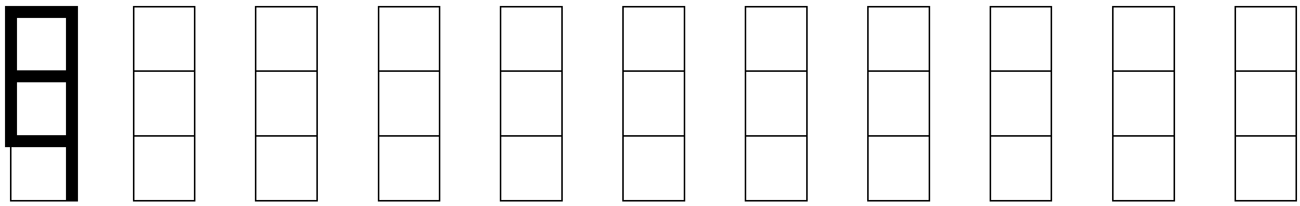


9

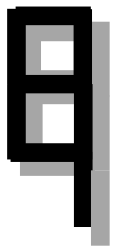
Say it

I I I I I I I I I	IIIIIIIII	I	IX		9
Nine 1s	1 nines	Greek	Roman	Icon	Arabic

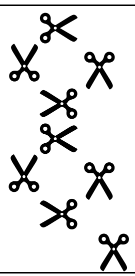
Write it



Stamp and Shade it



Form it

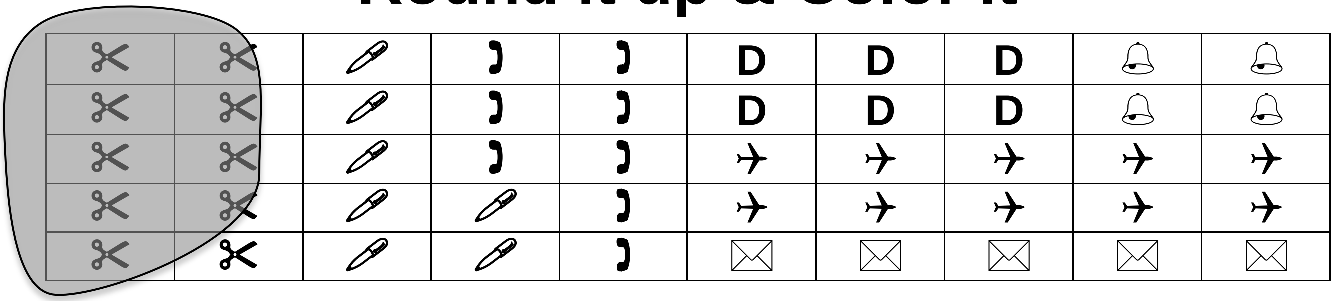


Repeat it

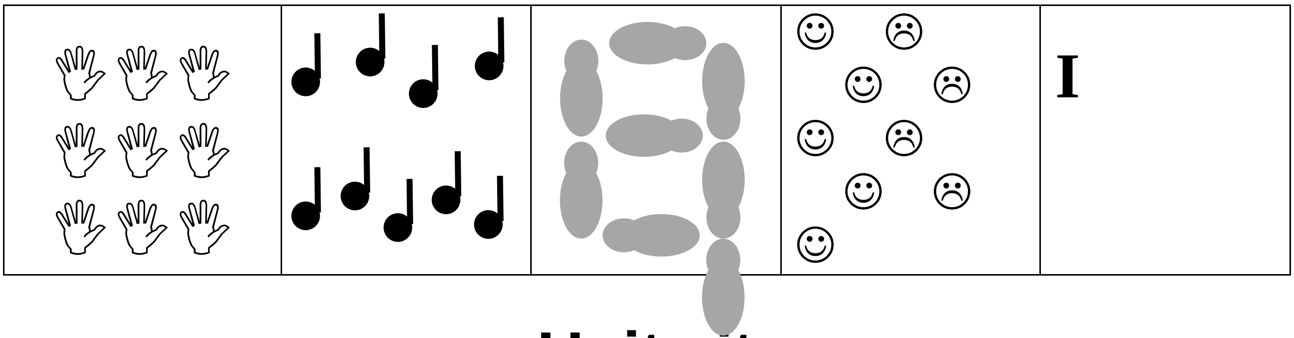
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

9

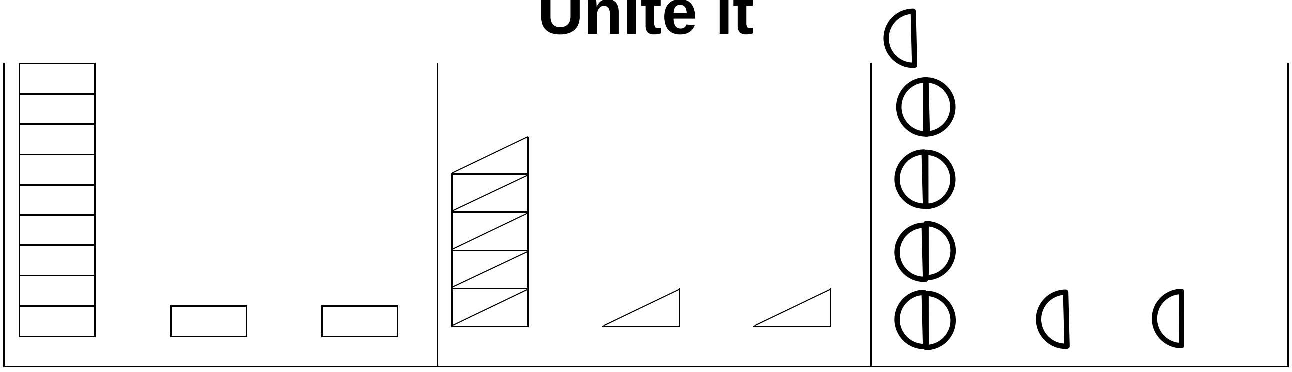
Round it up & Color it



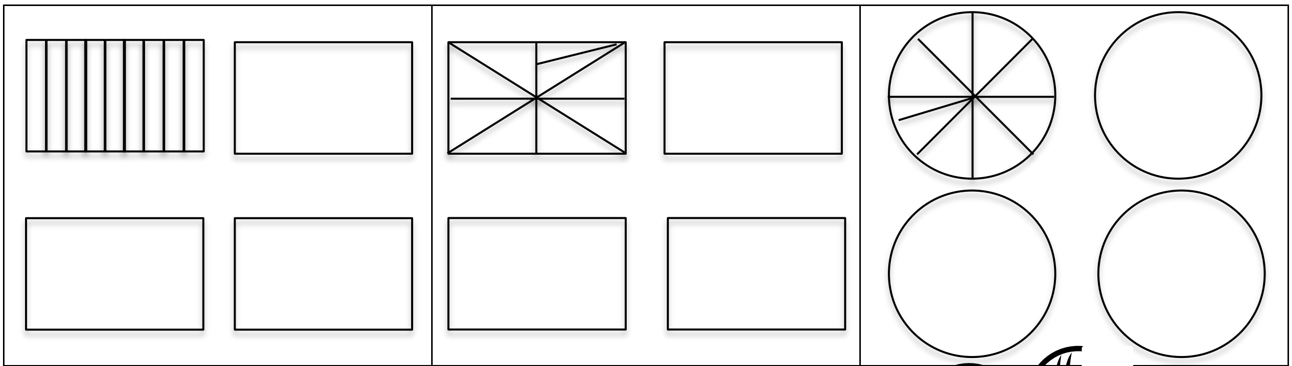
Clap, Sing, Walk, Act & Letter it



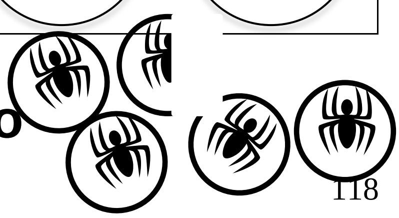
Unite it



Split it



Reward: Stickers, each counting two



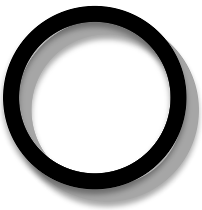
0

Say it

				o	0
None	Zero	Greek	Roman	Icon	Arabic

Write it

Stamp and Shade it

Form it

Repeat it

1	2	3	4	5	6	7	8	9

0

Round it up & Color it

✂	✂	✎	✆	✆	D	D	D	🔔	🔔
✂	✂	✎	✆	✆	D	D	D	🔔	🔔
✂	✂	✎	✆	✆	✈	✈	✈	✈	✈
✂	✂	✎	✎	✆	✈	✈	✈	✈	✈
✂	✂	✎	✎	✆	✉	✉	✉	✉	✉

Clap, Sing, Walk, Act & Letter it

Reward: Stickers, each counting two

M2 Counting in Bundles, Material for MicroCurriculum 2

Curricula

Micro-curriculum 1. Creating Icon-numbers

Micro-curriculum 2. Counting in Bundles

Micro-curriculum 3. Re-counting Bundle-numbers in the Same Bundle

Micro-curriculum 4. Re-counting Bundle-numbers in a Different Bundle

Micro-curriculum 5. Adding Bundle-numbers OnTop

Micro-curriculum 6. Adding Bundle-numbers NextTo

Micro-curriculum 7. Reversing Adding Bundle-numbers OnTop

Micro-curriculum 8. Reversing Adding Bundle-numbers NextTo

Preface

M2, Counting in Icons, is the second of a series of 8 micro-curricula in mathematics for preschool and home schooling. They present mathematics as MANYmatics, a natural science about the physical fact Many. They all build upon the following observation:

”How old will you be next time?” I asked the child. “Four”, he answered and showed me four fingers. “Four, you said?” I asked and showed him four fingers held together two by two. “No, that is not four, that is two twos!” the child replied thus insisting upon what exists, bundles of twos, and two of them. Likewise, preschool children have no difficulties counting in other units than ten, even if they only learn how to count in tens.

The micro-curricula use activities with concrete material to obtain its learning goals in accordance with Piaget’s principle ‘greifen vor begrifen’ (grasp to grasp).

In the first, children learn to use sticks to build the number icons up to nine, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second, children learn to count a given total in icons manually, using an abacus and by using a calculator. In the third, children learn to recount a total in the same unit. In the fourth, children learn to recount a total in a different unit. In the fifth, children learn to add two bundle-numbers on top of each other. In the sixth, children learn to add two bundle-numbers next to each other. In the seventh, children learn to reverse on-top addition. And in the eights, children learn to reverse next-to addition

As to concrete material, anything goes in the first micro-curriculum. The others will use sticks and strokes, beads on an abacus, LEGO-like blocks and squares, and a calculator respecting the priority of the operations. Fingers, pegs on a pegboard and other concrete material might also be used.

Some of the curricula can use silent education where the instructor is allowed to demonstrate and guide through actions, but not through words; or by using words form a foreign language not understood by the child.

G stands for Geometry and A stands for Algebra.

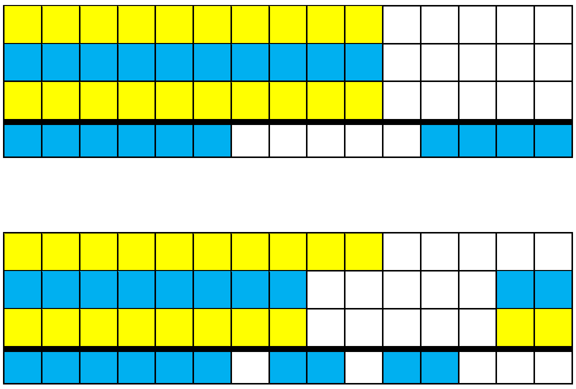
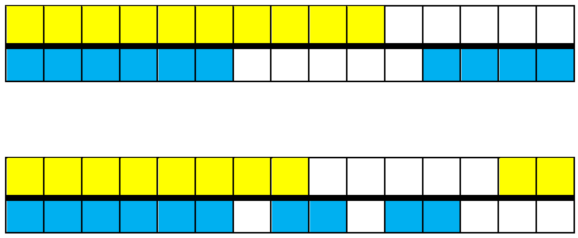
A total of 4 2s can be written as 5.~~2~~ 2s where ~~2~~ means ‘less 2’.

4 Counted in 2s

Sticks

G-counting		A-counting	
I I I I	*lay out*	I I I I	*lay out*
H H	*bundle*	H H	*bundle*
H		**2B**	*bundle-writing*
H	*stack*		
T = 2 2s	Total	**T = 2 2s**	*Total*

Abacus

G-mode	A-mode
	

Calculator

4 / 2	2
4 – 2 x 2	0

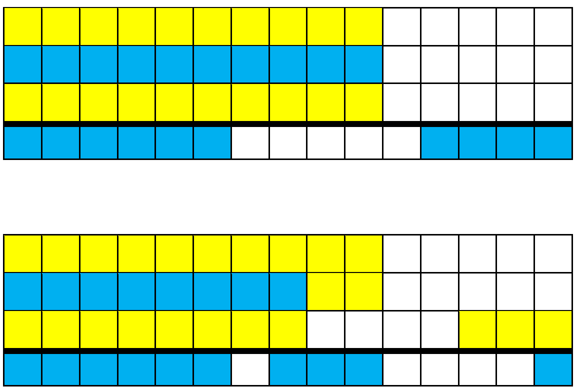
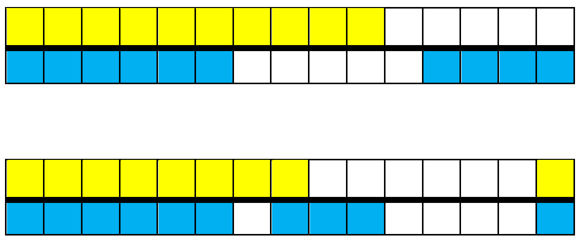
T = 4 = 2 2s

4 Counted in 3s

Sticks

G-counting		A-counting	
I I I I	*lay out*	I I I I	*lay out*
~~I I I~~ I	*bundle*	~~I I I~~ I	*bundle*
		1B1	*bundle-writing*
[~~I I I~~] I	*stack*		
T = 1.1 3s	Total	**T = 1.1 3s**	*Total*

Abacus

G-mode	A-mode
	

Calculator

4 / 3	1.some
4 – 1 x 3	1

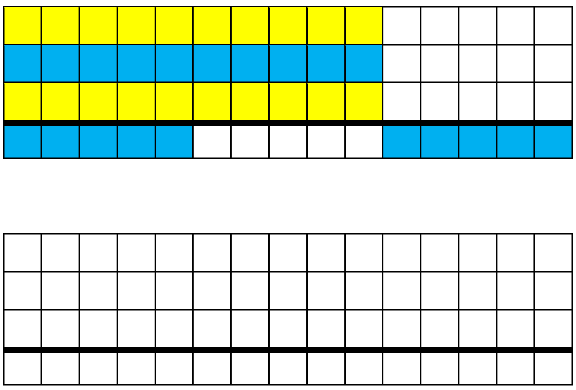
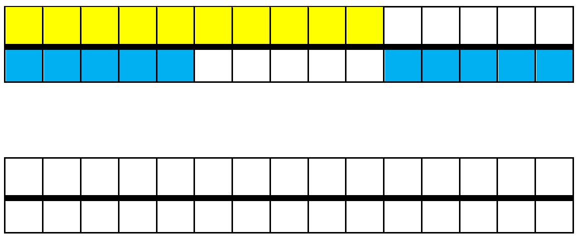
T = 4 = 1.1 3s

5 Counted in 2s

Sticks

G-counting		A-counting	
I	*lay out*	I	*lay out*
~~II~~	*bundle*	~~II~~	*bundle*
		B	*bundle-writing*
	stack		
T =	Total	**T =**	*Total*

Abacus

G-mode	A-mode
	

Calculator

5 / 2

5 –

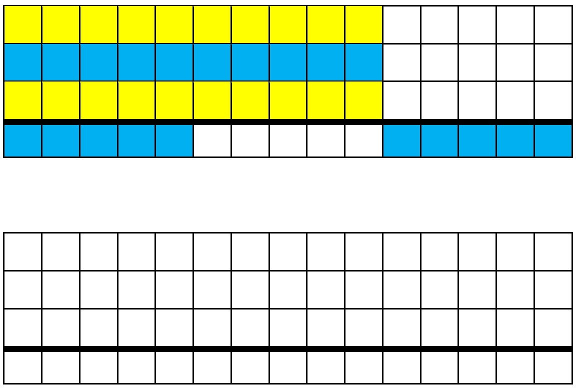
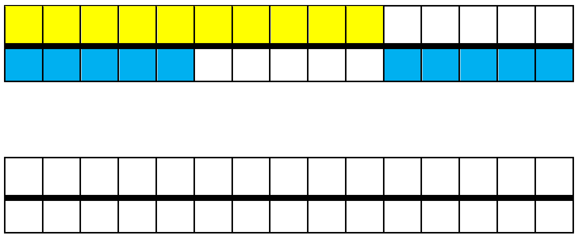
T = 5 = 2s

5 Counted in 3s

Sticks

G-counting	A-counting
lay out	*lay out*
bundle	*bundle*
	B *bundle-writing*
stack	
T = Total	**T =** *Total*

Abacus

G-mode	A-mode
	

Calculator

5 / 3
5 –

T = 5 = 3s

09. CATS: Learning Mathematics through Counting & Adding Many in Time & Space

CATS shows that mathematics is simple if respecting its nature as a natural science about MANY.

To deal with MANY we simply count and add, as demonstrated when writing a Total as we say it:

T = 456 = 4*10^2 + 5*10 + 6*1.

This shows why algebra means re-unite in Arabic: We count MANY by uniting the ones in bundles, in this case ten-bundles, ten-bundles of ten-bundles etc. Also we see that

All numbers carry units: ones, tens, ten-tens etc.

There are 4 ways to add: +, *, ^, integration, where

+ adds unlike numbers,

* adds like numbers,

^ adds like factors, and

integration adds stacks with different units next-to each other as areas.

Thus, adding next-to roots integration, adding on-top roots linearity forcing totals to be recounted in the same unit, and reversing addition roots equations.

Mathematics is as simple as that, when introduced as a natural science about MANY by CATS.

The unhappy-student problem

Today mathematics education creates many unhappy students. Why? And could it be otherwise?

One understanding of the unhappy-student problem comes from the fairy tales. The unhappy students are sleeping beauties that may be wakened, to live happily ever after. They are sleeping because they have touched a thorn or taken a bite of something that seemed healthy but carried poison inside. To awaken the thorns must be changed to roses or the poison must be taken away. But what are the thorns of mathematics or the poison inside mathematics? And is there a different mathematics inside the living room of mathematics that makes the Prince happy? Or will we have to look for a hidden Cinderella mathematics outside?

Words may be poisonous

The thorns and the poison might be the words used in mathematics education. This suggestion is inspired by French post-structuralist thinking opposing structuralist thinking saying that the words and sentences representing outside phenomena and relations can be used as an enlightened rational basis for institutions created to cure humans for e.g. ‘uneducatedness’. Contrary to this Derrida warns us against words, they are not representing but installing what they describe. Lyotard warns us against sentences, they are not representing but installing relations. And Foucault warns us against institutions, they are not curing but installing the patients.

This French scepticism towards words is easily verified by the ‘number&word’ pencil-paradox: placed between a ruler and a dictionary a pencil can point to numbers but not to words, thus numbers are reliable and words are unreliable; numbers are installed by the described based on the physical property ‘extension in space’, and words are installed by the describer based on an agenda.

In this way of thinking the poison inside mathematics are the words and sentences installed by ‘meta-matics’ and ‘mathe-matism’.

Meta-matics or set-ism

Meta-matics or set-ism is introducing words that are not derived from the outside world. It defines the words of mathematics as, not abstractions from examples, but as examples of abstractions; e.g. the word 'function' is defined as an example of a set-relation and not, as it was done historically, as a name for a calculation with a variable quantity. 'Function' is from around 1750 and 'set' is from around 1900; basing the definition of a 1750 word on a 1900 word is turning mathematics upside down transforming mathe-matics to meta-matics that is deduced from above instead of induced from below. Today's mathematics is basing its definition on the word set. But the history of mathematics shows that the word set is highly controversial and can only be rationally introduced as an undefined term, i.e. as something you believe in, thus making set and set-based mathematics an 'ism', set-ism.

Mathe-matism or add-ism

Mathematism or add-ism is introducing sentences that are not derived from the outside world. It claims that numbers can be added and multiplied without regard to the units thus portraying 2+3 = 5 and 2*3 = 6 as universal truths.

That 2*3 = 6 can easily be validated in the laboratory since 2*3 is physically present as a stack of 2 3s that can be recounted as 6 1s (2*3 = 2 3s = 6 1s = 6*1): *** *** -> * * * * * *.

Thus in multiplication the unit is automatically present as the last number.

2+3 = 5 depends on the unit as easily seen by examples: 2*week+3*week = 5*week, but 2* week + 3*day = 17*day = 2 3/7*week. Thus if units are not included before adding, addition becomes an 'ism', mathematism or add-ism, i.e. something that you have to believe since it seldom takes place in real life. Thus in the case of adding fractions 1/2 of 2 bottles + 2/3 of 3 bottles is 3/5 of 5 bottles making 1/2 + 2/3 = 3/5 and not 7/6 as add-ism teaches in spite of the fact that 7/6 of 6 bottles is a physical absurdity. So also when adding fractions the units must be included before adding.

Non-poisonous words

Words could be otherwise and should be debated before chosen; all democratic thinking recognizes this. The ancient Greek sophists taught about the democratic difference between information and debate by distinguishing between necessity and choice. Later the Enlightenment revived democratic thinking and installed two democracies in the late 1700s, the American and the French. Where the American still has its first republic France now has its 5^{th} republic always being threatened by 'pastoral power' as Foucault calls it. So where the French scepticism towards words is desperate the American scepticism towards words is pragmatic developing grounded theory as a research method, only allowing the words that comes from the data and only allowing the sentences coming from the tales that has been validated by their ability to survive for countless generations, the fairy tales, where rational agents pursuing a goal are being helped and hindered by helpers and opponents.

So where French scepticism is looking for hidden irrationality behind apparently rational institutions, American scepticism is looking for hidden rationality behind apparently irrational agents.

Combining American and French scepticism towards words

The MATHeCADEMY.net is designed by a postmodern 'sceptical Cinderella research' combining American and French scepticism towards words. It uses French scepticism to identify the poison inside mathematics in the form of set-ism and add-ism. And it uses American scepticism to look for new words that are Cinderella-differences, i.e. differences that make a difference to the unhappy-student problem and that cannot be accepted in the official room but has be found outside.

These new words must grow out of the root of mathematics, i.e. the human wish to be able to deal with many-ness by counting and adding Many in time and space. In this way an ism-free mathematics can be rebuild from below in a Many laboratory where mathematics is learned automatically through performing the activities of counting and adding.

The Count&Add Laboratory

A Many-Based Count&Add Laboratory is one example of building mathematics from below only allowing for the words that come out of the two fundamental competences needed to deal with Many, counting and adding. This laboratory could be named after the two mathematicians that were sceptical towards sets, Kronecker and Russell. In the Count&Add laboratory two different numbers exist, stack-numbers and per-numbers. This gives a whole new approach to mathematics. Primary school is about learning to deal with stack-numbers. Constructivist mathematics says that developing number sense is more important than learning calculation algorithms.

A Kronecker-Russell Many-Based Mathematics: The Count&Add Laboratory

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with Many in space (1 beat <->1 stroke).

3. Many in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Many can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ **T = (T/B)*B** is a prediction of the result when counting T in bs to be tested by performing the counting and stacking:

T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T = 3*4 = 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Many can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T	= 7 kg	T	= 7 kg	T	= 7 kg
	= (7/2)*2kg		= (7/2)*2kg		= (7/2)*2kg
	= (7/2)*6 litres		= (7/2)*100 %		= (7/2)*5 $
	= 21 litres		= 350 %		= 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:

C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ **T = (T–B) + B**):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67

Or 5ten17 = 5ten(17-10+10) = 6ten7)

Or with *cup-writing* and *internal trade*: T = 38+29 = 3B8 + 2B9 = 5B17 = 5+1B-10+17 = 6B7 = 67

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days @ 6$/day + 3days @ 8$/day = 5days @ (2*6+3*8)/(2+3)$/day = 5days @ 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= $\sum$ n*b	ΔT/Δn	= b
ΔT	= $\int$ b*dn	dT/dn	= b

Only in case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers:

4 years @ 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell Many-based mathematics can be summarised as a 'count&add-laboratory' adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting. (Respecting a Kronecker-principle of building on natural number; and a Russell-principle of not talking about sets of sets e.g. fractions)

ADDING/ Splitting	**Constant**	**Variable**
Unit-numbers m, s, kg, $	**T = n*b** *T/n = b*	**T2 = T1 + n*b** *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	**T = b^n** $\sqrt[n]{T} = b$ $\log_b T = n$	**T2 = T1 + ∫b*dn** *dT/dn = b*

The Count&Add Laboratory

Reinventing Mathematics as a Natural Science about MANY

Presentation at the MATHeCADEMY.net booth at the ICME 11 Congress in Mexico in 2008

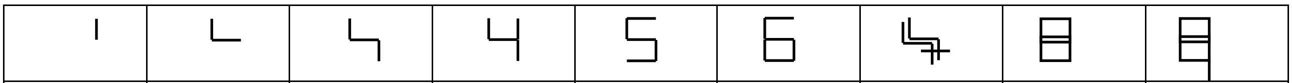
0. Let us reinvent mathematics as a natural science grounded in the study of the natural fact many. And let us see if this grounded natural mathematics will be the same as the mathematics we know from the textbooks. If not, textbook-mathematics might be called orthodox mathematics, where orthodox means claiming that things can only be as in the Book, there are no alternatives, in which case the Book presents its choices as nature.

So an anti-orthodox question is: Can things be otherwise than in the Book? Does the Book hide alternatives?

Uncovering hidden alternatives to orthodox choices presented as nature might be called anti-orthodox research. Or sophist-research since the ancient Greek sophists were the first to warn, that to practice democracy people must know nature from choice to prevent being patronized by choices presented as nature.

1. So let us try to reinvent mathematics as a natural science dealing with the natural fact many. What do we do when we meet many? Two things, first we Count, then we Add, and we do that where we live, in Time and Space. So this approach can be called the CATS-approach to mathematics: Count&Add in Time&Space.

With a pile of sticks there are 3 ways of counting: 1.order-counting, 2.order-counting and 3.order-counting. A 1.order-counting means rearranging the sticks in icons, so that there are five sticks in the five-icon 5 etc. So an icon contains the degree of many it describes. 1.order-counting stops after nine, thus ten has no icon.



1	2	3	4	5	6	7	8	9

A 2.order-counting counts by bundling and stacking in icon-bundles, i.e. counting in e.g. 5s, but not in tens. A 3.order-counting counts in tens, a very special number: the only number with a name but without an icon.

2. As an example of 2.order-counting let us count 7 1s in 3s, 5s and 2s.

I I I I I I I -> III III) I -> II) I -> 2) 1 -> 2B1 3s

I I I I I I I -> IIIII) II -> I) II -> 1) 2 -> 1B2 5s

I I I I I I I -> II II II) I -> III) I -> II I) I -> I) I) I -> 1) 1) 1 -> 1BB1B1 2s

Counting 7 1s in 3s, we take away a 3-bundle 2 times leaving 1 stick unbundled. The unbundled is placed in a right single-cup, and the 3-bundles are placed in a left bundle-cup, either as actual bundles, or as sticks counting bundles by being placed in the left bundle-cup. Thus the counting result is 2B1 3s, using a decimal point to separate the bundles to the left from then unbundled to the right; and including the unit 3s. Likewise counting 7 1s in 5s gives 1B2 5s.

Counting 7 1s in 2s gives 3.1 2s. However, in the bundle-cup we also have a bundle of bundles that can be moved to a new cup to the left, counting the bundles of bundles. Thus counting 7 1s in 2s gives 1BB1B1 2s.

Counting 3 8s in tens gives 2.4 tens, only this time we have no icon for ten: 3 8s = 2B4 tens.

In all cases, counting means bundling in a chosen bundle-size, and counting always produces decimal numbers carrying units. So natural numbers are decimal numbers carrying units.

3. Is this what the Book says? No. The book says: we only count in tens, and we do not write 2.4 tens. First we throw away the unit tens; then we misplace the decimal point one to the right. So instead of 2.4 tens we just write 24, which we call a natural number. Thus what the Book calls natural numbers are instead orthodox numbers hiding its natural alternative and creating problems to learners.

Counting in different bundle-sizes might also be called counting in different bases. However, base is a orthodox term hiding its alternative ‘counting in different bundle-sizes’. The term ‘bundle’ is grounded in experience, a bundle can be grasped. The term ‘base’ is not, it comes from the Book and it can’t be grasped.

4. Since orthodox numbers create learning problems by being unnatural, we ask: If ten is a cognitive bomb by having no icon but needing 2digits, how much mathematics can be learned from 1digit numbers alone?

Surprisingly the answer is that the core of mathematics can be learned as 1digit mathematics.

An example: My sister has 3.2 4s, and I have 2.3 5s. Now we would like to add them. However, to add, the units must be the same, so I must recount my 5s the 4s, or my sister must recount her 4s in 5s. Or we can add them as 9s by uniting the bundle-sizes.

Double-counting a given quantity in two different units, e.g. 4s and 5s, or kgs and £ is called proportionality, normally learned in middle school; and adding in the combined bundle-size is called integration, normally learned late in high school if ever. But using 1digit mathematics, both core concepts are learned in grade 1.

5. Furthermore, recounting 3.2 4s in 5s can be predicted by a calculator. We enter ‘(3*4+2*1)/5’ since counting in 5s means taking away 5s many times, which is iconised as division. The answer is 2.more 5s. To find the more we pull away the 2 5s by subtracting 2*5. Entering ‘(3*4+2*1)-2*5’ gives 4, so the recounting result can be predicted to be 2B4 5s. To test this prediction, we perform the actual recounting by de-bundling the 3B2 4s in 1s and the re-bundling the 1s in 5s:

3B2 4s -> 3)2 -> ~~IIII~~ ~~IIII~~ ~~IIII~~ II -> IIII IIII IIII II -> ~~IIIII~~ ~~IIIII~~ IIII -> 2)4 -> 2B4 5s

So the prediction holds. So from now on we don’t have to perform the actual recounting by de-bundling and re-bundling since we can predict the result on a calculator thus becoming a number-predictor.

6. When the units are the same we can add the two ‘stocks’ using cup-writing:

3B2 4s + 2B3 5s = 2B4 5s + 2B3 5s = 4B7 5s = 4)7 = 4+1) 7-5 = 5)2 = 1) 5-5)2 = 1)0)2 = 1BB0B2 5s

Here the 7 1s can be recounted in 1B2 5s transferring 5 1s as 1 5s form the singles to the bundle-cup. Here the 5 5s can be recounted to 1 5*5s transferring the 5 5s as 1 5*5 from the bundle-cup to the bundles of bundles-cup, thus giving the total of 1 bundle of 5 5s and 0 bundle of 5s and 2 unbundled 1s.

7. With 2.3 5s, what happens if I add an extra cup to the right?

2B3 5s = 2)3 <adding a cup to the right> 2)3) = 2BB3B0 5s.

Apparently adding an extra cup to the right means that the 3 1s becomes 3 5s, and that the 2 5s becomes 2 5*5s, i.e. means multiplying with the bundle-number and moving the decimal point 1 place to the right.

Likewise, removing 1 cup from the right means dividing with the bundle-number and moving the decimal point 1 place to the left:

2)3) 5s <removing a cup from the right> 2)3 = 2B3 5s.

8. Thus we see that 1digit mathematics respects the Piaget ‘through the hands to the head’-principle of natural learning: to grasp with the head, first grasp with the hand.

9. The CATS approach using 1digit mathematics conflicts with the traditional orthodox approach that introduces 2digit numbers in grade 1 by claiming that 10 is the follower of 9.

Now ten is the follower of nine by nature, but to say that 10 IS the follower of 9 is an orthodox choice hiding the alternatives.

With 8 as the bundle-number, 10 is the follower of 7, and the follower of nine is 12.

Thus the CATS approach treating mathematics as a natural science cannot be learned at traditional orthodox academies.

Hence a web-based anti-orthodox academy www.MATH**e**CADEMY.net has been set up to teach the CATS approach to mathematics as an anti-orthodox natural science respecting the huge learning potential of 1digit mathematics.

10. The MATH**e**CADEMY.net offers free master degrees to teachers having learned orthodox mathematics at orthodox academies but wanting to learn mathematics as a natural science investigating the natural fact many.

The learners are organized in groups of 8 using PYRAMID**e**DUCATION: the 8 learners are organized in 2 teams of 4 learners choosing 3 pairs and 2 instructors by turn.

The teacher coaches the instructors instructing the rest of their team.

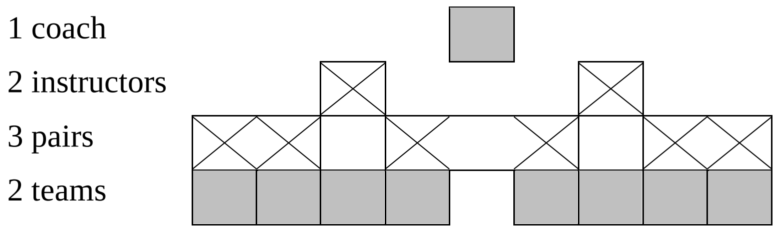
Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.

The coach assists the instructors in correcting the count&add assignments. In each pair each learner corrects the other learner's routine-assignment.

Each pair is the opponent on the essay of another pair.

Each learner pays for the education by coaching a new group of 8 learners.

It is not difficult to be a coach since the learners are educated, not by books but by counting and adding in time and space.



8 learners and 1 coach

The activities are divided into 2x4 parts, Count&Add in Time&Space 1 for primary school, C1, A1, T1 and S1; and Count&Add in Time&Space 2 for secondary school, C2, A2, T2 and S2.

The study units are activity-based and very short.

They are accessible at the MATH**e**CADEMY.net website.

The content is given in the summary below.

11. As research method the MATH**e**CADEMY.net uses anti-orthodox sophist research, following the ancient Greek warning 'in a democracy people must know nature from choice to prevent patronization by orthodox choices presented as nature' by uncovering hidden alternatives to orthodox choices presented as nature.

To demonstrate the power of this new research paradigm 12 papers was written to the ICME 11 conference, for Topic Study Groups (TSG) and Discussion Groups (DG) and most were accepted:

Paper	*Forum*
Avoiding Ten, a Cognitive Bomb	TSG 1: New developments and trends in mathematics education at preschool level. Acc.
A Fresh Start Presenting Mathematics as a Number-predicting Language	TSG 3: New developments and trends in mathematics education at lower secondary level. Rej.
Decimal-Counting, Disarming the Cognitive Bomb Ten	TSG 10: Research and development in the teaching and learning of number systems and arithmetic. Rej.
Orthodox Algebra Deconstructed	TSG 11: Research and Development in the Teaching and Learning of Algebra. Acc.
Orthodox Calculus Deconstructed	TSG 16: Research and development in the teaching and learning of calculus. Acc.
Applying Orthodox Metamatism or Re-Applying Grounded Mathematics	TSG 21: Mathematical applications and modelling in the teaching and learning of mathematics. Acc.
Orthodox Humboldt Mathematics Deconstructed	TSG 25: The role of mathematics in the overall curriculum. Rej.
CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher	TSG 27: Mathematical knowledge for teaching. Acc.
Orthodox Words in mathematics education	TSG 31: Language and communication in mathematics education. Rej.
Deconstructing the Mathematics Curriculum: Telling Choice from Nature	TSG 35: Research on mathematics curriculum development. Acc.
Mathematics Education: Orthodox Bildung - Or Anti-Orthodox Enlightenment	DG 5: The role of philosophy in mathematics education. Acc.
Concealing Choices to Teachers	DG 7: Dilemmas and controversies in the education of mathematics teachers. Acc.
Workshop in 1digit Mathematics, Cup-writing & Decimal-counting	Workshop. Acc.
The 12 Blunders of Orthodox Mathematics	Poster. Acc.

SUMMARY

	QUESTIONS	ANSWERS
C1 COUNT	How to count Many? How to recount 8 in 3s: T= 8 = ? 3s How to recount 6kg in $: T=6kg=?$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T= 8= ?*3= ?3s, T= 8=(8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3 If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle+2Bundle+3 = 4tenten2ten3 = 4*B^2+2*B+3
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
A1 ADD	How to add stacks concretely? T=27+16= 2ten7+1ten6= 3ten13=? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T= (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
A2 ADD	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10=2fold5. Prime-numbers cannot: 5=1fold5 Per-numbers occur when counting, when pricing and when splitting. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b
T1 TIME	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2=14 is reversed to x = (14-2)/3 Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x=b/a, x^a=b is reversed to x=a√b, a^x=b is reversed to x=logb/loga
T2 TIME	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
S1 SPACE	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)
QL	What is quantitative literature? Does quantitative literature also have the 3 different genres: fact, fiction and fiddle?	Quantitative literature tells about Many in time and space The word and the number language share genres: Fact is a since-so calculation or a room-calculation Fiction is if-then calculation or a rate-calculation Fiction is so-what calculation or a risk-calculation

CONTENTS

This website contains 2*4 study units in 'mathematics from below, the LAB-approach', organised as lab-activities where the learner learns 'CATS', i.e. learns to count and add in time and space. The study units CATS1 are for primary school and the study units CATS2 are for secondary school. The units were developed for a web-based teacher-training course in mathematics at a Danish teacher college.

Counting C1. We look at ways to count Many. In space, Many is representing temporal repetition through strokes. Many strokes can be rearranged in icons so that there are four strokes in the icon 4 etc. Then a given total T can be counted in e.g. 4s by repeating the process 'form T take away 4', which can be iconised as 'T-4' where the repeated process 'form T take away 4s' can be iconised as 'T/4' making it possible to predict the counting result through a calculation using the 'recount-equation' T = (T/b)*b, where the number T/b is called a per-number describing the total and the bundle-size. Thus counting a total of 8 in 2s there are T/b = 8/2 = 4 = 4 per 1.

Changing units is another example of a recounting first telling how a given total can be counted into different units e.g. T = 4$ = 5kg producing a per-number 4$/5kg.Thus to answer the question ' 7kg=?$' we just have to recount the 7 in 5s: T = 7kg = (7/5)*5kg = (7/5)*4$ = 5 3/5$.

In most cultures ten is chosen as a standard bundle-size thus bundling and stacking in bundles of tens. In this way a total T becomes a many-stack, a polynomial, consisting of a number of unbundled, a number of bundled, a number of bundles of bundled, etc.

Counting C2. We look at ways to count numbers that change unpredictably as e.g. in surveys. Through counting we can set up a table accounting for the frequency of the different numbers. From this we can calculate the average level and the average change. The average level can then be used as the winning probability p in a game that is repeated n times. By counting the different possibilities it turns out that there is a 95% probability that future numbers lie within an interval determined by the average level and change.

Addition A1. We look at how stacks can be added by removing the overload that often appears when one stack is placed on top of another stack. The stack can be added concretely through a principle of internal trade where a full stack of 10 1s is traded with one 10-bundle. And the result can be predicted by a calculation on paper using either a vertical way of writing the stacks using carrying to symbolise the internal trade; or using a horizontal way of writing the stacks using the FOIL principle. In both cases the overload can be recounted by the recount-equation T=(T/b)*b or transferred by the restack-equation T=(T-b)+b.

Addition A2. We look at how we add per-numbers by transforming them to totals. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b. 2days @ 6$/day + 3days @ 8$/day = 5days @ 7.2$/day. And 1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans. Repeated and reversed addition of per-numbers leads to integration and differentiation:

T2 = T1+a*b; T2-T1 = +a*b; $\Delta T = \sum a*b = \int y*dx$

T2 = T1+a*b; a = (T2-T1)/b = $\Delta T/\Delta b = dy/dx$

Time T1. We look at how addition can be reversed by moving numbers to the other side reversing their signs: x*3+2=14 is reversed to x = (14-2)/3. This enables us to do both forward and backward calculations. Thus we can consider the classical quantitative literature consisting of word-problems from especially economy and physics.

Time T2. We look at how a stack can change in time by adding a constant number or a constant percentage. Or by adding a variable predictable number.

Space S1. We look at how to describe plane properties of stacks as area and diagonals by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations: sinA=a/c, cosA=b/c and tanA=a/b. Then we look at how to describe spatial properties of solids such as surface and volume by formulas and by a 2-dimensional representation of 3-dimensional shapes.

Space S2. We look at how to calculate the position of points and lines by using a coordinate-system: If Po(x,y) = (3,4) and if $\Delta y/\Delta x$ = (y-4)/(x-3) = 2, then P1(8,y) = (8,2*(8-3)+4) = (8,14). Then we look at how to use the new calculation technology such as computers to calculate a set of numbers, vectors, and a set of vectors, matrices.

Where do concepts come from? The present form of presentation has been chosen in order to allow learning to take place in all four learning rooms coming from the four different answers to the question: 'Where do concepts come from? From above or from below? Form the outside or from the inside?' The two traditional learning rooms, the transmitter room and the constructivist room, say 'above&outside' and 'above&inside'. The two hidden alternatives, the fairy-tale room and the apprentice room, say 'below&outside' and 'below&inside'. The traditional rooms take mathematics for granted and see the world as applying mathematics. The hidden rooms have the opposite view seeing the Manyness of the world for granted and as a creator of mathematics through the principle 'coming to grips through gripping' or 'grasping by grasping'. The transmission/fairy-tale room arranges sentence-loaded educational meetings with sentences having abstract/concrete subjects.

The constructivist/apprentice room arranges sentence-free educational meetings with abstract/concrete subjects.

Historically mathematics arose from below as abstractions from examples. Today mathematics is turned upside down by being presented as examples of abstractions. Today's mathematics thus ought to be called 'meta-matics' to be distinguished from the historical 'mathe-matics', which could be called 'mathematics from below - the natural way'.

Where did the LAB-approach come from? Mathematics from below came out of a postmodern study searching for a solution to the global relevance-problem in mathematics by asking the 'Cinderella-question': 'Are there other hidden alternatives? Is there a postmodern mathematics?' The study was based upon the hypothesis that today's postmodern students like mathematics but reject meta-matics. To test this hypothesis an alternative postmodern mathematics from below was developed inspired by the historical development of mathematics. When tested in the laboratory, the difference turned out to be a genuine 'Cinderella-difference' making a difference in the mathematics classrooms in schools and teacher colleges. These study units are built upon this version of postmodern mathematics.

Word-language and Number-language. A ruler and a dictionary help us to assign numbers and words to things using our number-language and our word-language. Thus we have word-sentences containing a subject, a verb and an object; and we have number-sentences, equations, containing a quantity, an equation sign and numbers and calculations. Both sentences are describing the world and are being described by a meta-language. The meta-language of the word-language is called grammar. The meta-language of the number-language is called mathematics.

Our two languages and their meta-languages constitute a language-house with two floors. In the lower floor the language is used to describe the world, and in the upper floor the meta-language is used to describe the language. Syntax errors occur if the meta-language is used to describe the world: 'the verb got drunk'. So mathematics does not describe the world, mathematics describes the number-language, and the number-language describes the world.

		THE LANGUAGE HOUSE		
META-LANGUAGE	GRAMMAR	Subject	Constants and variables	MATHEMATICS
LANGUAGE	WORD-LANGUAGE	The pencil is red	Area = length*height	NUMBER-LANGUAGE
WORLD		THINGS IN TIME AND SPACE		

STUDY GUIDE

The learning principle is expressed by the formulation: GRASP by grasping & Learn from GOSSIP– the LAB approach. This means that learning has to come through the hands, both as objects that can be grasped and as actions that can be performed. Thus a lab approach means that the learning material is brief since the learning takes place not by reading but by doing, i.e. by grasping and by moving.

For a student a learning process has five steps: Do, name, write, reflect and communicate.

For a teacher a learning process has six steps: Do, name, write, reflect, communicate and design experiments.

A teacher is able to design a learning experiment for the students, and able to learn from observing it being carried out.

The experiment is performed three times, first by the designer, then by a student communicating with the teacher, finally by two students communicating with each other while the teacher is observing. During all experiments the teacher looks for examples of cognition, both existing recognition and new cognition. Afterwards the teacher works out a learning report reporting the three experiments observed. The report finally formulates a hypothesis based on what has been learned from observing these three experiments.

For each of the 2*4 final CATS-reports this hypothesis is validated by arranging a new learning experiment to be tested on one student and on two students; and by comparing the prediction from the hypothesis with the observations.

Example:

DO: take five matches from a matchbox and arrange them, first next to each other, then as the symbol 5.

SAY: five matches can be rearranged to the number symbol or the number icon 5.

WRITE: T = 5.

REFLECT. That five matches are called five is old cognition. That five matches can be rearranged as the number symbol 5 is new cognition. That the number-symbols are icons containing the number of matches they describe is new cognition. Also it is new cognition that this makes a fundamental difference between the ability of numbers and letters to represent the world.

COMMUNICATE. Write a postcard: Dear Paul. I have just participated in an experiment where I was asked to take out five matches from a matchbox and arrange them as the number symbol 5. All of a sudden I realised the difference between the symbols ‘5’ and ‘five’, the first representing what it describes an the second representing four sounds. See you next week. Best wishes.

DESIGN LEARNING EXPERIMENT: To build the first twelve number-symbols by rearranging matches.

HYPOTHESIS. This experiment will help Peter, who has problems to understand two digit numbers. Once he tries to build a number symbol for ten, eleven and twelve, he will realise how smart it is to stop inventing new symbols and instead perform a double counting, one counting bundles and one counting unbundled.

TEST. After having finished reporting what Peter did and said, it is my impression that constructing the number icon for ten was what broke the ice for Peter. It seems as if this enabled him to separate number names from number icons, and made him later ask, ‘Why don’t we say one-ten-seven’ instead of seventeen. It would make things much easier.’ This resonates with what Piaget writes:

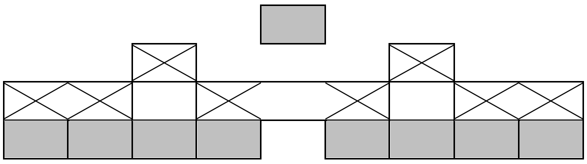
Similarly, it is possible to say that thought is well adapted to a particular reality when it has been successful in assimilating that reality into its own framework while also accommodating that framework to the new circumstances presented by the reality. Intellectual adaptation is thus a process of achieving a state of balance between the assimilation of experience into the deductive structures and the accommodation of those structures to the data of experience. Generally speaking, adaptation presupposes an interaction between subject and object, such that the first can incorporate the second into itself while also taking account of its particularities; and the more differentiated and the more complementary that assimilation and that accommodation are, the more thorough the adaptation. (Quote from J. Piaget (1970) Science of Education of the Psychology of the Child, New York: Viking Compass p. 153-154).

2. PYRAMIDeDUCATION

In PYRAMID**e**DUCATION 8 teachers are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. The coach teaches the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In each pair each student corrects the other student's routine-assignment. Each pair is the opponent on the essay of another pair.

Each teacher pays for the education by coaching a new group of 8 teachers.

1 coach
2 instructors
3 pairs
2 teams



8 teachers and 1 coach

Due dates for essays:

C1: February 25 A1: March 25 T1: April 25 S1: May 25	C2: September 25 A2: October 25 T2: November 25 S2: December 25

3. HOW TO ENROL

Please fill out the enrolment form on the website. Late date November 1.

4. CERTIFICATE

PMM certificate (postmodern master).

1st year programme resulting in a 2x4 reports totalling around 40000 words: 4 educational reports on CATS1 and 4 educational reports on CATS2.

Success criteria: Acceptance by your pyramid.

2nd year programme resulting in an extended 2 paper essay totalling around 40000 words. 1 paper reporting coaching CATS1, 1 paper reporting coaching CATS2.

Success criteria: Two presentations at a teacher union conference.

The study is organised as PYRAMID**e**DUCATION consisting of 8 students being coached by a 2nd year person. Fee: Coaching one 1st year pyramid.

Successful master essays are published on MATH**e**CADEMY.net

A local university might recognize a PMM degree as a basis for a local master degree or a basis for a PhD study.

COUNTING MANY

Questions	Answers
How to count Many?	By bundling and stacking the total T predicted by T = (T/b)*b.
How to recount 8 in 3s, T = 8 = ? 3s	T = 8 = ?*3 = ?3s, T = 8 = (8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3
How to recount 6 kg in $: T= 6 kg= ?$	If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$
How to count in standard bundles?	Bundling bundles gives a multiple stack, a stock or polynomial: T= 423 = 4BundleBundle2Bundle3 = 4tenten2ten3 = 4*B^2+2*B+3

1 REPETITION BECOMES MANY

Question. How can repetition in time be represented in space?
Answer. By iconisation: put a finger to the throat and add a match or a stroke for each beat of the heart.

Example: . . . . -> I I I I I I I I

Exercise. Find other examples of spatial representation of temporal repetition

2 MANY BECOMES BUNDLES

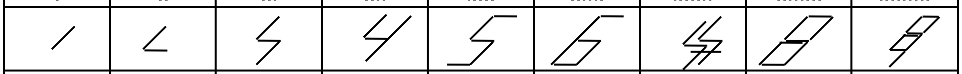
Question. How can we organise Many?
Answer. By bundling: line up the total and divide it into bundles.

Examples: I I I I I I I I I -> II II II II I or I I I I I I I I I -> IIIII IIII or I I I I I I I I I -> III III III or …

Exercise. Take a lot of matches and bundle them in 2s, then in 3s, then in 4s, etc.

3 BUNDLES BECOME ICONS

Question. How can we represent the different degrees of Many?
Answer. By iconisation: the strokes of the different degrees of Many are rearranged as icons, realising that there would be four strokes in the number-icon 4, etc., if written in a less sloppy way.
Example:

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Exercise. Find other ways to build icons for the numbers above. Invent icons for ten, eleven and twelve.

4 MANY IS COUNTED AS A STACK OR AS A STOCK

Question. How can we arrange the different degrees of Many?
Answer. By counting, by bundling and by stacking: First the total is lined up, then it is bundled and equal bundles are stacked and finally the height is counted as e.g. T = 3 4s = 3*4.

Examples:
IIIIIIIIIIII -> IIIIIIII [IIII] -> IIII [IIII] [IIII] -> [IIII] [IIII] [IIII] -> [IIII] stacked three high — T = 3 4s = 3*4 (a stack)

Leftovers are arranged in a separate stack creating a stock:
IIIIIIIIIIIIIII -> IIIIIIIIIII [IIII] -> IIIIIII [IIII] [IIII] -> III [IIII] [IIII] [IIII] -> [IIII] stacked three high with I I I beside — T = 3*4 + 3*1 (a stock)

Or the 3 leftovers are counted in 4s: 3 = ¾*4:
IIIIIIIIIIIIIII -> IIIIIIIIIII [IIII] -> IIIIIII [IIII] [IIII] -> III [IIII] [IIII] [IIII] -> [III] on top of [IIII] stacked three high — T= 3*4 + ¾ *4 = 3 ¾ *4

We count in 4s by taking away 4s. The process ‘from T take away 4’ may be iconized as ‘T-4’ and worded as ‘T minus 4’. The 4 taken away does not disappear, they are just put aside so the original total T is divided into two totals, one containing T-4 and the other containing 4:

[IIIIIIIII] -> [IIIII] [IIII] -> [IIIII] [IIII]
9 = (9-4) + 4 = 5 + 4 as predicted by the ‘restack-equation’ or ‘readd-equation’ T = (T-b)+b.

The repeated process ‘from T take away 4s’ may be iconized as ‘T/4’ and worded as ‘T counted in 4s’. So the ‘recount-equation’ or ‘rebundle-equation’ T = (T/4)*4 predicts the result of recounting the total T in 4-bundles: T =(T/4)*4 =3*4+3*1 =3 ¾*4. T/4 is called a per-number, T a stock-number or a total, and 4 a base.
Exercise. Take a lot of matches and count and stack them in 2s, then in 3s, then in 4s, etc.

5 STOCKS ARE RECOUNTED

Question. How can we change the bundle-size in a stack (T = 3 4s = ? 5s).
Answer. By de-stacking, de-bundling, re-bundling and re-stacking: First the stack is de-stacked into separate bundles, then the bundles are de-bundled into a total, then the total is bundled and equal bundles are stacked and finally the heights are counted. Recounted in 2s even numbers give a stack and odd numbers give a stock.
Example:

```
IIII
IIII                                                                    IIIII    I
IIII  ->  IIII IIII IIII  ->  IIIIIIIIIIII  ->  IIIIIII IIIII  ->  II IIIII IIIII  ->  IIIII    I
3*4  =  4+4+4    =    T    =  (T-5)+5    =    (T/5)*5    =    2*5 +  2*1
```

Again the counting result can be predicted by the recount-equation T = (T/5)*5 = (3*4/5)*5 = 2*5 + 2*1 and displayed on a calculator able to do integer division, e.g. the Texas Instruments' Math Explorer.
Exercise1. Recount a 2-stack in 3s, in 4s, in 5s, etc. Recount a 3-stack in 2s, in 4s, in 5s, etc.
Exercise2. Recount a 2-stack in ½ s, in 1/3 s, in ¼ s, etc. Show that T = (T*n)*1/n.

6 STOCKS ARE CODED

Question. How can we code a stock?
Answer. By using codes as C and S to code different bundle sizes, e.g. C = 4 and S = 5.
Then the recount-question 'T = 3*4 = ?*5' is reformulated to 'T = 3*C = ?*S '.
Example:

```
C
C                                                                S    I
C   ->  C C C  ->  IIIIIIIIIIII  ->  IIIIIII  S  ->  II  S S  ->  S    I
3*C  =  C+C+C  =   T   =  (T-S) +S   =   (T/S)*S   =   2*S + 2*1
```

Also the recounting can be done as in trade by skipping the 1s: Thus if 1*C = 2*S we have

```
     CCC       ->  SS  SS SS     ->   SSSSSS
T =  3*C       =    3*2*S        =      6*S        predicted by 3*C = 3*2*S = 6*S

     SSSSSS    ->  SS  SS  SS    ->   CCC
T =    6*S     =      3*2*S      =    3*C          predicted by 6*S = ((6/2)*2)*S = 3*2*S = 3*C

     SSSSSSS   ->  SS  SS  SS  S   ->  CCC   S
T =    7*S     =      3*2*S  + S   =  3*C  + 1*S   predicted by 7*S=((7/2)*2)*S=3*2*S+1*S=3*C+1*S
```

With codes a stack can by counted in Cs to practise tables. Thus C C C C can be counted as '2 4 6 8' or ' 3 6 9 twelve'. Here 'twelve' regains its original meaning 'two left' making it possible to count from ten to 'two-ten' as: ten, one left, two left, three left, …, nine left, two ten, etc.
Exercise. Using symbols, recount a 2-stack in 3s, in 4s, in 5s, etc. Recount a 3-stack in 2s, in 4s, in 5s, etc.

7 STOCKS ARE CODED

Question. How can we code a stock?
Answer. By using a container-symbol ')' to leave out the symbols for bundle and 1: T= 2*C+3*1 -> 2)3).
Thus we use the container-symbol '))' or '0' for 'none' to distinguish T = 2*C = 2)) = 20 from T = 2*1 = 2.
Later we leave out the containers and get many-digit numbers: T = 2)3) = 23, and T = 2)) = 2)0) = 20
Example1.
If C = 3, then T = 18 = 25 = 32 — C = 4, then T = 30 = 24 = 18

```
 C IIIIIIII -> C IIIII III -> CC  IIIII -> CC  III II -> CCC II  |  CCC  ->  CC IIII  ->  C IIII IIII
 T = 1   8    =          2    5  =          3    2                |  T =  30   =   2  4   =  1   8
```

Or with containers

T = 18 = 1)8)
T = 1)8) = 1+1)-3+8) = 2)5) = 25
T = 2)5) = 2+1)-3+5) = 3)2) = 32

I	IIIII III
II	II III
III	II

T = 30 = 3)) = 3)0)
T = 3)0) = 3-1)+4+0) = 2)4) = 24
T = 2)4) = 2-1)+4+4) = 1)8) = 18

III	
II	IIII
I	IIII IIII

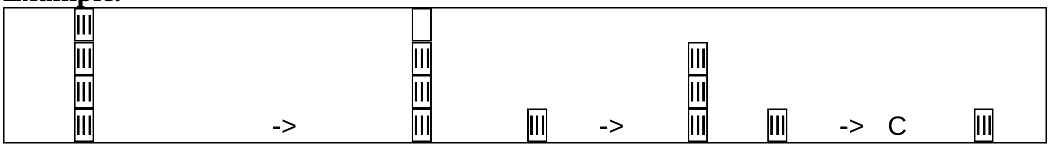
Example2. With many-digit numbers we can still practise recounting stocks by reformulating the recount-question 'T = 3*C + 1 = ?*S' to 'T = 31 = ?*S' , e.g. with C = 4 and S = 5:

```
C                                                          I
C                                                        S I   T  = 3*C + 1     = 31
C I  ->  C C C I  ->  IIIIIIIIIIIII  ->  IIIIIIII  S  ->  III  S S  ->  S I   T  = 2*S + 3*1  = 23
3 1   =         T     =      (T/S)*S           =     2  3
```

Example3. When the icons stop we use symbols. Thus the Romans used the symbol X for the number ten. The recount-question 'T = 3*8 = ?*X' recounts 3 8s as 2 tens and 4 1s: T = 3*8 = 2*X+4*1 = 24. This leads to traditional multiplication. However 3*8 is 3 8s; only if recounted in tens, it is 24.
The recount-question 'T = 3*X = ?*8' recounts 3 tens as 3 8s and 6 1s: T= 3*X= 30/8*8= 3*8+6*1. This connects traditional division with recounting, not with sharing, which is applying and not creating division.
Exercise1. If C=2 rewrite T=13 and T=30, etc. If C=3 rewrite T=15 and T = 41. If C=4, etc.
Exercise2. If C=2 and S=3 recount 31 in Ss, etc.
Exercise3. Recount a 3-stack in tens. Recount a ten-stack in 2s, in 3s, in 4s, etc.
Exercise4. Recount numbers on an abacus with 3 containers for 1s, 5s and tens, e.g. 2)4)8) = 2)5)3)= 4)1)3).

8 STACKS ARE OVERLOADED

Question. What do we do about an overload, i.e. if a stack is higher than its unit?
Answer. The overload then can be restacked to a new stack leaving a full stack C.
Example.



T = 4*3 = overload = (4*3 – 1*3) + 1*3 = 3*3 + 1*3 = 1*C + 1*3

So also bundles can be bundled and stacked in bundles-of-bundles, bundles & unbundled:
T = 234 = 2 bundles-of-bundles + 3 bundles + 4 unbundled (e.g. T = 2tenten3ten4)
In short a given degree of Many can always be rearranged as a multiple stack, a stock or a polynomial:
T = 2345 = 2)3)4)5 = 2)))+3))+4)+5 = 2*B^3 + 3*B^2 + 4*B + 5*1
Exercise. Recount the overloads: T= 7 5s, T= 18 8s, T= 34 tens, T= 562 hundreds, T= 562 tens, T= 562 1s

9 STOCKS ARE COUNTED IN TENS

Question. How can we code a stock counted in tens?
Answer. By many-digit numbers leaving out the symbols for bundle and 1: T = 2*X+3*1 -> 2)3) -> 23
Example. Multiplication shows the result of a standard recounting in tens and 1s:
T = 3 6s = 3*6 = 18 = 1*10 + 8*1.
Recounting in tens leads to decimals and percentages:
T = 3 6s = 3*6 = 18 = (18/10)*10 = (1 8/10)*10 = 1.8*10
T = 3 6s = 3*6 = 18 = (18/100)*100 = 18%*100
T = 3.6 = (3.6/10)*10 = 0.36*10
T = 3.6 = (3.6/(1/10))*1/10 = 36*1/10 = 36 tenths = (3.6*10)*1/10, so T/(1/b) = T*b
Remark. T = 3*6 = 3 6s and T = 3*6 = 18 only if recounted in tens. Thus multiplication is a special division.
Exercise. Recount a 3-stack in tens, in tenths, in hundreds, in hundredths, in thousands, in thousandths.

10 STOCKS ARE COUNTED IN UNITS

Question. How can we recount a stock in a different unit?
Answer1: Recount the number using the key-numbers.
Answer2: Recount the unit using the key-numbers
Example: Sugar can be bundled in kilos, litres, dollars and %.
Key-numbers: 2 kg = 5 $ = 6 litres = 100 %, T = 7 kg = ?

Recount the number	Recount the unit
T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $	$ = ($/kg)*kg = (5/2)*7 = 17.5
T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	litres = (litres/kg)*kg = (6/2)*7 = 21
T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	% = (%/kg)*kg = (100/2)*7 = 350
P = 5% = (5/100)*100% = (5/100)*2 kg = 0.1 kg	kg = (kg/%)*% = (2/100)*5 = 0.1

Exercise. 4 kg = 6 $ = 7 litres = 100 %, T = 8 kg = ?

11 STOCKS ARE COUNTED IN DIFFERENT BASES

Question. How can we recount a stock in a different base?
Answer1. Recount the number using the key-numbers.
Example1. T = 2 7s. The 'ten-counter' counts 'bundle + 4', i.e. 14, and the 'twelve-counter' counts 'bundle + 2', i.e. 12. Hence T = 14(10) = 12(12).
Example2. T = 31(4) = ? (5)

Key-numbers: 4 5s = 5 4s
Recount the number
31(4) = 3 4s+1 = (3/5)*5 4s + 1 = (3/5)*4 5s + 1 = 12/5 5s + 1 = 2 5s + 2+1 = 2 5s + 3 = 23(5)
Check by recounting in tens: 31(4) = 3*4+1 = 13 and 23(5) = 2*5+3 = 13
Exercise1. Count your ten fingers as a 5-counter, a 4-counter, a 3-counter and a 2-counter.
Exercise2. 23(4) = ?(5) = ?(6) , 23(6) = ?(5) = ?(4) , 23(10) = ?(9) = ?(8) = ? , etc.

12 STOCKS ARE DECODED

Question. How can we decode a coded stock?
Answer. By recounting and restacking.
Examples.

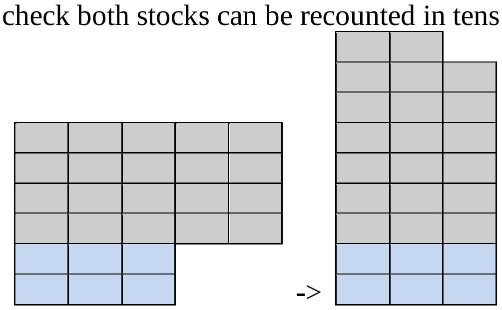
C III = IIIIIIII = IIIII III
C + 3 = 8 = (8-3)+3 — So C = 5 as predicted by C+3 = 8 = (8-3)+3 = 5+3

C
C = IIIIII -> II II II II -> II II II / III III
2*C = 6 = (6/2)*2 = 3*2 = 2*3 — So C = 3 as predicted by 2*C = 6 = (6/2)*2 = 3*2

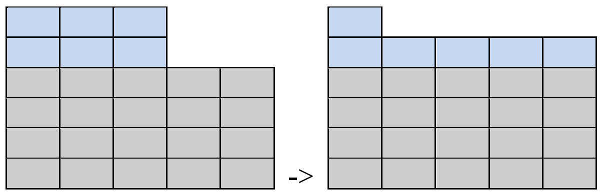
CC I = IIIIIIIII = IIIIIIII I = II II II II I = IIII IIII I
2*C + 1 = 9 = (9-1)+1 = 8+1 = (8/2)*2 +1 = 4*2 + 1 = 2*4 + 1
So C = 4 as predicted by 2*C + 1 = 9 = (9-1)+1 = 8+1 = (8/2)*2+1 = 4*2 + 1.
Exercise1. Decode C+1=3, C+1=4, etc. Decode C+2=4, C+3=5, etc.
Exercise2. Decode 2*C=4, C+1=4, etc. Decode C+2=4, C+3=5, etc.
Exercise3. Decode C+1=3, C+1=4, etc. Decode C+2=4, C+3=5, etc.

13 STACKS ARE ADDED UPWARD

Question. How can we add stacks upward?
Answer. Stacks can be added upward in space by recounting and restacking.
Examples. T = 2 3s + 4 5s = ? 3s, T = 2 3s + 4 5s = ? 5s. The result can be predicted by the recount-equation.
The 4 5s are recounted in 3s: T = 2 3s + 4 5s = 2 3s + (4*5/3)*3 = 2 3s + 6*3 +2 = 2 3s + 6 3s + 2 = 8 3s + 2
The 2 3s are recounted in 5s: T = 2 3s + 4 5s = (2*3)/5*5 + 4 5s = 1*5 + 1 + 4 5s = 1 5s + 1 + 4 5s = 5 5s + 1
To check both stocks can be recounted in tens: T = 8 3s + 2 = 8*3+2 = 26 and T = 5 5s + 1 = 5*5+1 = 26



T = 2 3s + 4 5s = 8 3s + 2

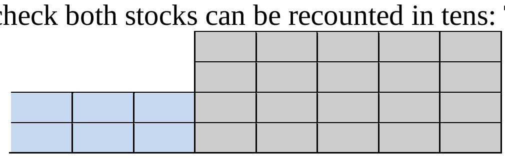


T = 2 3s + 4 5s = 5 5s + 1

Exercise. Add 1 2s + 3 4s upward and predict the result. Add 2 3s + 2 4s upward and predict the result, etc.

14 STACKS ARE ADDED SIDEWARD

Question. How can we add stacks sideward?
Answer. Adding stacks sideward in time is called integration. It can be done by recounting and restacking.
Example. T = 2 3s + 4 5s = ? 8s. The result can be predicted by the recount-equation.
The 3s and 5s are added as 3+5 = 8s: T = 2 3s + 4 5s = (2*3 + 4*5)/8*8 = 3*8 +2 = 3 8s + 2
To check both stocks can be recounted in tens: T = 2 3s + 4 5s = 2*3 + 4*5 = 26 and T = 3*8 +2 = 26



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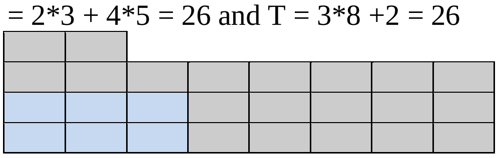


T = 2 3s + 4*5 = 3 8s + 2

Thus integration adds the per-numbers 2 and 4 as heights in stacks: 2 + 4 = 3 2/8.
Thus 2 + 4 can give many different results, unless the units are the same:
T= 2*3 + 4*3 = 6*3 if added in time; and T= 2*3 + 4*3 = (2*3 + 4*3)/6*6 = 3*6 if added in space.
Exercise1. Add 1 2s + 3 4s sideward and predict the result. Add 2 3s + 2 4s sideward and predict the result, etc.
Exercise2. Reverse integration to get differentiation: T = 2 3s + ? 5s = 3 2/8 8s, etc.

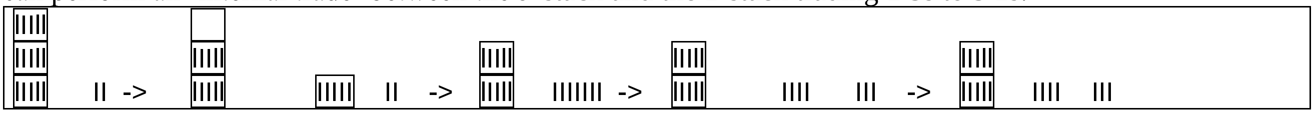
ADDING MANY

A1

Questions	Answers
How to add stacks concretely? T = 27+16= 2ten7+1ten6= 3ten13 =?	By restacking overloads predicted by the restack-equation T = (T-b)+b 3ten13 = 3ten(13-10+10) = 3ten1ten3 = 4ten3 = 43
How to add stacks abstractly?	Vertical calculation uses carrying. Horizontal calculation uses FOIL.

1 STACKS ARE SOLD

Question. How can we sell more from a stack than we have?
Answer. Create an overload by recounting and doing internal trade.
Example. From the stock T = 3 5s + 2 1s we want to sell 3 1s, but we only have 2 1s in stock. However we can perform an 'internal trade' between the 5-stack and the 1-stack trading 1 5s to 5 1s:

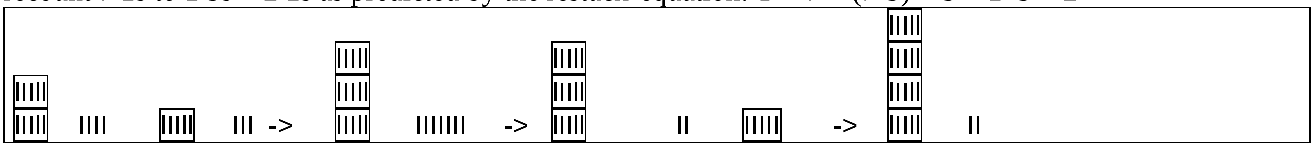


3*5 + 2 = (3*5-1*5) + 1*5 + 2 = 2*5 + 7 = 2*5 + (7-3) + 3 = 2*5 + 4 + 3

After the matches we use cups and internal trade to write T= 32-3 = 3)2)-3)= 3-1)5+2)-3)= 2)7)-3)= 2)4)= 24
Or: T = 32 –3 = 3five2 –3 = 2five1five2 –3 = 2five7 –3 = 2five4 = 24
In case of tens we have T = 32-3 = 3)2)-3) = 3-1)10+2)-3) = 2)12)-3) = 2)9) = 29 (- & + : outside - & +)
Or: T = 32 –3 = 3ten2 –3 = 2ten1ten2 –3 = 2ten12 –3 = 2ten9 = 29
Exercise1. Sell 3 from 41. Sell 34 from 421. Sell 342 from 4231. Count in fives. First use matches, then write.
Exercise2. Sell 3 from 41. Sell 34 from 421. Sell 342 from 4231. Count in tens. First use matches, then write.

2 STACKS ARE BOUGHT

Question. How can stocks be added?
Answer. Remove the overload by recounting and doing internal trade.
Example. To the stock T = 2 5s + 4 1s we add the stock T' = 1 5s + 3 1s. After adding the 1s we are able to recount 7 1s to 1 5s + 2 1s as predicted by the restack-equation: T = 7 = (7-5) + 5 = 1*5 + 2



2*5 + 4 + 1*5 + 3 = 3*5 + 7 = 3*5 + (7-5) + 5 = 4*5 + 2

After the matches we use cups and internal trade to write
T = 24 + 13 = 2)4) + 1)3) = 3)7) = 3)7-5+5) = 3)5+2) = 3+1)2) = 4)2) = 42
Or: T = 24 + 13 = 2five4 + 1five3 = 3five7 = 3five1five2 = 4five2 = 42
In case of tens we have T = 24 + 17 = 2)4) + 1)7) = 3)11) = 3)11-10+10) = 3)10+1) = 3+1)1) = 4)1) = 41
Or: T = 24 + 17 = 2ten4 + 1ten7 = 3ten11 = 3ten1ten1 = 4ten1 = 41
Exercise1. Add 3 to 24. Add 43 to 34. Add 241 to 444. Count in fives. First use matches, then write.
Exercise2. Add 8 to 24. Add 79 to 34. Add 879 to 444. Count in tens. First use matches, then write.

3 STOCKS ARE SPLIT

Question. How can stocks be split?
Answer. Create an overload by recounting and doing internal trade.
Example. The stock T = 3 5s + 4 1s is split in two parts.

VVV IIII -> VV V IIII -> VV IIIII IIII ->VV IIIIIIIII -> VV IIII IIII I -> VIIII VIIII I

34=(3*V+4)=(3/2)*2*V+ 4 = 2*V + 1*V + 4 = 2*V+5 + 4 = 2*V + 9 = 2*V+2*4 +1 = 2*(V+4) + 1= 2*14+1
After the matches we use cups and internal trade to write
34 = 3)4) = 3/2*2)4) = 1*2+1)4) = 2*1)+5+4) = 2*1)9) = 2*1)9/2*2) = 2*1)4*2+1) = 2*1)2*4) + 1 = 2*14+1
Or: 34 = 3five4 = 2five1five4 = 2five9 = 2*1five2*4 +1 = 2*1five4 +1 = 2*14 +1
In case of tens we write
34 = 3)4) = 3/2*2)4) =1*2+1)4) = 1*2)+10+4) = 2*1)14) = 2*1)14/2*2) = 2*1)7*2) = 2*1)2*7) = 2*17
Or: 34 = 3ten4 = 2ten1ten4 = 2ten14 = 2*1ten2*7 = 2*1ten7 = 2*17
Remark. 8 counted in 2s: 8 = (8/2)*2 = 4*2 = 4 2s. 8 split in 2: 8 = (8/2)*2 = 4*2 = 2*4 = 2 4s. So division is used both when counting and when splitting.
Exercise1. Split 43 in 2. Split 43 in 3. Split 34 in 4. Split 43 in 7. Count in fives. First use matches, then write.
Exercise2. Split 43 in 2. Split 43 in 3. Split 34 in 4. Split 43 in 12. Count in tens. First use matches, then write.

4 STOCKS ARE MULTIPLIED

Question. How can stocks be multiplied?
Answer. Remove the overload by recounting and doing internal trade.
Example. The stock T = 2 5s + 4 1s is bough three times.

VVIIII VVIIII VVIIII	-> VV VV VV IIIIIIIIIII	-> VVVVVV IIIII IIIII II	-> VVVVV VVV II	-> W VVV II
3*24 = 3*(2*V+4)	= 3*2*V + 3*4	= 6*V + 5 + 5 + 2	= 5*V + 3*V+ 2	= 1*W + 3*V + 2 = 132

After the matches we use cups and internal trade to write
T = 3*24 = 3*2)4) = 6)12) = 6)12/5*5) = 6)2*5+2) = 6+2)2) = 8)2) = 8/5*5)2) = 1*5+3)2) = 1)3)2) = 132
Or: T = 3*24 = 3*2five4 = 6five12 = 6five2five2 = 8five2 = (1five3)five2 = 1fivefive3five2 = 132
In case of tens we write
T = 3*24 = 3*2)4) = 6)12) = 6)12/10*10) = 6)1*10+2) = 6+1)2) = 7)2) = 72
Or: T = 3*24 = 3*2ten4 = 6ten12 = 6ten1ten2 = 7ten2 = 72
Remark. 3*11 = 33 both when counting in tens and fives; but 3*12 = 36 or 41.
Exercise1. Do 2*34. Do 3*34. Do 4*34. Do 7*34. Count in fives. First use matches, then write.
Exercise2. Do 2*34. Do 3*34. Do 4*34. Do 12*34. Count in tens. First use matches, then write.

5 STOCKS ARE ACCOUNTED FOR

Question. How can we account for selling and buying?
Answer. By ledger accounts.
Example. From the stock 4)2) we sell S: 2)3) and buy B: 1)3). We set up two ledger accounts, one for the bundles, and one for the ones to account for the IN (debit) and OUT (credit) of the internal and external trade.

Bundle = 5 *Fives*
Ones

	IN	OUT	TOTAL	IN	OUT	TOTAL
Stock			**4**			**2**
Internal trade		1		5		
External trade S: 2)3		2			3	
Stock			**1**			**4**
External trade B: 1)3	1			3		
Internal trade	1				5	
Stock			**3**			**2**

Bundle = 10 *Tens*
Ones

	IN	OUT	TOTAL	IN	OUT	TOTAL
Stock			**4**			**2**
Internal trade		1		10		
External trade S: 2)3		2			3	
Stock			**1**			**9**
External trade A: 1)3	1			4		
Internal trade	1				10	
Stock			**3**			**2**

Exercise1. From the stock 3)1) we sell S: 2)3) and add A: 2)4). Count in fives. First use matches, then write.
Exercise2. From the stock 3)1) we sell S: 2)3) and add A: 2)4). Count in sevens. First use matches, then write.
Exercise3. From the stock 3)1) we sell S: 2)3) and add A: 2)4). Count in tens. First use matches, then write.

6 TWO STOCK ACCOUNTS I

Question. How can we account for selling and buying between two traders?
Answer. By ledger accounts.
Example. The TwoTrader-game: Two traders A and B throw two dices in turn, a red and a white. Red means bundles, white means ones and throwing means buying. The dices are reduced so that 4->1, 5->2 and 6->3 in case of 5-bundles. We set up two ledger accounts for each trader, one for the bundles, and one for the ones to account for the IN and OUT of the internal and external trade.

Bundle = 5 *Fives* *Ones* *Fives*
Ones

(red,white)	IN	OUT	TOT	IN	OUT	TOT	**A** | **B**	IN	OUT	TOT	IN	OUT	TOT
A(2,3)			**4**			**2**				**4**			**2**
Internal									1		5		
External	2			3					2			3	
Internal	1				5								
B(3,1)			**7**			**0**				**1**			**4**
Internal		1		5									

	IN	OUT	TOT	IN	OUT	TOT		IN	OUT	TOT	IN	OUT	TOT
External		3			1			3			1		
Internal								1				5	
Stock			**3**			**4**				**5**			**0**
Bundle = 10			*Tens*			*Ones*				*Tens*			
Ones													
(red,white)	IN	OUT	TOT	IN	OUT	TOT	**A** | **B**	IN	OUT	TOT	IN	OUT	TOT
A(5,4)			**4**			**2**				**8**			**2**
Internal									1		10		
External	5			4					5			4	
Internal													
B(4,3)			**9**			**6**				**2**			**8**
Internal													
External		4			3			4			3		
Internal								1				10	
Stock			**5**			**3**				**7**			**1**

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.
Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.
Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

7 TWO STOCK ACCOUNTS II

Question. How can we simplify the account for the selling and buying between two traders?
Answer. By using many-digit numbers instead of one-digit numbers, and carrying instead of internal trade.
Example. The same example as in 6.

Bundle = 5

(red,white)	IN	OUT	TOT	**A** | **B**	IN	OUT	TOT
A(2,3)			**12**				**32**
Internal					5	10	
External	23					23	
Internal	10	5					
B(3,1)			**40**				**4**
Internal	5	10					
External		31			31		
Internal					10	5	
Stock			**4**				**40**

Bundle = 10

(red,white)	IN	OUT	TOT	**A** | **B**	IN	OUT	TOT
A(5,4)			**42**				**82**
Internal					10	10	
External	54					54	
Internal							
B(4,3)			**96**				**28**
Internal							
External		43			43		
Internal					10	10	
Stock			**53**				**71**

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.
Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.
Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

8 TWO STOCK ACCOUNTS III

Question. What is the simplest way to account for the selling and buying between two traders?
Answer. By using number signs instead of ledger accounts, i.e. by replacing the medieval 3|5 by 3-5.
Example. The same example as in 6.

Bundle = 5

(red,white)	CHANGE ΔT	TOTAL T	**A** | **B**	CHANGE ΔT	TOTAL T
		42			**42**
A(2,3)	+23	**70**		-23	**14**
B(3,1)	-31	**34**		+31	**50**

Bundle = 10

(red,white)	CHANGE ΔT	TOTAL T	**A** | **B**	CHANGE ΔT	TOTAL T
		42			**82**
A(5,4)	+54	**96**		-54	**28**
B(4,3)	-43	**53**		+43	**71**

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.
Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.
Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

9 STOCKS MAY CHANGE

Question. In what way can a total stock change?
Answer. Predictably by a change equation ΔT = +2. Unpredictably by throwing a dice ΔT = 1, 2 or 3
Example. A gets his income from working (ΔT = +2). B gets his income from trading (ΔT = 1, 2 or 3)

Bundle = 5	CHANGE ΔT	TOTAL T	A	B	CHANGE ΔT	TOTAL T
Day 0		**12**				**12**
Day 1	+2	14			+1	13
Day 2	+2	21			+3	21
Day 3	+2	23			+3	24
Day 4	+2	30			+3	32
Day 5	+2	32			+2	34
Day 6	+2	34			+1	40
Day 7	+2	41			+1	41
Day 8	+2	**43**			+1	**42**

Bundle = 10	CHANGE ΔT	TOTAL T	A	B	CHANGE ΔT	TOTAL T
Day 0		**12**				**12**
Day 1	+2	14			+1	13
Day 2	+2	16			+3	16
Day 3	+2	18			+3	19
Day 4	+2	20			+3	22
Day 5	+2	22			+2	24
Day 6	+2	24			+1	25
Day 7	+2	26			+1	26
Day 8	+2	**28**			+1	**27**

Exercise1. Continue the Income-game. Count in fives. First use matches, then write.
Exercise2. Continue the Income-game. Count in tens. First use matches, then write.
Exercise3. Repeat the Income-game. Count in sixes. First use matches, then write.
The two-dice income-game: A and B get their income from throwing two dices, a red and a white. An even number on the red means +, and an odd number on the red means -, so ΔT = ±1, ±2, ±3, ±4, ±5 or ±6.
Exercise4. Play the two-dice income-game. Count in fives. First use matches, then write.
Exercise5. Play the two-dice income-game. Count in tens. First use matches, then write.
Exercise6. Play the two-dice income-game. Count in sixes. First use matches, then write.

10 STOCKS ARE ADDED ABSTRACTLY

Question. How can many-digit numbers be added abstractly?
Answer. Vertically in arithmetic or horizontally in mathematics.
Example1. Addition
Verbal: T = 283 + 78 = 2 tenten 8 ten 3 + 7 ten 8 = 2 tenten 15 ten 11 = 2 tenten 16 ten 1 = 6 tenten 6 ten 1 = 361

Horizontal calculation (mathematics)

T	= 283 + 78	*By restacking*
	= 2*B^2 + 8*B + 3*1 + 7*B + 8*1	
	= 2*B^2 + (8*B + 7*B) + (3*1 + 8*1)	
	= 2*B^2 + (8 + 7)*B + (3 + 8)*1	restacking
	= 2*B^2 + 15*B + 11*1	overload!
	= 2*B^2 + (15-B+B)*B + (11-B+B)*1	restacking
	= 2*B^2 + (5+B)*B + (1+B)*1	
	= 2*B^2 + 5*B + B*B + 1*1 + B*1	
	= 2*B^2 + 5*B + 1*B^2 + 1*1 + 1*B	recounting
	= 2*B^2 + 1*B^2+ 5*B+ 1*B + 1*1	
	= (2+1)*B^2 + (5+1)*B + 1*1	restacking
	= 3*B^2 + 6*B + 1*1	
	= **361**	
T	= 283 + 78	*By recounting*
	= 2*B^2 + 8*B + 3*1 + 7*B + 8*1	
	= 2*B^2 + (8*B + 7*B) + (3*1 + 8*1)	
	= 2*B^2 + (8 + 7)*B + (3 + 8)*1	restacking
	= 2*B^2 + 15*B + 11*1	overload!
	= 2*B^2 + (15/B*B)*B + (11/B*B)*1	recounting
	= 2*B^2 + (1*B+5)*B + (1*B+1)*1	
	= 2*B^2 + 1*B^2 + 5*B + 1*B + 1*1	
	= (2+1)*B^2 + (5+1)*B + 1*1	restacking
	= 3*B^2 + 6*B + 1*1	

Vertical calculation (arithmetic)

B^2	**B**	**1**
By carrying		
1	*1*	
2	8	3
	7	8
3	**6**	**1**
By overload		
2	8	3
	7	8
2	*15*	*11*
2+1	*5+1*	*1*
3	**6**	**1**

Remark: We restack when we are moving common multipliers inside and outside of parentheses

= **361**	

Example2. Subtraction

Verbal: T = 263 - 87 = 2 tenten 6 ten 3 - 8 ten 7 = 2 tenten -2 ten -4 = 2 tenten -3 ten 6 = 1 tenten 7 ten 6

T	= 263 - 87	*By restacking*
	= 2*B^2 + 6*B + 3*1 - (8*B + 7*1)	
	= 2*B^2 + 6*B + 3*1 - 8*B - 7*1	
	= 2*B^2 + (6*B - 8*B) + (3*1 - 7*1)	
	= 2*B^2 + (6 - 8)*B + (3 - 7)*1	restacking
	= 2*B^2 -2*B -4*1	underload!
	= (2-1+1)*B^2 -2*B -4*1	restacking
	= (1+1)*B^2 -2*B -4*1	
	= 1*B^2 +1*B^2 -2*B -4*1	
	= 1*B^2 +10*B -2*B -4*1	recounting
	= 1*B^2 + (10 -2)*B -4*1	restacking
	= 1*B^2 + 8*B -4*1	
	= 1*B^2 + (8-1+1)*B -4*1	restacking
	= 1*B^2 + (7+1)*B -4*1	
	= 1*B^2 + 7*B + 1*B -4*1	
	= 1*B^2 + 7*B + 1*10 -4*1	recounting
	= 1*B^2 + 7*B + 10*1 -4*1	
	= 1*B^2 + 7*B + (10 – 4)*1	
	= 1*B^2 + 7*B + 6*1	
	= 176	

	B^2	**B**	**1**
Borrowing:			
	-1	*+10 -1*	*10*
	2	6	3
		8	7
	1	**7**	**6**
Adding from below:			
	2	6	3
		8	7
	1	**7**	**6**
	1	*1*	
Deficits:			
	2	6	3
		8	7
	2	*-2*	*-4*
	2-1	*+10-2-1*	*+10-4*
	1	**7**	**6**

Example3. Multiplication

Verbal: T= 8*246= 8 * 2tenten4ten6= 16tenten32ten48= 16tenten36ten8= 19tenten6ten8=1tententen9tenten6ten 8= 1968

T	= 8*246	*By restacking*
	= 8*(2*B^2 + 4*B + 6*1)	
	= 8*2*B^2 + 8*4*B + 8*6*1	
	= 16*B^2 + 32*B + 48*1	overload!
	= (16-10+10)*B^2 + (32-30+30)*B + (48-40+40)*1	restacking
	= (6+B)*B^2 + (2+3*B)*B + (8+4*B)*1	
	= 6*B^2 + 1*B*B^2 + 2*B + 3*B*B + 8*1 + 4*B*1	
	= 1*B^3 + 6*B^2 + 3*B^2 + 2*B + 4*B+ 8*1	recounting
	= 1*B^3 + (6+ 3)*B^2 + (2 + 4)*B+ 8*1	restacking
	= 1*B^3 + 9*B^2 + 6*B+ 8*1	
	= 1968	
T	= 8*246	*By recounting*
	= 8*(2*B^2 + 4*B + 6*1)	
	= 8*2*B^2 + 8*4*B + 8*6*1	
	= 16*B^2 + 32*B + 48*1	overload!
	= (16/B*B)*B^2 + (32/B*B)*B + (48/B*B)*1	recounting
	= (1*B+6)*B^2 + (3*B+2)*B + (4*B+8)*1	
	= 1*B*B^2 + 6*B^2 + 3*B*B + 2*B + 4*B*1+ 8*1	restacking
	= 1*B^3 + 6*B^2 + 3*B^2 + 2*B + 4*1*B+ 8*1	
	= 1*B^3 + (6+ 3)*B^2 + (2 + 4)*B+ 8*1	restacking
	= 1*B^3 + 9*B^2 + 6*B+ 8*1	
	= 1968	

B^3	**B^2**	**B**	**1**
Carrying:			
	3	*4*	
*8**	2	4	6
1	**9**	**6**	**8**
Overloads:			
*8**	2	4	6
	16	*32*	*48*
0+1	*6+3*	*2+4*	*8*
1	**9**	**6**	**8**

The harmonica:

8*	243	= ?

8*	243	= **1944**
	200	*1600*
	40	*320*
	3	*24*

Example4. Division

Verbal: T = 253/8 = 2tenten5ten3 /8 = 25ten3 /8 = 3ten + 1ten3 /8 = 3ten + 13 /8 = 3ten + 1 +5/8 = 31 5/8

T	= 253	
	= 2*B^2 + 5*B + 3*1	
	= 2*B*B + 5*B + 3*1	
	= 20*B + 5*B + 3*1	recounting
	= (20 + 5)*B + 3*1	restacking
	= 25*B + 3*1	
	= 25/8*8*B + 3*1	recounting
	= (3*8 + 1)*B + 3*1	
	= 3*8*B + 1*B + 3*1	restacking
	= 3*B*8 + B*1 + 3*1	recounting
	= 30*8 + (10+3)*1	restacking
	= 30*8 + (13/8*8)*1	recounting

Finding the remainder:

8	253	31
	24	
	13	
	8	
	5	

Reversed harmonica:

8*	?	= 253

	8*	= 253
	30	*240*
		13

	= 30*8 + (1*8+5)*1	
	= 30*8 + 1*8*1 + 5*1	restacking
	= 30*8 + 1*1*8 + 5*1	
	= (30+1)*8 + 5*1	restacking
	= 31*8 + 5*1	
	= 31*8 + 5*1/8*8	recounting
	= 31*8 + 5/8*8	
	= (31 + 5/8)*8	restacking
	= 31 5/8 * 8	
T	= 253 = 253/8*8 = 31 5/8 * 8	
	so 253/8 = 31 5/8	

8*		= 253
	30	*240*
		13
	1	*8*
		+5

8*	**31**	**+5**	**= 253**
	30		*240*
			13
	1		*8*
			+5

Example5. Multiplying many-digit numbers. Verbal: T = 42*68 = 4ten2*6ten8 = 24tenten(32+12)ten16 = 24tenten44ten 16 = 24tenten45ten6 = 28tenten5 ten 6 = 2 tententen8tenten5ten 6 = 2856

T	= 42*68	
	= 42 * (6*B + 8*1)	restacking
	= 42*6*B + 42*8*1	
	= (4*B + 2*1)*6*B + (4*B+ 2*1)*8*1	
	= (4*B*6*B + 2*1*6*B) + (4*B*8*1+ 2*1*8*1)	restacking
	= 24*B^2 + 12*B + 32*B + 16*1	overload!
	= 24*B^2 + (12 + 32)*B + 16*1	restacking
	= 24*B^2 + 44*B + 16*1	overload!
	= (24/B*B)*B^2 + (44/B*B)*B + (16/B*B)*1	recounting
	= (2*B+4)*B^2 + (4*B+4)*B + (1*B+6)*1	
	= 2*B^3 + 4*B^2 + 4*B^2 + 4*B + 1*B + 6*1	restacking
	= 2*B^3 + (4 + 4)*B^2 + (4 + 1)*B + 6*1	restacking
	= 2*B^3 + 8*B^2 + 5*B + 6*1	
	= 2856	

FOIL: First+Outside+Inside+Last

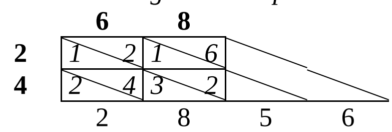
T = 42*68
= (40+2)*(60+8)
= 2400+320+120+16
= 2856

T = 54^2
= (50+4)*(50+4)
= 2500 + 200 + 200 + 16
= 2916

*Multiplying double stacks T = 42 * 68*

	60	**8**
2	*120* *I*	*16* *L*
40	*2400* *F*	*320* *O*

Medieval or diagonal multiplication



Multiplying using carrying

			6	**8**
2		*1*	*3*	*6*
40	*2*	*7*	*2*	*0*
	2	8	5	6

Spreadsheet

	60	**8**	
40	*2400*	*320*	2720
2	*120*	*16*	136
	2520	336	**2856**

Example6. Dividing many-digit numbers the medieval way

T = 2856 = 42*?

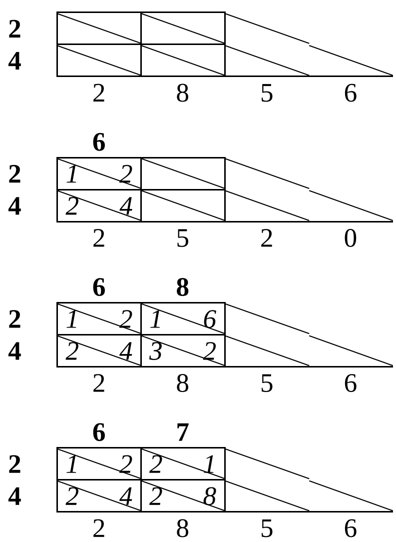
?: 4*5 = 20, 4*6 = 24, 4*7 = 28

!: 4*6 = 24
2*6 = 12

1+4+? = 8, ? = 3
We choose 2 expecting to carry because of 8

?: 4*7 = 20, 4*8 = 32
7 is a possibility
?: 4*7 = 28
!: 4*8 = 32

Answer: **D = 2856 = 42*? = 42*68**



Exercise1. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (verbal calculation)
Exercise2. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (horizontal calculation)
Exercise3. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (vertical calculation)

10 CODED STOCKS ARE MULTIPLIED

Question. How can two double-stacks be multiplied? (a+b)*(c+d) = ?
Answer. By the FOIL-method: (a+b)*(c+d) = a*c (First) + a*d (Outside) + b*c (Inside) + b*d (Last)
Example1. T = (4x+3y)*(6x+7y) = ? and (4x+3y)*? = 24x^2 + 46 xy + 21y^2

<u>Multiplying stocks horizontally</u>

FOIL: First+Outside+Inside+Last

T = (4x+3y)*(6x+7y)
= 4*x*6*x + 4*x*7*y + 3*y*6*x + 3*y*7*y
= 24x^2 + 28xy + 18xy + 21y^2
= 24*x^2 + 46*xy + 21*y^2

<u>Multiplying special stocks</u>
The square of a two-term sum:

T = (a+b)^2
= (a+b)*(a+b)
= (a*a + b*a) + (a*b + b*b)
= a^2 + 2*a*b + b^2
Example:
13^2 = (10+3)(10+3) = 10*10 + 2*3*10 + 3^2 = 169*

The square of a two-term difference:

T = (a-b)^2
= (a-b)*(a-b)
= (a – b)*a - (a-b)*b)
= a*a – b*a –(a*b – b*b)
= a^2 – a*b – a*b + b^2
= a^2 - 2*a*b + b^2
Example: 9^2 = (10-1)*(10-1) = 10*10 - 2*1*10 + 1^2 = 81
Remark. We see that +*- = - ; and -*- = +; and that numbers moving outside a negative parenthesis have their signs changed.

A two-term sum multiplied with their difference

T = (a+b)*(a-b)
= (a+b)*a -(a+b)*b
= (a*a + b*a) - (a*b + b*b)
= a^2 + a*b – a*b - b^2
= a^2 - b^2
Example: 9^2 = (10-1)*(10-1) = 10*10 - 2*1*10 + 1^2 = 81

<u>Multiplying stocks vertically</u>

Method 1 (stacking) T = (4x+3y)(6x+7y)*

	6x	**7y**
3y	*18xy*	*21y^2*
4x	*24x^2*	*28xy*

Method 2: Medieval calculation

	6x	**+ 7y**	
+ 3y	*18xy*	*21y^2*	
4x	*24x^2*	*28xy*	
	24x^2	**+46xy**	**+21y^2**

	a	**b**
b	*ab*	*b^2*
a	*a^2*	*ab*

	a-b	**b**
b	*ab*	*b^2*
a-b	*(a-b)^2*	*ab*

	a	**+b**
b	*ab*	*b^2*
a-b	*(a-b) * (a+b)*	

Exercise1. Do (4x+5y)*(6x+8y). Do (4x+2y)*(9x+7y).
Exercise2. Do (x+2y)^2, (x-2y)^2 and (x+2y)*(x-2y). Do (3x+2y)^2, (3x-2y)^2 and (3x+2y)*(3x-2y).

11 ADDING WITH AN ABACUS

Question. How to add on an 1&4-abacus?
Answer. By recounting both in 2s and 5s.
An abacus is a double-cup calculation board counting e.g. both ones and fives: 8 = 1]3), 37 = 3)1]2)

Horisontal:

			I		I	I		I	II		I	I	
		III	II	+		III	=	III	IIIII	=	IIII		

37 + 58 = 3) 1]2) + 1]0) 1]3) = 1]3) 2]5) = 1]4) 1]0) = 95

Vertical:

37 ->		3	1	2
<u>58 -></u>	<u>1</u>	<u>0</u>	<u>1</u>	<u>3</u>
	1	3	2	5
95 <-	1	4	1	0

Exercise1. Do 28+46. Do 86-58. Etc.
Exercise2. Do 3*46. Do 58/4. Do 24*36. Etc.

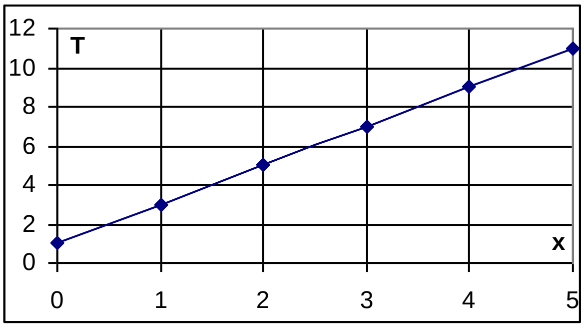
COUNT&ADD IN TIME

Question	Answer
How can counting & adding be reversed?	By calculating backward moving a number to the other side reversing its calculation sign.
Counting ? 3s and adding 2 gives 14.	x*3+2 = 14 is reversed to x = (14-2)/3
Can all calculations be reversed?	Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x =b/a, x^a =b is reversed to x =a√b, a^x =b is reversed to x =logb/loga

1 REVERSED CODING

Question. How can we decode a coded number?
Answer. Use reversed calculations, also called solving equations. **x**
Example. **::: .** **x .**
Coding hides the bundle-size: T=2*3+1 -> T=2*x+1.
A table can be used to guess the Total when coded.
The table can be drawn as a graph.

x	0	1	2	3	4	5			
T = 2*x+1	1	3	5	7	9	11			



A decoding can take place in three steps:
1. First the coding x + 3 = 5 is decoded by restacking: From the 5-stack we take away 3 to a new stack leaving 5-3 = 2 in the original stack as predicted by the restack-equation T = (T-3)+3: T = 5 = (5-3)+3 = 2+3

o o o o o o o o o x o = o -> o o	x + 3 = 5 = (5-3) + 3 = 2+3 x = 2	or quicker:	x + 3 = 5 x = 5-3 = 2

x + 3 = 5 = (5-3) + 3 = 2+3

So the question x+3 = 5 is answered by restacking 5 to (5-3)+3 making x = 5-3. Thus an equation x+b = T is solved by x = T-b to be found by moving the number b across the equation sign and reversing its calculation sign from + to -.
2. Next the coding 2*x = 6 is decoded by recounting: The 6 is recounted to 3 2s and overturned to 2 3s as predicted by the recount-equation T = (T/2)*2: T = 6 = (6/2)*2 = 3*2

oo x oo ooo x = oooooo -> oo oo oo -> oo -> ooo	2*x = 6 = (6/2)*2= 3*2 x = 3	or quicker:	2*x = 6 x = 6/2 = 3

2*x = 6 = (6/2)*2 = 3*2 = 2*3

So the question 2*x = 6 is answered by recounting 6 to (6/2)*2 making x = 6/2. Thus an equation b*x = T is solved by x = T/b to be found by moving the number b across the equation sign and reversing its calculation sign from * to /.
3. Finally the coding 2*x+1 = 7 is decoded. First we restack 7 by taking away 1: 7 = (7-1)+1 = 6+1. Then the 6 is recounted in 2s and overturned.

oo x oo ooo x o = ooooooo -> oooooo o -> oo oo oo o -> oo o -> ooo o	2*x+1 = 7 = (7-1)+1 = 6+1 2*x = 6 = (6/2)*2 = 3*2 x = 3	2*x+1 = 7 2*x = 7-1 = 6 x = 6/2 = 3

2*x+1 = 7 = (7-1)+1 = 6+1 = (6/2)*2 + 1= 3*2 + 1 = 2*3 + 1

Here the result is predicted by applying both the restack-equation and the recount-equation.
Remark. The recount-equation and the restack-equation show directly that equations are solved when moving a number to the other side of the equation sign reversing its calculation sign:

Recounting:	T	= (T/4) * 4	Restacking:	T	= (T-4) + 4
Equation	T	= x * 4	Equation	T	= x + 4
Solution	T/4	= x	Solution	T-4	= x

Exercise1. Decode 2*x+4=10, 3*x+5=17, 4*x+1=9, 5*x+2=17. First use matches, then write.
Exercise2. Decode 2*x-4=6, 3*x-5=7, 4*x-1=11, 5*x-2=18. First use matches, then write.
Exercise3. Decode x*2=6, x*2=7, x*2+1=6, x*2-1=6, x*3+2=16, x*3-4=12. First use matches, then write.

2 REVERSED VERTICAL CALCULATIONS

Question. How can a calculation be reversed vertically?
Answer. Use arrows to illustrate the forward and backward calculation steps (dance the equation).
Example. The equation 3*x + 2 = 14 is a story about two calculations that took place after each other. FIRST the number x was multiplied by 3, THEN 2 was added producing a total of 14.
This sequence can be reversed to produce x: FIRST 2 is subtracted from 14; THEN this is divided by 3. So 3*x + 2 = 14 makes x = (14-2)/3. Finally, to check, the forward calculation can be repeated.

x --*(*3)*--> 3*x --*(+ 2)*--> 3*x + 2 (forward)

4 <--*(/3)*-- 12 <--*(- 2)*-- 14 (backward)

Exercise1. Dance 2*x+4=10, 3*x+5=17, 4*x+1=9, 5*x+2=17. First on the floor, then write.
Exercise2. Dance 2*x-4=6, 3*x-5=7, 4*x-1=11, 5*x-2=18. First on the floor, then write.
Exercise3. Dance x*2=6, x*2=7, x*2+1=6, x*2-1=6, x*3+2=16, x*3-4=12. First on the floor, then write.

3 REVERSED HORIZONTAL CALCULATIONS

Question. How can a calculation be reversed horizontally?
Answer. Use arrows to illustrate upward and downward calculation steps (climb the equation)
Example. The equation 3*x + 2 = 14 is a story about two calculations that took place after each other. FIRST, on the forward side, the calculation is built up to give a total: x is multiplied by 3, and 2 is added giving a total of 14.
THEN, on the backward side, the result is broken down to produce the initial number: 2 is subtracted from 14 and the result is divided by 3. So 3*x + 2 = 14 makes x = (14-2)/3. Finally, to check, the upward calculation can be repeated. If we leave out the arrows the move&change-method becomes visible.

Forward-side		*Backward-side*
3*x + 2	=	14
	+2 ↑↓ -2	
3*x	=	14-2 = 12
	**3 ↑↓ /3*	
x	=	12/3 = 4

Forward	*Backward*
3*x + 2	= 14
3*x	= 14-2 = 12
x	= 12/3 = 4

Forward	*Backward*
a*x + b	= c
a*x	= c-b
x	= (c-b)/a

Exercise1. Climb 2*x+4=10, 3*x+5=17, 4*x+1=9, 5*x+2=17. First from the floor, then write.
Exercise2. Climb 2*x-4=6, 3*x-5=7, 4*x-1=11, 5*x-2=18. First from the floor, then write.
Exercise3. Climb x*2=6, x*2=7, x*2+1=6, x*2-1=6, x*3+2=16, x*3-4=12. First from the floor, then write.

4 CALCULATION TABLES

Question. How can we report solving an equation?
Answer. Use a calculation-table showing both what we know and don't know and the equation to be solved.
Example. In the equation 3*x+2 = 14 the double-calculation 3*x+2 is split up into two calculations by the 'invisible' parenthesis: 3*x+2 = (3*x)+2.

Calculating numbers

x = ?	3*x + 2 = 14
	(3*x) + 2 = 14
	3*x = 14-2
	x = 12/3
	x = 4
Check:	*3*4 + 2 = 14*
	14 = 14 ☺

Calculating letters

x = ?	a*x + b = c
a = 3	(a*x) + b = c
b = 2	a*x = c-b
c = 14	x = $\frac{(c-b)}{a}$
	x = $\frac{(14-2)}{3}$
	x = 4
Check:	*3*4 + 2 = 14*
	14 = 14 ☺

Exercise1. Solve in a calculation-table 2*x+4=10, 3*x+5=17, 4*x+1=9, 5*x+2=17.
Exercise2. Solve in a calculation-table 2*x-4=6, 3*x-5=7, 4*x-1=11, 5*x-2=18.
Exercise3. Solve in a calculation-table x*2=6, x*2=7, x*2+1=6, x*2-1=6, x*3+2=16, x*3-4=12.

5 APPLYING CALCULATION TABLES WITH FORMULAS

Question. Where can we use calculation-tables?
Answer. Calculation-tables can be used with formulas.

Example1. Recounting units

3 $ for 4 pieces: 21$ for ? pieces

pieces = ?	**pieces = (pieces/$)*$**
pieces/$ = 4/3	pieces = 4/3*21
$ = 21	pieces = 28

3 $ for 4 pieces: ?$ for 24 pieces

$ = ?	**$ = ($/pieces)*pieces**
$/pieces = 3/4	$ = ¾*24
pieces = 24	$ = 18

Example2. Percentages I

30 $ = 40%, 21$ = ?%

% = ?	**% = (%/$)*$**
%/$ = 40/30	% = 40/30*21
$ = 21	% = 28

30 $ = 40%, ?$ = 24%

$ = ?	**$ = ($/%)*%**
$/% = 30/40	$ = 30/40*24
% = 24	$ = 18

Example2. Percentages II

25% af 200 $. = ? $.

A = ?	**p = a/T**
p = 25%	p*T = a
T = 200	25%*200 = a
	50 = a

25% af ? $. = 40 $

T = ?	**p = a/T**
p = 25%	p*T = a
a = 40	T = a/p
	T = 40/25%
	T = 160

?% af 200 $. = 40 $

p = ?	**p = a/T**
a = 40	p = 40/200
T = 200	p = 0.20
	p = 20/100
	p = 20%

Example3. Adding percentages

200 + 25% = ?

K = ?	**K = Ko*(1+r)**
Ko = 200	K = 200*(1+0.25)
r = 25%	**K = 250**
= 0.25	

? + 25% = 400

Ko = ?	**K = Ko*(1+r)**
K = 500	K/(1+r) = Ko
r = 25%	500/(1+0.25) = Ko
= 0.25	**400 = Ko**

200 + ?% = 280

r = ?	**K = Ko*(1+r)**
K = 280	K/Ko = 1+r
Ko = 200	(K/Ko)-1 = r
	(280/200)-1 = r
	0.40 = r
	40% = r

Example4. Per-numbers

25kg. à 4 $./kg. = ? $.

T = ?	**k*p = T**
k = 25	25*4 = T
p = 4	100 = T

25kg. à ? $./kg. = 200$.

p = ?	**k*p = T**
k = 25	p = T/k
T = 200	p = 200/25
	p = 8

?kg. à 4 $./kg. = 300$.

k = ?	**k*p = T**
p = 4	p = T/k
T = 300	p = 300/4
	p = 75

Exercise. Repeat the calculations with different numbers. Remember to check.

6 THE LEVER METHOD FOR NEUTRALISING

Question. Are there other ways to solve equations?

Answer. Modern mathematics introduced the lever-method to make solving equations understandable being under the false assumption that the move&change method was only accessible through rote learning.

The equation sign is considered an example of an equivalence relation. And an equation 3*x+2=14 is considered an example of an open statement expressing the equivalence between two numbers 3*x+2 and 14. The equation is solved by determining the statement's truth-set, i.e. the set of numbers that make the open statement a true statement. This is done by neutralising the numbers with their inverse numbers that has to be included on both sides of the equal sign to preserve the equivalence as indicated by the equivalent arrows.

L = {x| 3*x + 2 = 14}	L is the truth-set making the open statement 3*x + 2 = 14 true
3*x + 2 = 14	The open statement
↕ (3*x) + 2 = 14	A hidden parenthesis is added according to priority
↕ ((3*x) + 2)+(-2) = 14+(-2)	To neutralise +2 its inverse under +, –2, is added on both sides
↕ (3*x) + (2+(-2)) = 12	+ parentheses can be removed or added (the associative law)
↕ (3*x) + 0 = 12	+2 and –2 neutralise each others, and +'s neutral number is 0
↕ 3*x = 12	The definition of the neutral number says that a+0 = 0+a = a
↕ (3*x)*(1/3) = 12*(1/3)	To neutralise *3 its inverse under *, 1/3, is multiplied on both side
↕ (x*3)*(1/3) = 4	*numbers (and +numbers) may be commuted (the commutative law)
↕ x*(3*(1/3)) = 4	*parentheses can be removed or added (the associative law)
↕ x*1 = 4	*3 and 1/3 neutralise each others, and *'s neutral number is 1
↕ x = 4	The definition of the neutral number says that a*1 = 1*a = a
L = {x | 3*x + 2 = 14} = {4}	The truth-set L. Because of the arrows the check is not needed.

Exercise. Repeat the lever or neutralising method with a different equation as e.g. 3+2*x = 11.

7 BACKWARD CALCULATION WITH CODED STOCKS I

Question. What about coded stacks?
Answer. Backward calculation can also be used with coded stocks using the reversed FOIL-method.
Example1. T = (4*x+3*y)*? = 24*x^2 + 46*x*y + 21*y^2

<u>Dividing stocks (polynomials)</u>	
(4*x + 3*y) * ? = 24*x^2 + 46*x*y + 21*y^2	**? ?** **+ 3*y** [ ] [ ] **4*x** [ ] [ ] **24*x^2 +46*x*y +21*y^2**
4x * ? = 24*x^2 *<u>? = 6*x</u>* 3y * 6x = 18*x*y	**6*x ?** **+ 3*y** *18*x*y* [ ] **4*x** *24*x^2* [ ] **24*x^2 +46*x*y +21*y^2**
18*x*y + ? = 46*x*y *<u>? = 28*x*y</u>*	**6*x ?** **+ 3*y** *18*x*y* [ ] **4*x** *24*x^2* *28*x*y* **24*x^2 +46*x*y +21*y^2**
4*x * ? = 28*x*y *<u>? = 7*y</u>* 3*y * 7*y = 21*y^2 **So 24*x^2 + 46*x*y + 21*y^2 = (4*x+3*y)*(6*x+7*y)**	**6*x + 7*y** **+ 3*y** *18*x*y* *21*y^2* **4*x** *24*x^2* *28*x*y* **24*x^2 +46*x*y +21*y^2**

Example2. (x+a)(x-a) = x^2-a^2

Forward	*Backward*
(x + 6)(x - 6) = x^2 – 6^2 = x^2 – 36	x^2 – 36 = x^2 – 6^2 = (x + 6)(x - 6)
(x + √20)(x - √20) = x^2 – (√20)^2 = x^2 – 20	x^2 – 20 = x^2 – (√20)^2 = (x + √20)(x - √20)

Example3. (x±a)^2 = x^2+a^2±2*a*x

Forward	*Backward*
(x + 6)^2 = x^2 + 6^2 + 2*6*x = x^2 + 36 + 12x	x^2 + 36 + 12x = x^2 + 6^2 + 2*6*x = (x + 6)^2
(x - 6)^2 = x^2 + 6^2 - 2*6*x = x^2 + 36 - 12x	x^2 + 36 - 12x = x^2 + 6^2 - 2*6*x = (x - 6)^2

Example4. A second degree equation: If a*x^2 + b*x + c = 0 then x = ?

Forward	*Backward*	*Summary*
T = (x + k)^2 T = x^2 + k^2 + 2*k*x 0 = x^2 + (2*k)*x + (k^2 – T) 0 = x^2 + p*x + q p = 2*k og q = k^2 – T p/2 = k og T = k^2 – q = (p/2)^2 – q	(x + k)^2 = T x + k = ± √T x = -k ± √T ------- If x^2 + p*x + q = 0 then **x = - p/2 ± √((p/2)^2 – q)**	a*x^2 + b*x + c = 0 x^2 + (b/a)*x + (c/a) = 0 x^2 + p*x + q = 0 p = b/a og q = c/a x = - b/(2a) ± √((b/(2a))^2 – c/a) **x = (-b ± √D)/(2*a) , D = b^2 – 4*a*c**

Exercise1. Do (8x^2 + 22 xy + 15y^2):(2x+3y). Do (6x^2 + 26 xy + 24y^2):(3x+4y)
Exercise2. Factorise x^2-25, 3x^2-48, x^2+6x+8, 2x^2-4x-30. Solve x^2-8x+12=0, 4x^2-24x+20

8 A SUMMARY

Question. Is there a common principle when solving equations?
Answer. Move a number to the other side of the equation sign and change its calculation sign.

	5+3 = 5+1+1+1	5*3 = 5+5+5	5^3 = 5*5*5	
Forward	5 + 3 = ?	5 * 3 = ?	5^3 = ?	5^3 = ?
Backward	? + 3 = 8 *? = 8-3*	? * 3 = 15 *? = 15/3*	?^3 = 64 *? = 3√64*	5^? = 64 *? = log64/log5*
Definitions	8-3 is the +number that together with 3 gives 8	15/3 is the *number that together with 3 gives 15	3√64 is the base that together with the exponent 3 gives 64	Log64/log5 is the exponent that together with the base 5 gives 25
Move&change	x + 3 = 8 x = 8-3	x * 3 = 15 x = 15/3	x^3 = 64 x = 3√64	x^2 = 25 x = √25
Move-rules	+numbers <--> -numbers	*numbers <--> /numbers	^3 <--> 3√	^2 <--> √

Exercise1. Solve the following equations using both the move-method and the neutralising-method.

1	T = a+(1+b)*c	T = a+(1-b)*c	T = a+b*(1+c)	T = a+b*(1-c)
2	T = a-(1+b)*c	T = a-(1-b)*c	T = a-b*(1+c)	T = a-b*(1-c)
3	T = a+(1+b)/c	T = a+(1-b)/c	T = a+b/(1+c)	T = a+b/(1-c)
4	T = a-(1+b)/c	T = a-(1-b)/c	T = a-b/(1+c)	T = a-b/(1-c)

Exercise2. Solve the following equations using both the move-method and the neutralising-method.

1	T = a+b*c^d	T = a+(1+b)*c^d	T = a+(1-b)*c^d	T = a+b*(1-c)^d
2	T = a-b*c^d	T = a-(1+b)*c^d	T = a-(1-b)*c^d	T = a-b*(1-c)^d
3	T = a+b/c^d	T = a+(1+b)/c^d	T = a+(1-b)/c^d	T = a+b/(1-c)^d
4	T = a-b/c^d	T = a-(1+b)/c^d	T = a-(1-b)/c^d	T = a-b/(1-c)^d

Exercise3. Solve the following equations.

	*Find **a, b** and **c*** *Answer:*		a	b	c
1	$T = a + b\cdot c$		$T - b\cdot c$	$\frac{T-a}{c}$	$\frac{T-a}{b}$
2	$T = a - b\cdot c$		$T + b\cdot c$	$\frac{a-T}{c}$	$\frac{a-T}{b}$
3	$T = a + \frac{b}{c}$		$T - \frac{b}{c}$	$(T-a)\cdot c$	$\frac{b}{T-a}$
4	$T = a - \frac{b}{c}$		$T + \frac{b}{c}$	$(a-T)\cdot c$	$\frac{b}{a-T}$
5	$T = (a + b)\cdot c$		$\frac{T}{c} - b$	$\frac{T}{c} - a$	$\frac{T}{a+b}$
6	$T = (a - b)\cdot c$		$\frac{T}{c} + b$	$a - \frac{T}{c}$	$\frac{T}{a-b}$
7	$T = \frac{a+b}{c}$		$T\cdot c - b$	$T\cdot c - a$	$\frac{a+b}{T}$
8	$T = \frac{a-b}{c}$		$T\cdot c + b$	$a - T\cdot c$	$\frac{a-b}{T}$
9	$T = \frac{a}{b+c}$		$T\cdot (b+c)$	$\frac{a}{T} - c$	$\frac{a}{T} - b$
10	$T = \frac{a}{b-c}$		$T\cdot (b-c)$	$\frac{a}{T} + c$	$b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$		$(T-c)\cdot b$	$\frac{a}{T-c}$	$T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$		$(T+c)\cdot b$	$\frac{a}{T+c}$	$\frac{a}{b} - T$
13	$T = a\cdot b^{c}$		$\frac{T}{b^{c}}$	$\sqrt[c]{\frac{T}{a}}$	$\frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^{c}}$		$T\cdot b^{c}$	$\sqrt[c]{\frac{a}{T}}$	$\frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a\cdot b)^{c}$		$\frac{\sqrt[c]{T}}{b}$	$\frac{\sqrt[c]{T}}{a}$	$\frac{\log T}{\log(a\cdot b)}$
16	$T = (\frac{a}{b})^{c}$		$\sqrt[c]{T}\cdot b$	$\frac{a}{\sqrt[c]{T}}$	$\frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^{c}$		$\sqrt[c]{T} - b$	$\sqrt[c]{T} - a$	$\frac{\log T}{\log(a+b)}$
18	$T = (a - b)^{c}$		$\sqrt[c]{T} + b$	$a - \sqrt[c]{T}$	$\frac{\log T}{\log(a-b)}$
19	$T = a + b^{c}$		$T - b^{c}$	$\sqrt[c]{T-a}$	$\frac{\log(T-a)}{\log b}$
20	$T = a - b^{c}$		$T + b^{c}$	$\sqrt[c]{a-T}$	$\frac{\log(a-T)}{\log b}$
21	$T = a^{(b+c)}$		$\sqrt[(b+c)]{T}$	$\frac{\log T}{\log a} - c$	$\frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$		$\sqrt[(b-c)]{T}$	$\frac{\log T}{\log a} + c$	$b - \frac{\log T}{\log a}$

COUNT&ADD IN SPACE

S1

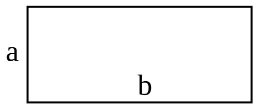
Question	**Answer**
How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 recount-equations: sinA = a/c, cosA = b/c, tanA = a/b. By adding squares by the 3 Pythagoras', mini, midi & maxi.

1 COUNTING A LENGTH

Question. How can we count a one-way extension, i.e. length?
Answer. By a ruler constructed by parallel lines dividing the distances between two points in equal lengths.
Example. In a 4*8 stack on quadratic paper connect the horizontal distances 2, 4, 6 and 8 with the vertical distances 1, 2, 3 and 4. The lines are called parallel.
Exercise1. Use parallel lines to divide 10cm in 2 equal lengths. Check by measuring.
Exercise2. Use parallel lines to divide 10cm in 5 equal lengths. Check by measuring.
Exercise3. Use parallel lines to divide 12cm in 6 equal lengths. Check by measuring.
Exercise4. Use parallel lines to divide 15cm in 10 equal lengths. Check by measuring.

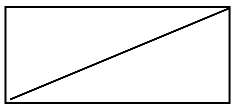
2 COUNTING A SURFACE

Question. How can we count a two-way extension, i.e. area?
Answer1. By dividing the surface in squares. Or by dividing the surface in triangles.



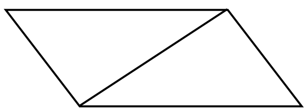
An a*b stack is called a rectangle; or a square if a = b.

An a*b stack can be recounted as (a*b) 1s, thus having the surface-number or area T = a*b = altitude*base.



A rectangle is divided in two equal right triangles by its diagonal. Thus the area of a right triangle is T = ½*a*b = ½*altitude*base.

By moving the two outside triangles in an skew stack (a parallelogram) it is transformed into an a*b stack with the same area, thus having the area T = altitude*base.

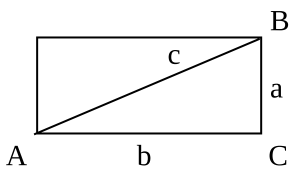


A triangle is half of a parallelogram thus having the area T = ½ * altitude*base.

Exercise1. Cut out two similar triangles and find their area by transforming them into a stack.
Exercise2. Draw a triangle and find its area by transforming it into a stack.

3 COUNTING AN ANGLE

Question. How can we count the size of an angle?
Answer1. By counting with a protractor, or by recounting one side by another.



In an a*b stack a is turned ¼ round or 90 degrees from b.
A full round is 360 (days) = (360/4)*4 = 90*4.
An acute angle is less than 90 degrees and an obtuse angle is greater than 90 degrees.
In an a*b stack the diagonal is called c. The corners or vertices are called angles A, B and C.

Recount a in bs: a = a/b*b = tanA*b
Recount a in cs: a = a/c*c = sinA*c
Recount b in cs: a = b/c*c = cosA*c

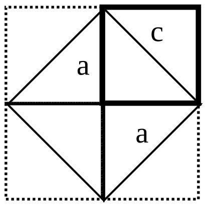
Exercise1. Draw a diagonal in a stack. Measure the angles A and B by a protractor. Predict the result by backward calculation.

tan A = a/b	(A = (tan-1)(a/b))	tan B = b/a	(B = (tan-1)(b/a)).
sin A = a/c	(A = (sin-1)(a/c))	sin B = b/c	(B = (sin-1)(b/c))
cos A = b/c	(A = (cos-1)(b/c))	cos B = a/c	(B = (cos-1)(a/c))

Exercise2. Test by calculating that B+A = 90 and tan A = (sin A)/(cos A) and (sin A)^2 + (cos A)^2 = 1

4 ADDING SQUARES

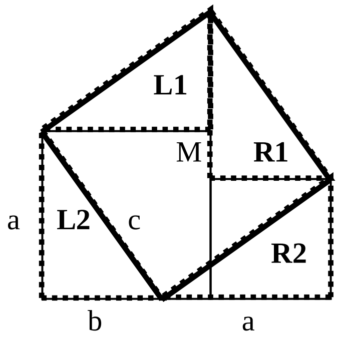
Question. How can we add squares?
Answer1. By using one of the 3 Pythagorean theorems.
Examples.



Mini-Pythagoras:

c is the diagonal in an a*a stack. Turning the stack around the diagonal's endpoints produces a c*c stack. By rearranging the triangles inside and outside the c*c stack we see that

a^2 + a^2 = c^2

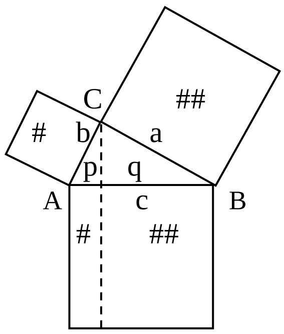


Midi-Pythagoras:

c is the diagonal cutting a paper sheet (an a*b stack) in two, L & R. Turning L and R from position 1 to 2 changes the area from c^2 to a^2 + b^2. So

a^2 + b^2 = c^2 (the Pythagoras' theorem)

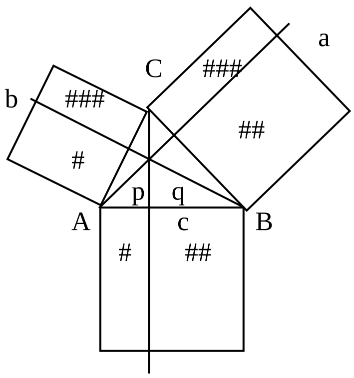
Remark. c can also be calculated as c = a /(sin(tan-1(a/b)))



Calculating I:
c^2 = M + 4*½*a*b where M = a-b
= (a-b)^2 + 2*a*b
= a^2 + b^2 - 2*a*b + 2*a*b
= a^2 + b^2

Calculating II:
b = c*cosA and p = b*cosA, so # = c*p = (b*cosA)*(b/cosA) = b^2
a = c*cosB and q = a*cosB, so ## = c*q = (a*cosA)*(a/cosA) = a^2

c^2 = q*c + p*c = a^2 + b^2



Maxi-Pythagoras:

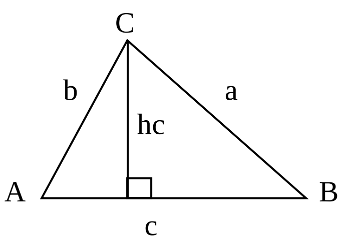
In an acute triangle the altitudes divide the outside squares in pieces corresponding pair wise:

= c*p = c*b*cosA = b*c*cosA, ## = a*c*cosB, ### = a*b*cosC

c^2 = ## + # = (a^2 - ###) + (b^2 - ###) = a^2 + b^2 – 2*###

Or, expressed as the **Cosine-relations**:

c^2 = a^2 + b^2 – 2*a*b*cosC
b^2 = a^2 + c^2 – 2*a*c*cosB
a^2 = b^2 + c^2 – 2*b*c*cosA



The **Sine-relations** come from the altitudes:

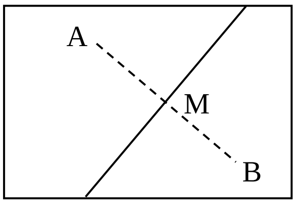
hc: a*sinB = b*sinA, so a/sinA = b/sinB
hb: a*sinC = c*sinA, so a/sinA = c/sinC
ha: b*sinC = c*sinB, so b/sinB = c/sinC

Thus $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

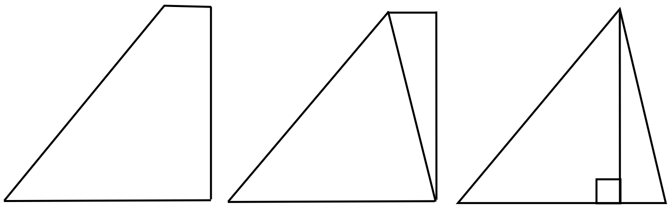
Exercise1. Predict the length of the diagonal in a stack and test by measuring.
Exercise2. Predict the length of the three unknown sides or angles in a triangle and test by measuring. Try all kinds of triangles, SideSideSide, SideSideAngel, SideAngelSide, SideAngelAngel
Exercise3. Do the sine- and cosine-relations also hold for obtuse triangles?

5 DIVIDING THE LAND

Question. How can we divide a piece of land between two persons?
Answer. By using perpendicular bisectors.



A piece of land is to be divided between two settlements A and B. They decide upon the dividing-principle 'equal distance to a border-point'. This makes the border-points form a perpendicular bisector, i.e. a straight line perpendicular to the midpoint M of the connecting line AB.



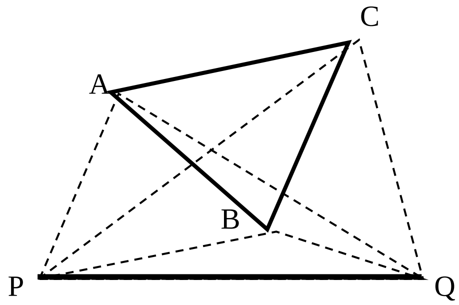
To measure it a piece of land can be divided into triangles (triangulation). And a triangle can be divided into two right triangles. Thus we are especially interested in right triangles.

Geo-metry means 'earth-measuring' in Greek.

Exercise1. Draw a piece of land with two settlements. Divide it so AM/AB = 1/1. Divide it so AM/AB = 1/2.
Exercise2. Draw a piece of land with three settlements. Divide it so there is equal distance to a border-point.

6 REESTABLISHING THE LAND

Question. How can we re-establish a piece of land where the fences disappeared under a flooding?
Answer. By using a baseline to measure distances.



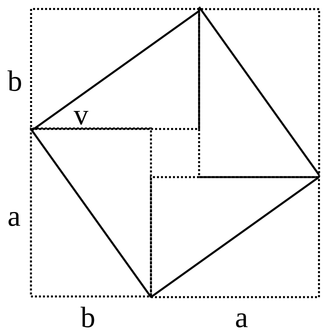
A piece of land is divided into triangles. From the endpoints P and Q on a permanent baseline the distances are measured to A B and C (triangular coordinates).

After the flood of the Nile the point A is re-established by laying out the triangle PAQ. Or by calculating the angle PAQ. Or as the intersection point between circles from P and Q having the distances as the radius.

Exercise1. Draw a triangle on the floor or in the sand. Remove and re-establish the triangle from a base line.

7 THE SLOPES OF STRAIGHT LINES

Question. How can we count the steepness of a straight line?
Answer1. By the slope counting the rise over the run.
Example.



The four diagonals are parallel pair wise.

The slope is the rise recounted in runs.

The upward diagonals have the slope b/a.

The downward diagonals have the slope -a/b.

The slopes of two perpendicular lines are reciprocal with different signs:

Linie l ⊥ linie m: slope for l * slope for m = -1

The steepness-angle v is determined by tan v = slope

Exercise1. What is the slope of a 20-degree line? What is the angle of a 10% slope?

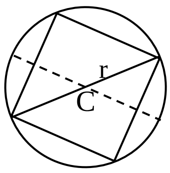
8 DIMENSIONS

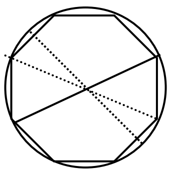
Question. What is the difference between the line, the plane and the space?
Answer. The line has one direction, the plane has two and space has three directions or dimensions.
Examples.
A line segment is part of a line having 1 dimension. It's points are identified by 1 number, the out-number x.

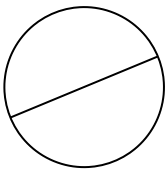
A stack is part of a plane having 2 dimensions. It's points are identified by 2 numbers, the out-number x and the up-number y.
A box is part of a space having 3 dimensions. It's points are identified by 3 numbers, the out-number x and the up-number y, and the back-number z.
Exercise1. Take a random walk at a line by throwing a dice where odd means – and even means +.
Exercise2. Take a random walk at a plane by throwing 2 dices where odd means – and even means +.
Exercise3. Take a random walk in a space by throwing 3 dices where odd means – and even means +.

9 CIRCUMFERENCE AND AREA OF A CIRCLE

Question. How can we count the length and the area of a circle?
Answer1. By approximating the circle with a regular polygon to develop formulas.



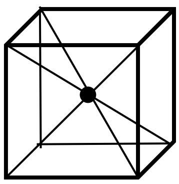


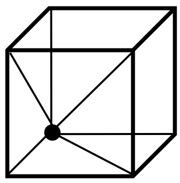


A circle has a centre C, a width or diameter d and a radius r = d/2.
A square is wrapped in a circle with radius 1.
The length of the side (the cord) is k4 = 2* sin(180/4).
The circumference of the square is C4 = 4 * 2 sin(180/4).
The area of the square is A4 = 8*½*alt.*base = 4*sin(180/4)*cos(180/4).
Pulling the midpoints to the circle transforms the square from a regular 4sided polygon to a regular 8sided polygon.
The length of the side (the cord) is k8 = 2* sin(180/8).
The circumference of the polygon is C8 = 8 * 2 * sin(180/8).
The area of the polygon is A8 = 8 * sin (180/8) * cos(180/8).
Continuing to pull the midpoints we get approximate formulas:
Circle circumference ≈ Cn = n * 2 * sin(180/n).
Circle area ≈ An = n * sin (180/n) * cos(180/n).
For n = 1000 we get:
Circle circumference = C ≈ 1000 * 2 * sin(180/1000) = 6,28317 ≈ 2*π
Circle area ≈ 1000*sin (180/1000)*cos(180/1000) = 3,14157 ≈ π
We see that the circumference and the area of a circle can be described by a special number π = 3,141592654; and that this number can be calculated as a limit-number
n*sin(180/n) -> π for n -> ∞, or $\lim_{n \to \infty} (n*\sin(180/n)) = \pi$.
With the radius r the formulas become C = 2*π*r and A = π*r^2
Exercise. Predict the circumference of a bottle and test by measuring.

10 SHAPES IN SPACE

Question. How can spatial shapes be counted?
Answer. A shape has a 1dimensional extension, a 2dimensional area and a 3dimensional volume.
Example1. Bundling 5 1s gives 1 5-bundle having the length 5 cm. Stacking 4 5s gives a 4*5 stack having the breadth √(4^2+5^2) and having the area 4*5=20 cm^2. Stacking 3 4*5 stacks gives a 3*4*5 box having the breath 3√(3^2+4^2+5^2) and having the surface area 2*(3*4+3*5+4*5) = 94 cm^2; and having the volume 3*4*5 = 60 cm^3.
Example2. From the centre point an a*a box can be divided into 6 pyramids each having the volume V = 1/6*a*a*a = 1/3*(1/2a)*(a*a) = 1/3*height*base-area. If the centre point is drawn to a corner, 3 pyramids disappear giving each of the 3 remaining pyramids the volume V = 1/3*a*a*a = 1/3*height*base-area





Example3. A circle has two 1dimensional extensions, its diameter d = 2*r, and its circumference C = 2*π*r. A circle has one 2dimensional extension, its area A = π*r^2. A circular-disk is called a cylinder having two 2dimensional extensions and one 3dimensional extension, its volume V = π*r^2*altitude. Constricted in the one end a cylinder becomes a cone having the volume V = 1/3*π*r^2*altitude. Constricted in both ends a cylinder becomes a ball having the volume V = 4/3*π*r^3*and the surface S = 4*π*r^2.
Exercise1. The volume of a liquid can be measured in a measuring glass. Pour water or sand from different shapes into a measuring glass to measure the volume. Predict the result from a formula.

C2

COUNTING EVENTS

Questions	Answers
How can we count possibilities?	By using the numbers in Pascal's triangle
How can we predict unpredictable numbers?	We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)

1 COUNTING UNSYSTEMATIC EVENTS

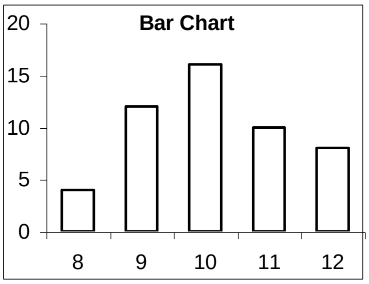
Question1. How to count events occurring randomly?
Answer. By working out statistics for the events, their frequencies and their average.
Exemple1. To the question 'How old are you?' the answers are arranged in a table showing the frequencies of the answers. From the distribution the average answer and the average deviation is calculated. From this 'post-diction' a pre-diction can be made saying that the following answer, with 95% probability, will fall in the 'confidence interval' I = average answer ± 2*average deviation.

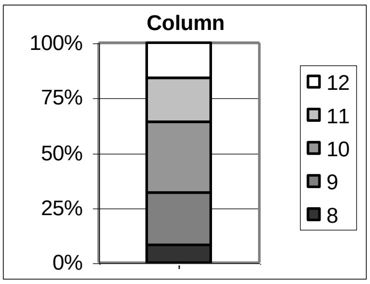
Observations x	Frequency f	Relative frequency p	Cumulative frequency	Total years lived	Deviation from the mean	Average deviation
8	4	4/50 = 0,08 = 8%	8%	8*4 = 32	10.1 - 8 = 2,1	2.1^2*4 = 17.64
9	12	12/50 = 0,24 = 24%	32%	9*12 = 108	10.1 - 9 = 1.1	1.1^2*12 = 14.52
10	16	16/50 = 0,32 = 32%	64%	10*16 = 160	10.1 - 10 = 0.1	0.1^2*16 = 0.16
11	10	10/50 = 0,20 = 20%	84%	11*10 = 110	11 - 10.1 = 0.9	0.9^2*10 = 8.1
12	8	8/50 = 0,16 = 16%	100%	12*8 = 96	12 - 10.1 = 1.9	1.9^2*8 = 28.88
Total	50	1,00 = 100%	-	506	-	69.3
Average, mean:				506/50 = 10.1		(√69.3)/50 = 1.2

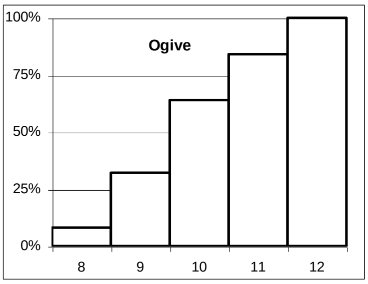
The frequency-table can be illustrated graphically both in case of grouped and non-grouped observations:

OBS	FRQ
8	4
9	12
10	16
11	10
12	8

DoubleClick &Edit



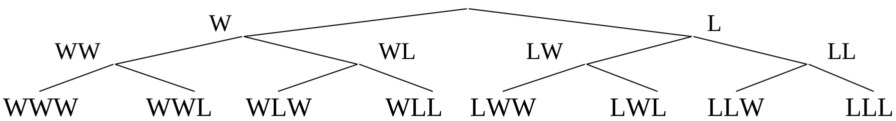




Exercise. Observe the sum of 2 dices 10 times. Set up a frequency table and calculate the confidence interval. Repeat the exercise 20 times, and 30 times. Repeat the exercise with the theoretical frequencies p(x=2)= 1/36, p(x=3) = 2/36, p(x=4) = 3/36, … , p(x=12) = 1/36. x is called a random variable or unpredictable variable.

2 COUNTING SYSTEMATIC EVENTS

Question1. How to count events (pools) in 3 repetitions of a 2-option Win/Loose-game?
Answer. By repeated multiplying.



game	pools	
1	2:	W, L
2	2*2:	WW, WL, LW, LL
3	2*2*2:	WWW,WWL,…

Key to 3 2-option games: 2 options for game 1; each having 2 options for game 2 totalling 2*2 events; each having 2 options for game 3 totalling (2*2)*2 = 2^3 events.
Prediction. There are r^n different events in n repetitions of an r-option game.
Exercise. Check the prediction with 4 2-option games, 4 3-option games, 4 4-option games, etc.
Question2. How to count letter plates?
Answer. By counting ordered arrangements, permutations.

Letters:	1 (A)	2 (A,B)	3 (A,B,C)	4 (A,B,C,D)	5 (A,B,C,D,E)
Plates:	1	2	6	?	?
	A	AB	ABC		
		BA	ACB		
			BAC		
			BCA		
			CAB		
			CBA		

Key to 3-letter plates: 3 options for letter 1, 2 for letter 2, 1 for letter 3 totalling 3*2*1 = 3! plates.
Prediction. There are n! different n-letter plates (n! is read n factorial).

Exercise. Check the prediction with 4-letter plates and 5-letter plates, etc.

Question3. How to count restricted letter plates only consisting of 2 letters of 5.
Answer. By counting restricted permutations.

Letters:	1of2 (A,B)	2of3 (A,B,C)	2of4 (A,B,C,D)	2of5
Plates:	2	6	?	?
	A	AB		
	B	AC		
		BC		
		BA		
		CA		
		CB		

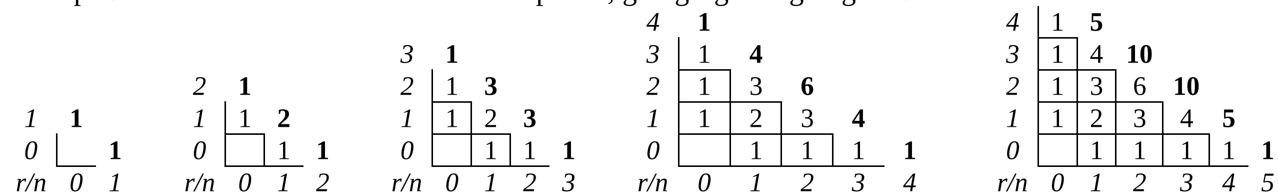
Key to 2of5 letter plates: 5 options for letter 1, 4options for letter 2 totalling 5*4=5*4*(3!/3!)=5!/3! Plates.
Prediction. There are n!/(n-r)! different n-of-r letter plates.
Exercise. Check the prediction with 2of4 and 3of5letter plates.

Question4. How to count lotto tickets in a lotto game guessing 2of5 letters.
Answer. By using the binomial numbers B(n,r) to count unordered arrangements, combinations.

Letters:	1of2 (A,B)	2of3 (A,B,C)	2of4 (A,B,C,D)	2of5
Plates:	2	6/2	12/2	?
	A	AB	AB	
	B	AC	AC	
		BC	AD	
		~~BA~~	BC	
		~~CA~~	BD	
		~~CB~~	CD	

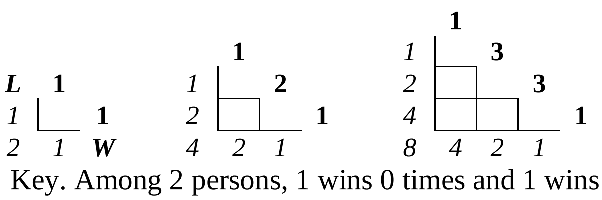
Key to 2of5 lotto tickets: 5 options for letter 1, 4options for letter 2 totalling 5*4=5*4*(3!/3!)=5!/3! tickets giving 2*the total number of tickets. Thus B(5,2) = 5!/3!/2! = 5!/(3!*2!) = 5!/[(5-2)!*2!]
Prediction. There are B(n,r) = n!/[(n-r)!*r!] different n-of-r lotto tickets.
Exercise. Check the prediction with 2of4 and 3of5letter lotto tickets.

Question5. Is there a pattern in the binomial numbers B(n,r)?
Answer. The binomial numbers B(n,r) form a triangle (Pascall's triangle).
Example. In a maze of blocks there are two options, going right & going left.



Prediction. B(n,0) = B(0,n) = n. B(n,r) = B(n-r,r). B(n,r) = B(n,r-1)+B(n-1,r). B(0,0) is defined as 1.
Exercise. Check the prediction with the numbers B(6,r) and B(8,r).

Question6. How to count the winners in 4 2-option Win/Loose-games?
Answer. By the binomial numbers B(n,r).



Key. Among 2 persons, 1 wins 0 times and 1 wins 1 time (1,1 persons win 0,1 times) in 1 2-option game.
Among 4 persons 1,2,1 persons win 0,1,2 times in 2 2-option games.
Among 8 persons 1,3,3,1 persons win 0,1,2,3 times in 3 2-option games.
Prediction1. Among 16 persons 1,4,6,4,1 persons win 0,1,2,3 times in 4 2-option games.
Or among 2^4 persons B(4,0), B(4,1), B(4,2), B(4,3), B(4,4) persons win 0,1,2,3 times in 4 2-option games.
Exercise. Check the binomial prediction with 4 2-option games.
Prediction2. Among 2^n persons B(n,r) persons win r times in n 2-option games.
Exercise. Check the binomial prediction with 6, 7 and 8 2-option games.

Question7. How to count the winners in 4 3-option games (1 Win and 2 Loose)?

Answer. By weighted binomial numbers.

L	**2**				**4**					**8**		
2		**1**		*4*		**4**			*8*		**12**	
3	*1*	***W***		*6*			**1**		*12*			**6**
				9	*3*	*1*			*18*			
									27	*9*	*3*	*1*

(last column of the third diagram: **1** beside row *18*)

Persons in 3 3-option games

1 * 2/3*2/3*2/3 * 27	B(3,3)*(1/3)^0*(2/3)^3 * 27
3 * 1/3*2/3*2/3 * 27	B(3,2)*(1/3)^1*(2/3)^2 * 27
3 * 1/3*1/3*2/3 * 27	B(3,1)*(1/3)^2*(2/3)^1 * 27
1 * 1/3*1/3*1/3 * 27	B(3,0)*(1/3)^3*(2/3)^0 * 27

Key. Among 3 persons 2 win 0 times and 1 wins 1 time (2,1 persons win 0,1 times) in 1 3-option game.
Among 9 persons 4,4,1 persons win 0,1,2 times in 2 3-option games.
Among 27 persons 8,12,6,1 persons win 0,1,2,3 times in 3 3-option games.
Prediction1. Among 81 persons 16,32,24,8,1 persons win 0,1,2,3,4 times in 4 3-option games.
Or Among 3^4 persons B(4,0)*16, B(4,1)*8, B(4,2)*4, B(4,3)*2, B(4,4)*1 persons win 0,1,2,3,4 times.
Or Among 3^4 persons B(4,0)*2^4, B(4,1)*2^3, B(4,2)*2^2, B(4,3)*2^1, B(4,4)*2^0 persons win 0,1,2,3,4 times.
Exercise1. Check the binomial prediction with 4 3-option games.
Prediction2. Among 3^n persons B(n,r)*1^r*2^(n-r) persons win r times in n 3-option 1W/2L-games
Exercise2. Check the binomial prediction with 5, 6 and 7 3-option games, etc.
Prediction3. Among 4^n persons B(n,r)*1^r*3^(n-r) persons win r times in n 4-option 1W/3L-games
Exercise3. Check the binomial prediction with 2, 3 and 4 4-option games, etc.
Prediction4. Among k^n persons B(n,r)*1^r*(k-1)^(n-r) persons win r times in n k-option 1W/(k-1)L-games.
Exercise4. Check the binomial prediction with 4 5-option games and 4 6-option games.
Prediction5. Among k^n persons B(n,r)*q^r*(k-q)^(n-r) persons win r times in n k-option qW/(k-q)L-games.
Exercise5. Check the binomial prediction with 3 5-option games and a 4 6-option games both with 2 wins.

Question8. How to predict the chance for winning?
Answer. By using probabilities.
Example1. A dice is manipulated changing the 6 and 5 to 1 and the 4 to 2. Thus out of 6 outcomes 1 occurs 3 times (3 = 3/6*6), 2 occurs 2 times (2 = 2/6*6) and 3 occurs 1 time (1 = 1/6*6). The number 3/6 is called the chance or probability for the event 1: p(x=1) = 3/6. Likewise p(x=2) = 2/6, p(x=3) = 1/6.
Example2. In a 3-option game the chance to win is p = 1/3 if the outcomes are equally likely. Thus the probabilities can be found from the binomial numbers:

	8			
8		**12**		
12			**6**	
18				**1**
27	*9*	*3*	*1*	

Persons		*Probabilities*
1 * 2/3*2/3*2/3 * 27	B(3,3)*(1/3)^0*(2/3)^3 * 27	p(x=0) = B(3,3)*(1/3)^0*(2/3)^3
3 * 1/3*2/3*2/3 * 27	B(3,2)*(1/3)^1*(2/3)^2 * 27	p(x=1) = B(3,2)*(1/3)^1*(2/3)^2
3 * 1/3*1/3*2/3 * 27	B(3,1)*(1/3)^2*(2/3)^1 * 27	p(x=2) = B(3,1)*(1/3)^2*(2/3)^1
1 * 1/3*1/3*1/3 * 27	B(3,0)*(1/3)^3*(2/3)^0 * 27	p(x=3) = B(3,0)*(1/3)^3*(2/3)^0

The binomial probabilities p(x = r) = B(n,r)* p^r*(1-p)^(n-r), where x counts the number of wins, are cumulated before being tabulated, e.g. in case of 5 3-option games with a winning chance p = 1/3:

x	p	∑p	P(x >= 3) at least 3W	P(x <= 4) at most 4W	P(x = 2) Precisely 2W	P(1 <= x <= 3) Between 1W & 3W
0	0.132	0.132				-0.132
1	0.329	0.461			-0.461	
2	0.329	0.790	-0.790		+0.790	
3	0.165	0.955				+0.955
4	0.041	0.996		+0.996		
5	0.004	1.000	+1.000			
			0.210	**0.996**	**0.329**	**0.832**

Thus there is a 21% chance for winning at least 3 times of 5, etc.
Example2. The average wins in a n=5 game depends on the winning chance (here p=1/3, p=1/4 and p=1/2):

x	p	x*p		x	p	x*p		x	p	x*p
0	0.132	0		0	0.237	0		0	0.031	0
1	0.329	0.329		1	0.396	0.396		1	0.156	0.156
2	0.329	0.658		2	0.264	0.527		2	0.313	0.625
3	0.165	0.495		3	0.088	0.264		3	0.313	0.938
4	0.041	0.164		4	0.015	0.059		4	0.156	0.625
5	0.004	0.020		5	0.001	0.005		5	0.031	0.156
		m=1.666				**m=1.25**				**m=2.5**

Prediction. The average wins in n games with the winning chance p is m = n*p
Exercise. Check the prediction with n=3 and n=4 games with the winning chance p=1/3, p=3/4, p=½, p=2/5.
Example3. The average deviation from the average depends on the winning chance (here p=1/4 and p=1/2):

x	p	x*p	c = x-m	c^2*p		x	p	x*p	c = x-m	c^2*p
0	0.237	0	-1.25	0.371		0	0.031	0	-2.5	0.195
1	0.396	0.396	-0.25	0.025		1	0.156	0.156	-1.5	0.352
2	0.264	0.527	0.75	0.148		2	0.313	0.625	-0.5	0.078
3	0.088	0.264	1.75	0.269		3	0.313	0.938	0.5	0.078
4	0.015	0.059	2.75	0.111		4	0.156	0.625	1.5	0.352
5	0.001	0.005	3.75	0.014		5	0.031	0.156	2.5	0.195
	*n*p =*	**m=1.25**		v = 0.938			*n*p =*	**m=2.5**		v = 1.25
			√*n*$\underline{p}$* =	√v= 0.968					√*n*$\underline{p}$* =	√v= 1.118

Prediction. The average deviation D in n games with the winning chance p is D = √(n*p*(1-p)) = √n*$\underline{p}$, where the average of p and 1-p is $\underline{p}$ = √(p*(1-p))

Exercise. Check the prediction with n=3 and n=4 games with the winning chance p=1/3, p=3/4, p=½, p=2/5.

Remark. Dividing the binomial numbers B(n,r)*q^r*(k-q)^(n-r) with k^n gives the binomial probabilities:

$$\frac{B(7,r) * 1^r * 3^(7-r)}{4^7} = B(7,r) * \frac{1^r * 3^(7-r)}{4^r * 4^(7-r)} = B(7,r) * \frac{1^r}{4^r} * \frac{3^(7-r)}{4^(7-r)} = B(7,r) * \frac{1}{4}^r * \frac{3}{4}^(7-r) = B(7,r) * p^r * (1-p)^(7-r)$$

Question9. How to predict the unpredictable when events are systematic?

Answer. The number of wins can be predicted as x' = x average ± 2*average deviation = n*p ± 2*√n*$\underline{p}$.
And the percentage of wins can be predicted as p' = x'/n = (n*p ± 2*√n*$\underline{p}$)/n = p ± 2*$\underline{p}$/√n.

Example1. In a n=7 game with the winning chance p=2/5 the numbers of wins is predicted to be 7*2/5 ± 2*√7*√(2/5*3/5) = 2.8 ± 2.6 = [0.2;5.4]. So winning 0, 6 or 7 times is less than 5% probable.

Exercise. What is the prediction for the sum when throwing 2 dices? And when throwing 3 dices? Check it.

Question10. How does the mean of a sample vary?

Answer. From a population with a mean m and a deviation D the experiment 'examine a member' is repeated n times. The mean of this sample is an unpredictable event having m as mean and D/√n as deviation. Its distribution is called a normal distribution which is tabulated as cumulated frequencies, e.g. Φ(1.220) = 0.889.

Example. Throwing a dice has 6 events with the mean 3,5 and the deviation 1,71. A sample is produced by throwing a dice 10 times (or throwing 10 dices once). The sample-mean is an unpredictable variable with the mean 3,5 and the deviation 1,71/√10.

Exercise. Check the prediction by 12 times throwing ten dices each time observing the mean.

Question11. Are there any short cuts to the binomial distribution?

Answer. A binomial distribution can be approximated by a normal distribution if n*p>5 and n*(1-p)>5.
If n=20 and p=0.3 then m=n*p=20*0.3=6 (and n*(1-p)=20*0.7=14) and D =√n*$\underline{p}$=√20*√(0.3*0.7)=2.05

Factual number	Approximated number
P(x<=8) = 0.887	P(x<=8) = Φ((8+0.5-6)/2.05) = Φ(1.220) = 0.889

Exercise. Check several examples of the normal approximation prediction of binomial probabilities.

Question12. How to win in a 2person game?

Answer. By mixing the strategies unpredictably.

Example. The players A and B can choose between two strategies, Paper and Stone. The gain paid from B to A depends on the combination (A,B): g(P,P) = -1, g(S,S) = -1, g(P,S) = 1, g(S,P) = 2. A hopes for the gain 2 and chooses S; B sees this and chooses S; A sees this and chooses P; B sees this and chooses P, etc.

A broker proposes the following solution: both A and B choose a mixed strategy P/S = x/(1-x) and P/S = y/(1-y). The gain then is g = -1*x*y + 1*x*(1-y) + 2*(1-x)*y - 1*(1-x)*(1-y) = -1 - 5*x*y + 2*x + 3*y.
y = 0 lets g = -1+2x; y = 1 lets g = 2-3x. The intersection point is x = 3/5 = 60% and g = 1/5.
x = 0 lets g = -1+3y; x = 1 lets g = 1-2y. The intersection point is y = 2/5 = 40% and g = 1/5.
Thus the fair result of this game is that B pays to A 1/5 per game. Else A weighs P 60%, B weighs P 40%.

Exercise. Check the prediction by playing the game with the weighed strategies determined by 10 cards.

3 REVERSING STATISTICS

Question1. How to do reversed calculation in statistics?

Answer. By testing hypothesis.

Example1. There is one red card, or is there?
I assume I got what I ordered, 1 red and 1 black card. To check I randomly draw 1 card 3 times, giving 0 red cards. The probability for this event is p(x=0) = B(n,r)* p^r*(1-p)^(n-r) = B(3,0)* 1/2^0*(1/2)^3 = 1/8.
To reject my hypothesis 1R1B the probability must be less than 5%, so I have to draw n=5 times since p(x=0) = B(n,r)* p^r*(1-p)^(n-r) = B(n,0)* 1/2^0*(1/2)^n = ½^n = 5%, gives n = log5%/log½ = 4.3.

Example2. There is only one red card, or is there?
I assume I got what I ordered, 1 red & 2 black cards. To check I randomly draw 1 card 10 times, giving 6 red cards. The probability for the event x>=6 with n,p = 10,1/3 is p(x>=6) = 1-p(<=5) = 1-0.923 = 0.077.

I cannot reject my hypothesis 1R2B since the probability is higher than 5%.
Next time I got 7 red cards. The probability for the event $x>=7$ in an $n,p = 10,1/3$ game is $p(x>=7) = 1-p(<=6) = 1-0.980 = 0.02$. Now I reject my hypothesis since the probability is less than 5%.
Example 3. I got 20 cards, how many are red? To check I randomly draw 1 card 10 times, giving 6 red cards. The following hypothesis can be rejected: The number of red cards is 0, 1, 2, …, 6, 18, 19, 20.
With $n,p = 10$, $1/20$ the probability for the event $x>=6$ is $p(x>=6) = 1-p(<=5) = 1-1.000 = 0$
With $n,p = 10$, $6/20$ the probability for the event $x>=6$ is $p(x>=6) = 1-p(<=5) = 1-0.953 = 0.047$
With $n,p = 10$, $17/20$ the probability for the event $x<=6$ is $p(x<=6) = 0.050$
With $n,p = 10$, $19/20$ the probability for the event $x<=6$ is $p(x<=6) = 0.001$
This result can be predicted by the confidence interval $I = p \pm 2*\underline{p}/\sqrt{n} = 6/10 \pm 2*\underline{0.6}/\sqrt{10} = 0.60 \pm 0.31 = [0.29;0.91]$. Thus the hypotheses that can be rejected are from 0/20 to 6/20 and from 18/20 to 20/20.
Exercise. Take 2 like bags. Put 2 reds & 1 black ball in one and 1 red & 2 blacks in the other. Choose 1 bag. Repeat n times the experiment 'take out 1 ball' until you can form a hypothesis. How sure can you be?

4 COMPARING REPRESENTATIONS

Question1. How to compare parts?
Answer. Through conditional percentages.
Example 1. We compare two decks of cards with and without red Kings (I and II).

I	Spades	NonSpades	Total
CourtCards	3	9	12
NumberCards	10	30	40
Total	13	39	52

II	Spades	NonSpades	Total
CourtCards	3	7	10
NumberCards	10	30	40
Total	13	37	50

I. The unconditional probability or percentage of CoutCards (among all) = p(C I all) = p(C) = 12/52 = 23.1%.
The conditional probability or percentage of CoutCards (among the spades) = p(CIS) = 3/13 = 23.1%.
Since p(CISpade)=p(C) we say that 'among the spades the CourtCards are neither over- or under-represented.'
The conditional probability or percentage can be shown in a contingency table or pivot table:

I after colour	Spades	NonSpades	Total
CourtCards	23.1%	23.1%	23.1%
NumberCards	76.9%	76.9%	76.9%
Total	100%	100%	100%

I after kind	Spades	NonSpades	Total
CourtCards	25%	75%	100%
NumberCards	25%	75%	100%
Total	25%	75%	100%

II. The unconditional probability or percentage of CoutCards (among all) = p(C I all) = p(C) = 10/50 = 20%.
The conditional probability or percentage of CoutCards (among the spades) = p(CIS) = 3/13 = 23.1%.
Since p(C I Spade) > p(C) we say that 'among the spades the CourtCards are over-represented.'
Since p(C I NonSpade) < p(C) we say that 'among the non-spades the CourtCards are under-represented.'
The conditional probability or percentage can be shown in a contingency table or pivot table:

II acc. colour	Spades	NonSpades	Total
CourtCards	23.1%	18.9%	20%
NumberCards	76.9%	81.1%	80%
Total	100%	100%	100%

II acc. kind	Spades	NonSpades	Total
CourtCards	30%	70%	100%
NumberCards	25%	75%	100%
Total	26%	74%	100%

Example 2. A big population is distributed according to gender and smoking habit. To get an estimate of the percentages we draw a sample from a population creating an uncertainty to the percentages $D=2*\underline{p}/\sqrt{n}$.
Thus p(Girl I Smoker) = 27.3% and n = 550 gives $D = 2*\sqrt{(27.3\%*72.7\%)}/\sqrt{550} = 3.8\%$.
We set up frequency tables and contingency tables:

Sample	Smoker	NonSmoker	Total	*D*
Female	150	270	420	*4,7%*
Male	400	500	900	*3,3%*
Total	550	770	1320	*2,7%*
D	*3,8%*	*3,4%*	*2,6%*	

DoubleClick &Edit

Acc. Gender	Smoker	NonSmoker	Total
Female	27,3%	35,1%	31,8%
Male	72,7%	64,9%	68,2%
Total	100,0%	100,0%	100,0%

Acc. Smoking	Smoker	NonSmoker	Total
Female	35,7%	64,3%	100,0%
Male	44,4%	55,6%	100,0%
Total	41,7%	58,3%	100,0%

A deviation is significant if it exceeds the uncertainty. From the distribution acc. to gender table we can say:
Among the smokers the females are significantly under-represented (31.8%-27.3% = 4.5% > 3.8%).
Among the non-smokers the females are not significantly over-represented (35.1%-31.8% = 3.3% < 3.4%).
In the population the females are significantly under-represented (50%-31.8% = 18.2% > 2.6%).
Exercise. What can be said about the males? What can be said from the distribution after smoking table?

ADDING PER-NUMBERS A2

Questions	Answers
What is a prime-number?	Fold-numbers can be folded: 10 = 2fold5. Prime-numbers cannot: 5 = 1fold5
What is a per-number?	Per-numbers occur when counting, when pricing and when splitting
How to add per-numbers?	The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b

1 FOLD-NUMBERS AND PRIME-NUMBERS

Question. What are there primary numbers?
Answer. The primary numbers are numbers that cannot be folded, prime-numbers.
Example1. The stack T = 1*4 can be recounted to another stack T = 2*2. The stack T = 1*3 cannot. Thus 4 = 2fold2 whereas 3= 1fold3. 4 is called a fold-number, and 3 is called a nonfold-number or a prime number.
Fold-numbers are built from prime numbers, e.g. by folding in tables: 2, 4, 6, etc. 3, 6, 9, etc. 5, 10, 15, etc.
Fold-numbers can be factorised in prime numbers by using a prime-number filter:

48:	48/2	24/2	12/2	6/2	3/3	1
	*2	*2	*2	*2	*3	

So 48 = 2*2*2*2*3 = 2^4*3. Likewise 72 = 2*2*2*3*3 = 2^3*3^2
Example2. A packet consists of a mix of different goods coded with different prime-numbers.
The packet with the code 42 contains the goods coded 2 and 3 and 7 but not the good coded 5.
The packets 30 and 42 have the goods coded 2 and 3 in common (the greatest common factor HCF).
HCF(30,42) = HCF(2*3*5, 2*3*7) = 2*3 = 6. The HCF can be found by Euclid's algorithm.
Exercise1. Factorise the fold-numbers 6, 16, 26, 36, 46, 56, 66, 76, 86, 96. Factorise other fold-numbers.
Exercise2. Find the HCF of 30 and 105. Find the HCF of 30 and 40, etc.

2 PER-NUMBERS IN COUNTING

Question. What does a per-number consist of?
Answer. A per-number consists of a stack-number and a bundle-number.

A total stack of 8 is counted in bundles of 5s as T= 8 = (8/5)*5 = 1 3/5*5. Thus the per-number (fraction) 8/5 consists of a stack-number (nominator) 8 and a bundle-number (denominator) 5.	IIIIIIII -> III / IIIII

Example1. Repeated recounting creates double per-numbers:
1 = (1/2)*2 = (1/2) 2s = (1/2/3)*3 2s = (1/2/3)*3*2 = (1/2/3)*6. Since also 1 = (1/6)*6: **1/(2*3) = 1/2/3**
1 = (1/(2/3))*(2/3). Since also 1 = (1/3)*3 = (((1/2)*2)/3)*3 = (1/2*3)*2/3: **1/(2/3) = 1/2*3**
*The bracket rule: a per-bracket can be added or removed if the signs are changed from / to * and reverse.*
a is counted in bs, c is counted in ds: a=(a/b)*b & c=(c/d)*d. Thus a*c = (a/b)*b*(c/d)*d = (a/b)*(c/d)*(b*d).
a*c is counted in (b*d)s: a*c = ((a*c)/(b*d))*(b*d). Thus (a/b)*(c/d) = (a*c)/(b*d)
The multiplication rule 'stack by stack & bundle by bundle': **(a/b)*(c/d) = (a*c)/(b*d)**
The division rule 'stack by bundle & bundle by stack: **(a/b)/(c/d)** = a/b/c*d = a*d/b/c = **(a*d)/(b*c)**
Exercise. Calculate 24/(2*3), 24/2/3 and 24/(2/3) in different ways. Likewise with x^5*y^3/(x^2*y).

3 PER-NUMBERS IN PRICING AND RATING

Question. How to use per-numbers in pricing?
Answer. The price 2$ per 3kg can be written as a per-number 2$/3kg or 2/3 $/kg.
Since the price is 5*2$ for 5*3kg the price could also be written as 5*2$ per 5*3kg or (5*2)/(5*3) $/kg or 10/15 $/kg. Thus 10/15 = (5*2)/(5*3) = 2/3 in accordance with (5*2)/(5*3) = 5*2/5/3 = 5/5 * 2/3 = 2/3.
Removing common prime-factors from the stack and the bundle (the nominator and the denominator) in a per-number is called *reducing* the per-number. Adding common prime-factors to the stack and the bundle (the nominator and the denominator) in a per-number is called *extending* the per-number.
Extending a per-number: 2/3 = (2*7)/(3*7) = 14/21 in accordance with 2/3*7/7 = 2*7/7/3 = (2*7)/(7*3) = 14/21.
Reducing a per-number: 14/21= (2*7)/(3*7) = 2 * 7 / 3 / 7 = 2 / 3 * 7 / 7 = 2/3 according to the bracket rule.
Example1. Reduce the per-number 48/72.
48/72 = (2*2*2*2*3)/(2*2*2*3*3) = 2*2*2*2*3/2/2/2/3/3 = 2/2 * 2/2 * 2/2 * 2 * 3/3 /3 = 2/3
Example2. Reduce the per-number (4*x^2*y)/(6*x*y^3).
(4*x^2*y)/(6*x*y^3) = (4*x*x*y)/(6*x*y*y*y) = 2*2*x*x*y/2/3/x/y/y/y = 2*x/3/y/y = (2*x)/(3*y*y)
Example3. Comparing prices. Which is the higher price, 2/3 or 4/7? Both per-numbers are extended to 3*7:
P1 = 2/3 = 2/3 * 7/7 = (2*7)/(3*7) = 14/21. And P2 = 4/7 = 4/7 * 3/3 = (4*3)/(7*3) = 12/21.
So P1>P2, and the difference is P1-P2 = 14/21-12/21 = 2/21.
Which is the higher price, 3/4 or 5/6? Since 4=2*2 and 6=2*3 both per-numbers are extended to 2*2*3 (the lowest common multiple, LCM):
P1 = 3/4 = 3/4 * 3/3 = (3*3)/(3*4) = 9/12. And P2 = 5/6 = 5/6 * 2/2 = (5*2)/(6*2) = 10/12.

So P1<P2, and the difference is P2-P1 = 10/21-9/12 = 1/12

Example 4. Adding prices. What is the total price consisting of a basis price 11/6 + a fee of 4/9?
Since 6 = 2*3 and 9 = 3*3, the LCM(6,9) = LCM(2*3,3*3) = 2*3*3
P1 = 11/6 = 11/6 * 3/3 = (11*3)/(6*3) = 33/18. And P1 = 4/9 = 4/9 * 2/2 = (4*2)/(9*2) = 8/18
So the total is P1 + P2 = 33/18 + 8/18 = 41/18.

Exercise1. Reduce the per-number 36/64. Etc.

Exercise2. Reduce the per-number (6*x^3*y^2)/(8*x^2*y^4).

Exercise3. What are the difference and the sum of 2/5 and 3/7? 3/8 and 4/10? 23/36 and 31/48?

4 PER-NUMBERS IN SPLITTING

Question. How to split a winning?

Answer. By returning the stake several times. Or by receiving a proportional part of the winning.

Example1. The players A, B and C split a winning of 400$ from putting 2$, 3$ & 5$ into a pool creating 10$.
Method 1. The winning is counted in pools to get the odds: W = 400$ = (400/10)*10 = 40*10. Thus the players get their stake back 40 times: A gets 2$ 40 times, i.e. 80$, etc.
Method 2. The winning is split in the ratio 2:3:5. A gets 2 ten parts of the winning: A = 2/10*W = W/10*2.
Together A and B get 2/10 and 3/10 of W i.e. 2/10*W+3/10*W = (2/10+3/10)*W = ((2+3)/10)*W = 5/10*W.

Example2. An apartment is sold in 2 ¼-shares, 2 1/8-shares and 4 1/16-shares.
There are no shares left since 2/4*A + 2/8*A + 4/16*A = 2/(2*2)*A + 2/(2*2*2)*A + 2*2/(2*2*2*2)*A = ½*A + ¼*A + ¼*A = (½+¼+¼)*A = 1*A. Here ½+¼+¼ = ½*2/2 + ¼ + ¼ = 2/4 + ¼ + ¼ = 4/4 = 1.
B buys 1 1/8-share and 3 1/16-shares giving B a total control of 1/8*A + 3/16*A = (1/8 + 3/16)*A = (1/8*2/2 + 3/16)*A = (2/16 + 3/16)*A = 5/16*A.

Example3. B receives 2/10 of a 200$-winning and 3/5 of a 300$-winning.
The total income for B is T = 2/10*200 + 3/5*300 = 200/10*2 + 300/5*3 = 40 + 180 = 220 = (220/500)*500 = (2*2*5*11)/(2*2*5*5) = 11/25*500. So in this case 2/10+3/5=11/25, and not 20/25 as in example 1 above.

Example4. ½ of 2 cokes + 2/3 of 3 cokes = 1 coke + 2 cokes = 3 cokes = (3/5)*5 cokes = 3/5 of 5 cokes.
So in this case ½+2/3 = 3/5 and not 7/6 as in the example 1 and 2 above.

Example5. The fraction-paradox:

Inside the classroom	20/100 = 20%	+ +	10/100 = 10%	= =	30/100 = 30%
Outside the classroom e.g. in the laboratory	20%	+	10%	= or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

20% + 10% = 30% only when taken of the same total: 20% of 300 + 10% of 300 = 30% of 300. In all other cases the sum is different from 30%, so there is no general rule saying that 20% + 10% = 30%.

Exercise1. Split the winning 1600$ between A, B and C in the ratio 3:1:4.

Exercise2. The king gets 2 7parts of the harvest, and the bishop get 1 9part. How much is left?

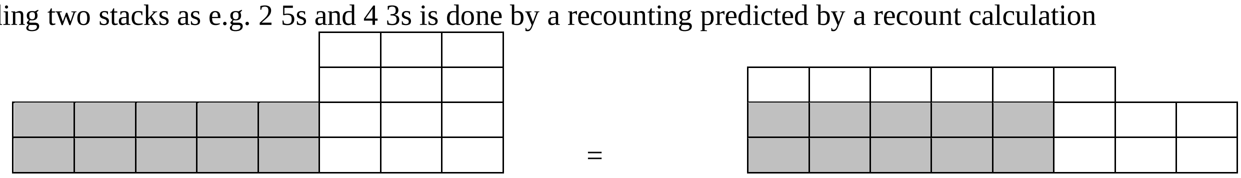
Exercise3. My two investments 200$ and 500$ gave 12% and 4% yield. What is the total yield percentage?

5 ADDING PER-NUMBERS I

Question. How to add per-numbers with number units?

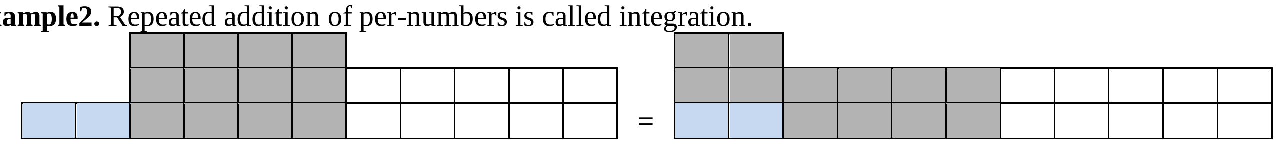
Answer. Per-numbers can be added upward keeping the bundle-size; or sideward extending the bundle-size.

Example1. Sideward addition of two per-numbers.
Adding two stacks as e.g. 2 5s and 4 3s is done by a recounting predicted by a recount calculation



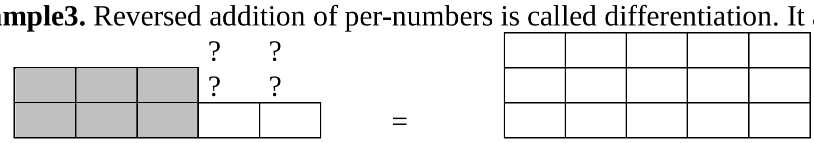
T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8

Example2. Repeated addition of per-numbers is called integration.



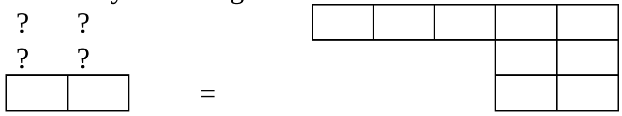
T = 1 2s + 3 4s + 2 5s = 1*2 + 3*4 + 2*5 = (1*2 + 3*4 + 2*5)/11*11 = 2 2/11*11

Example3. Reversed addition of per-numbers is called differentiation. It asks e.g. 2 3s+? 2s = 3 5s:



T = 2 3s + ? 2s = 3 5s

The answer can be obtained by removing the 2 3s from the 3 5s and then counting the remaining 9 in 2s.



? 2s = (3 5s – 2 3s)/2 * 2 = (9/2)*2 = 4 1/2 * 2

Or the answer can be obtained by a predication through a reversed calculation. In this way solving equations becomes another name for reversed calculations.

2 3s + ? 2s = 3 5s	The question	
2*3 + x*2 = 3*5	The equation	T1 + x*b = T2
x*2 = 3*5-2*3 = 9	The 2 3s are removed from the 3 5s leaving 9	x*b = T2 – T1 = ΔT
x*2 = (9/2)*2 = 4 1/2 * 2	The 9 is recounted as 2s	$x = \frac{\Delta T}{b} = \frac{\Delta T}{\Delta n}$
x = 1 1/5	The answer	

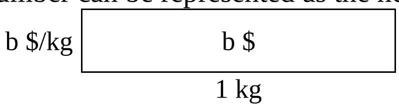
Exercise. Integrate 3 4s, 5 6s and 7 8s, etc. Differentiate 3 4s+? 5s = 6 7s, etc.

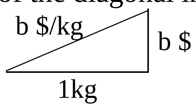
6 ADDING PER-NUMBERS II

Question. How to add per-numbers with letter units?

Answer. The per-number a is multiplied with the bundle-number b before added to the total stack T: T2=T1+a*b.

Per-numbers occur in Renaissance trade questions as price-numbers 4 $/kg or rent-numbers 4 $/day. Again the per-number can be represented as the height of a stack, or as the slope of the diagonal in a change-triangle.

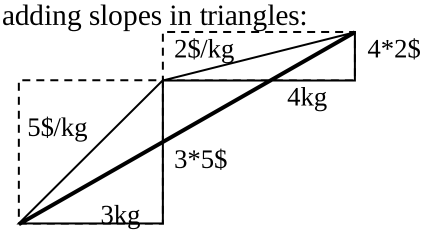




Adding per-numbers from trade takes place in a table

a kg @	*b $/kg*	*= a*b $*		
3 kg @	5 $/kg	= 3 * 5		= 15 $
4 kg @	2 $/kg	= 4 * 2		= 8 $
7 kg @	x $/kg	= 7 * x	= ∑a*b	= 23 $
	x	= 23/7		= 3 2/7 $

or by adding slopes in triangles:



So per-numbers are added by their totals:

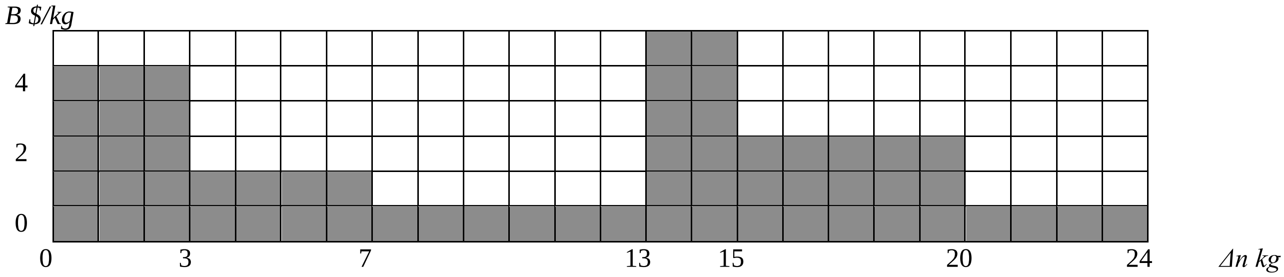
3 kg @ 5 $/ kg + 4 kg @ 2 $/ kg = (3+4) kg @ (∑a*b)/(3+4) $/ kg

The table can be supplemented with two columns showing the added values of both the kg-number Δn, and of the $-number ΔT, and of the per-number Σ b, as in this example where a teashop is adding different amounts with different prices to create a blending.

Δn kg	*b $/kg*	*Δn*b = ΔT*		*Σ Δn = Δn*	*Σ ΔT = ΔT*	*Σ b $/kg = ΔT/Δn*
3 kg @	5$/kg =	3 * 5 = 15		3	15$	15/3 = 5.0
4 kg @	2$/kg =	4 * 2 = 8		7	23$	23/7 = 3.3
6 kg @	1$/kg =	6 * 1 = 6		13	29$	29/13 = 2.2
2 kg @	6$/kg =	2 * 6 = 12		15	41$	41/15 = 2.7
5 kg @	3$/kg =	5 * 3 = 15		20	56$	56/20 = 2.8
4 kg @	1$/kg =	4 * 1 = 4		24	60$	60/24 = 2.5

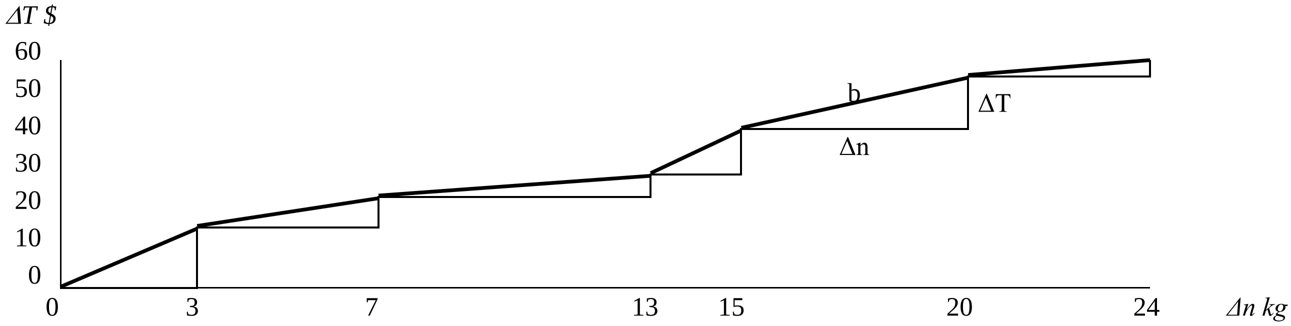
When plotting the per-number b $/kg against Δn kg in a coordinate system the total $-number is the area under the curve representing the sum of the stacks.

The Total as an area under the PerNumber curve



When plotting ΔT against Δn in a coordinate system the curve shows both the added kg-number Δn, the added total ΔT, and the single per-numbers b = ΔT/Δn as the slopes.

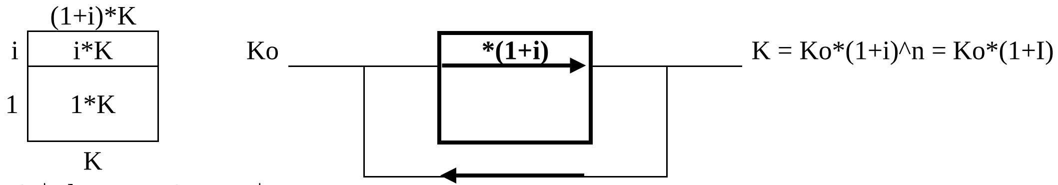
The PerNumber as the slope of the Total curve



Thus from blending tea in a shop we learn that:
The Total is the area under the PerNumber curve predicted by an integration formula: T= ∑$/kg*kg = ∑b*Δn.
The PerNumber is the slope of the Total curve predicted by a differentiation formula: b = Δ$/Δkg = ΔT/Δn.
Exercise. Travel, first 5 seconds @ 4m/s, then 3 seconds @ 6m/s, then 4 seconds @ 2m/s etc.

7 ADDING INTEREST PERCENTAGES

Question. How can we add stacks sideward?
Answer. Adding stacks sideward in time is called integration. It can be done by recounting and restacking.
Example. T = 2 3s + 4 5s = ? 8s. The result can be predicted by the recount-equation.
The interest i*K is added to a capital K giving K + i*K = 1*K + i*K = (1+i)*K. Repeated n times the terminal capital is K = Ko*(1+i)^n.



5 days @ 9 $/day = 5*9 = 45 $
5 days @ 9 %/day = 5*9% (i.e. 45 %) + CI = simple interest + compound interest, or I = n*i + CI, so
5 days @ 9 %/day = 5*9% (i.e. 45 %) + 8.9% = 53.9% since (1+9%)^5 = 1.539
Exercise. Predict 6 days @ 7%/day, 4 days @ -8%/day etc.

8 PER-NUMBERS IN WORD-PROBLEMS

Question. How do we treat per-numbers in word problems?
Answer. Also in word problems the per-number must be transformed to a stack-number before being added.
Example1: Train1 travels from A to B at 40 km/h. Two hours later train2 travels from A to B at 60 km/h. When do they meet?

Per-numbers	*Text*	*Stack-numbers*	*Prediction*		*ANSWERS*
	Hours	x = ?	40*(x+2)	= 60*x	*4*
40 km/h	Velocity1		40*x + 80	= 60*x	
60 km/h	Velocity2		80	= 60*x – 40*x = 20*x	
	Distance1	40*(x+2) km	80/20	= x	*240*
	Distance2	60*x km	4	= x	*240*

Example2: Train1 travels from A to B at 40 km/h. At the same time train2 travels from B to A at 60 km/h. When do they meet if the distance from A to B is 300km?

Per-numbers	*Text*	*Stack-numbers*	*Prediction*		*ANSWERS*
	Hours	x = ?	40*x + 60*x	= 300	*4*
40 km/h	Velocity1		100*x	= 300*x	
60 km/h	Velocity2		x	= 300/100	
	Distance1	40*x km	x	= 3	*120*
	Distance2	60*x km			*180*

Exercise1. Repeat the train problems with other numbers.
Exercise2. It takes a motor boat 2 hours downstream and 3 hours upstream to travel the same distance. The current is 5 km/h. What is the speed of the boat?
Exercise3. ? litre 40% alcohol + 3 litre 20% alcohol gives ? litre 32% alcohol
Exercise4. ? $ @ 3%/$ + ? $ @ 8%/$ = 200$/4000$
Exercise5. B can dig a ditch in 4 hours, C in 3 hours. How long time does it take if they work together?

COUNT&ADD IN TIME

T2

Question	Answer
How to predict the terminal number when the change is constant?	By solving constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7 = Ko*(1+r)^n = 30*1.02^7 = 34.46
How to predict the terminal number when the change is variable, but predictable?	By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx

1 COUNTING CHANGE

Question. How can we count change?
Answer. By three change-numbers restacking or recounting the terminal number:
Change, increment = ΔT = TerminalNumber-InitialNumber: T2 = (T2-T1)+T1 = ΔT + T1
Change-multiplier or index = I = TerminalNumber/InitialNumber T2 = (T2/T1)*T1 = I*T1
Change-percent, interest = r = Change-number/InitialNumber r = ΔT/T1 = (T2 - T1)/T1 = T2/T1-1 = I-1
From the level-numbers T the change-numbers are calculated directly. From the change-numbers the level-numbers are predicted by solving the change-equation telling how the change can be calculated. The change might be constant or variable.
Example.

Level		Single change				Total change		
T		**ΔT**	**I**	**r**		**ΔT**	**I**	**R**
200								
230		+ 30	*115 (%)	+ 15%		+ 30	*115 (%)	+ 15%
210		- 20	*91,3 (%)	- 8,70%		+ 10	*105 (%)	+ 5%
450		+240	*214,3 (%)	+ 114,30%		+ 250	*225 (%)	+ 125%

Exercise. Find the single and total change-numbers for T-levels 300, 360, 324, 420, etc.

2 CONSTANT CHANGE

Question. How can we predict the terminal number in constant change?
Answer. By solving the constant change-equations.
Example1. Constant change: Money in a box can change by adding a constant number (linear change by adding), by adding a constant percentage (exponential change by multiplying) or both (+&*change, saving).

+CHANGE (ΔK = a$)	*+&*CHANGE* **= a*R/r**	**CHANGE (ΔK/K = r%)*
Ko + A = Ko + a*n =	**K**	**= Ko*(1+r)^n = Ko*(1+R)**
	+ a $ + r %	
+ A = + a*n	**+ … + …**	+ R %
	+ a $ + r %	*1+R = (1+r)^n*
	+ a $ + r %	
Linear change by adding +5$+5$+5$ change, +change K*term* = K*init* +a+a+a n times: K = Ko + a*n = Ko + A, where A = a*n is the total change	**Ko**	**Exponential change by multiplying** +5%+5%+5% change, *change K*term* = K*init* +r%+r%+r% n times K*term* = K*init* +r%+r%+r% n times K = Ko*(1+r)*(1+r)*(1+r) = Ko*(1+r)^n = Ko*(1+R) where R is the total %change or compound interest

Example2. Compound interest: We cannot add percentages to dollars so Ko is recounted as 100% making
K1 100% + 5% = 105% of Ko: K1 = Ko*105% = Ko*1,05 = Ko*(1+r)
R is the total compound interest, i.e. the sum of the simple interest SR = n*r and compounded interest RR:
R = SR+RR = n*r + RR, where (1+r)^n = 1+R
7 times @ 5$ = 7*5 $ = 35 $
7 times @ 5% = 7*5% + RR = 35% + 5% = SR + RR = R = 40% (since 1+R=107%^5 = 140%)
Example3. Double-taxation:
CarPrice + 200% tax + 25% vat. = CarPrice*(1+2)*(1+0,25) = CarPrice*3,75 = CarPrice*(1+2,75) = CarPrice + 275% tax. And 275% tax = 200% CarTax + 75% Tax-of-Tax
Example4. Doubling:
R=100% gives change-multiplier 2 and the doubl-time predicted by: (1+r)^N = 2, N = log2/log(1+r) ≈ 70/r.

Example5. Saving, or +&*change, is change by adding&multiplying both a constant deposit a$ and a constant interest r% e.g. +(5%&7$)+(5%&7$). Thus the compound interest Ko*R is a saving generated by the deposit a = Ko*r$ and the interest r%. So the saving is K = Ko*R = a/r*R = a/r*((1+r)^n-1). Only the interest is taxed since the deposits have already been taxed.
Account II hold a debt G having grown exponentially to K from its interests: K = G*(1+R). In an instalment plan the goal is to balance the debt on II by the saving so G*(1+R)=a/r*R: G=a/r*R/(1+R)=a/r*(1-(1+r)^-n).
Example6. Seen as +change & *change, linear and exponential change is grouped with +&change or saving. Seen as ++change and +*change, linear and exponential change is grouped with **change or power-change.

Linear change (++ change) x: +1, y: +a, e.g. trade	*Exponential change (+* change)* x: +1, y: +a %, e.g. interest	*Power change (** change)* x: +1 %, y: +a %, e.g. dimensions
b + x times @ a $/time total T b + x*a = T	b + x times @ á r %/time total T b * (1+r%)^x = T	Dimension T = a*Dimension x b * x^a = T #
Δy/Δx = a, y = b+a*x Δy/Δx: slope ≈ dy/dx = y’	Δy/Δx = a*y, y = b*a^x (Δy/Δx)/y: interest	Δy/y = c*Δx/x, y = b*x^a (Δy/y)/(Δx/x): elasticity

Example7. Calculation tables: # b is a conversion number between units

	Linear change		*Exponential change*		*Power change*
K = ?	K = Ko+a*n	K = ?	K = Ko*a^n	K = ?	K = Ko*n^a
Ko = 40	K = 40+3*5	Ko= 40	K = 40*1.03^5	Ko= 40	K = 40*5^3
a = 3	**K = 55**	a= 1.03	**K = 46.371**	a= 3	**K = 5000**
n = 5		n = 5		n = 5	

Ko = ?	K = Ko+a*n	Ko = ?	K = Ko*a^n	Ko = ?	K = Ko*n^a
K = 70	K = Ko+(a*n)	K = 70	K = Ko*(a^n)	K = 70	K = Ko*(n^a)
a = 3	K - (a*n) = Ko	a = 1.03	K/(a^n) = Ko	a = 3	K/(n^a) = Ko
n = 5	70 – (3*5) = Ko	n = 5	70/(1.03^5) = Ko	n = 5	70/(5^3) = Ko
	55 = Ko		**60.383 = Ko**		**0.56 = Ko**
Check:	*70 =? 55+3*5* *70 =! 70*	*Check:*	*70 =? 60.383*1.03^5* *70 =! 70*	*Check:*	*70 =? 0.56*5^3* *70 =! 70*

a = ?	K = Ko+a*n	a = ?	K = Ko*a^n	a = ?	K = Ko*n^a
Ko = 40	K = Ko+(a*n)	Ko = 40	K = Ko*(a^n)	Ko = 40	K = Ko*(n^a)
K = 70	K-Ko = a*n	K = 70	K/Ko = a^n	K = 70	K/Ko = n^a
n = 5	(K-Ko)/n = a	n = 5	n√(K/Ko) = a	n = 5	log(K/Ko)/log n = a
	(70-40)/5 = a		5√(70/40) = a		log(70/40)/log 5 = a
	6 = a		**1.118 = a**		**0.348 = a**
Check:	*70 =? 40+6*5* *70 =! 70*	*Check:*	*70 =? 40*1.118^5* *70 =! 69.866*	*Check:*	*70 =? 40*5^0.348* *70 =! 70.032*

n = ?	K = Ko+a*n	n = ?	K = Ko*a^n	n = ?	K = Ko*n^a
Ko = 40	K = Ko+(a*n)	Ko = 40	K = Ko*(a^n)	Ko = 40	K = Ko*(n^a)
a = 3	K-Ko = a*n	a = 1.03	K/Ko = a^n	a = 3	K/Ko = n^a
K = 70	(K-Ko)/a = n	K = 70	log(K/Ko)/log a = n	K = 70	a√(K/Ko) = n
	(70-40)/3 = n		log(70/40)/log 1.03=n		3√(70/40) = n
	10 = n		**18.9 = n**		**1.205 = a**
Check:	*70 =? 40+3*10* *70 =! 70*	*Check:*	*70 =? 40*1.03^18.9* *70 =! 69.933*	*Check:*	*70 =? 40*1.205^3* *70 =! 69.988*

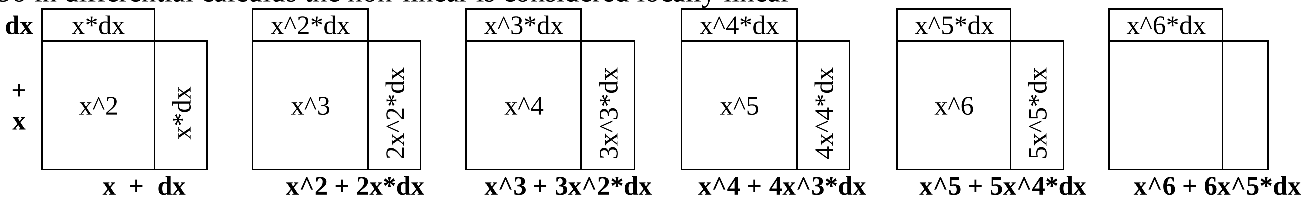
In savings there are 3 calculation tables since the interest r cannot be isolated appearing in both r and R:

K = ?	K = a/r*R	a = ?	K = a/r*R	n = ?	K = a/r*R
a = 100	K=100/0.05*0.796	K = 1000	K/R*r = a	K=1000	K*r/a = R =(1+r)^n-1
n = 12	**K = 1592**	n = 12	1000/0.796*0.05=a	a = 100	1 + K*r/a = (1+r)^n
r = 0.05		r = 0.05	**62.81 = a**	r = 0.05	log(1+K*r/a)/log(1+r)=n
1+R=1.05^12		1+R=1.05^12			**8.3 = n**
R = 0.796		R = 0.796			R = 1.05^8.3-1 = 0.499
		Check:	1000= 62.814/0.05*0.796 1000 =! 999.999	Check:	1000 =? 100/0.05*0.499 1000 =! 998

Example7. Graph papers:
On an equidistant +scale the same number is added for each step: 0,10,20,30,40
On a logarithmic *scale the same number is multiplied for each step: 1,2,4,8,16,32
Linear change gives a straight line on ++ paper (graph paper).
Exponential change gives a straight line on +* paper (logarithmic paper).
Power change gives a straight line on ** paper (double logarithmic paper).
Exercise. Make the calculation tables with different numbers. Make the graphs on graph paper too.

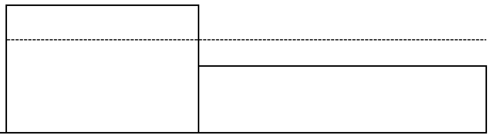
3 PREDICTABLE CHANGE

Question. How can we predict the terminal number in predictable change?
Answer. By solving the predictable change-equations.
Example. Totalling per-numbers:
5 seconds @ 6 m/s total 5*6 = 30 m.
5 seconds @ 6 m/s increasing to 8 m/s total ? m.
5 seconds @ 6 %/s total 5*6% = 30% + 3.8% = linear interest + compounded interest (since 1.06^5 = 1.338)
So compounded interest is what keeps exponential change from being linear.
For small interests dx the compounded interest can be neglected: 1.006^5 = 1.03036 ≈ 1.03 so 5*0.6% ≈ 3%
Neglecting compounded interest (or the upper right corner of the change stack) is called differential calculus.
So in differential calculus the non-linear is considered locally linear



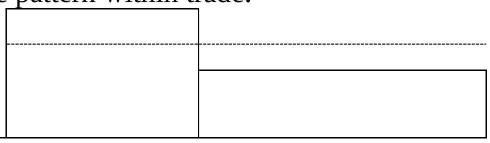
In differential calculus (x+dx)^2=x^2+2*x*dx; (x+dx)^3=x^3+3*x^2*dx etc.: (x+dx)^n = x^n+n*x^(n-1)*dx
If y = x^n, then a change in x, dx, produces a change in y, dy, and
y + dy = (x+dx)^n = x^n + n*x^(n-1)*dx = y + n*x^(n-1)*dx
So dy = + n*x^(n-1)*dx, or dy/dx = n*x^(n-1)
The elementary school introduces the practise of counting Many by bundling and stacking to be predicted by per-numbers, thus making the students acquainted with the geometrical representation of per-numbers as the height of a stack. Also simple additions as T = 3 4s + 2 5s = ? 9s are carried out both by recounting and by prediction thus realising that mathematics is our language of prediction:

T = 3 4s+2 5s = 3*4+2*5 = (3*4+2*5)/9*9 = 2 4/9 *9



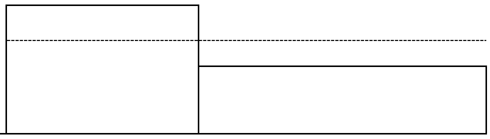
In secondary school addition of per-numbers follows the same pattern within trade:

T = 4kg @ 3$/kg + 5kg @ 2$/kg = 9kg @ 2 4/9$/kg



And so does addition of per-numbers within physics:

T = 4s @ 3m/s + 5s @ 2m/s = 9s @ 2 4/9m/s



This also applies if the m/s-number is locally constant and not just piecewise constant, i.e. if ε and δ changes places in the formal definition of constancy:

A variable y is globally constant c	∀ε>0:	|y-c|	< ε all over
A variable y is piecewise constant c	∃δ>0 ∀ε>0:	|y-c|	< ε in the interval δ
A variable y is locally constant c (continuous)	∀ε>0 ∃δ>0:	|y-c|	< ε in the interval δ

Thus per-numbers are added by the area under their curve. Since any smooth curve is locally constant, its area can be approximated by stacks that are summed (integrated): ∫ y*dx ≈ ∑ y*Δx. However if y*Δx can be written as the change of another variable z (y*Δx = Δz, or y = Δz/Δx) then the sum can be predicted since the sum of single changes = the total change = TerminalNumber - InitialNumber:

∑ y*Δx = ∑ Δz = Δz = z2 - z1.

This relation does not depend on the size of the change, so also ∫ y*dx = ∫ dz = z2 – z1.
Now we are able to predict the result of adding variable per-numbers through the calculation integration:

5 sec. @ 3m/sec increasing to 4 m/sec total $\int_0^5 (3+\frac{4-3}{5}x)\,dx = \int_0^5 (3+0.2x)\,dx$ = ? m

Since d/dx (3x+0.1x^2) = 3+0.2x we get that d (3x+0.1x^2) = (3+0.2x) dx, so

$$\int_0^5 (3+0.2x)\,dx = \int_0^5 d(3x+0.1x^2) = \Delta_0^5 (3x+0.1x^2) = (3\cdot5+0.1\cdot5^2) - 0 = 17.5\text{ m}$$

Exercise. Do other examples of integration from p. 6 and 7.

4 DRAWING GRAPHS

Question. How can we draw a non-linear graph?
Answer. By identifying its turning points.
A variable number is called x. A formula with a variable number is called a function f(x) e.g. f(x) = 3*x^2 – 4*x + 5. A function can be drawn as a f-curve y = f(x) in a coordinate system.

y = f(x)	predicts the level of the curve
y' = f'(x)	predicts the slope of the curve and of the tangent extending the local linearity of the curve
y" = f'(x)	predicts the curvature of the curve
∫y dx	predicts the area under the curve, and the average level of the curve

Example 1. An f-table has four rows with x, y, y' og y", telling that for e.g. x = 3 the level is 20, the slope is 14 and the curvature is 3. With a positive slope the curve is going upward. With a positive curvature the curve is bending upward.

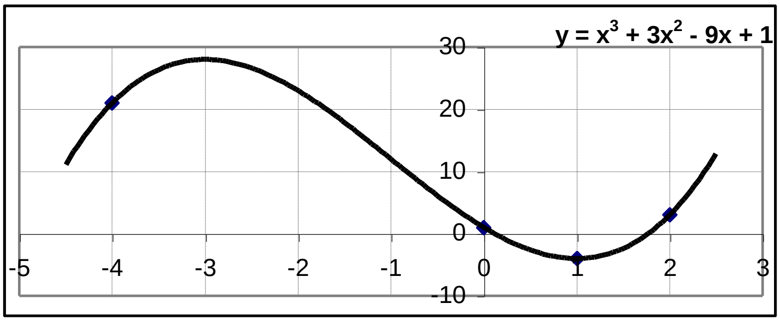
Position	x	3	5
Level	y = 3*x^2–4*x+5	y = 3*3^2–4*3+5 = 20	y =
Slope	y' = 6*x–4	y' = 6*3 – 4 = 14	y' =
Curvature	y" = 3	y" = 3	y" =

Example 2. At a curve tops and bottoms can be located from the drawing. Or from the curve's f-table:

x		-3		-1		1	
y =x^3+3*x^2–9*x+1	↗			↘			↗
y' =3*x^2+6*x–9 *y'=0 for x=-3 og x=1*	+	y' =0 top		-		y'= 0 bottom	+
y" = 6*x+6 *y"=0 for x=-1*		-		y" = 0 turning tangent		+	

The f-table shows that for x = -3 and x = 1 the curve has a horizontal tangent with slope y'= 0.
For x = -3 the curve is bending downward (y" is negative) giving a top. Also the signs of y' indicate a top being positive before and negative after x = -3.
For x = 1 the curve is bending upward (y" is positive) giving a bottom. Also the signs of y' indicate a top being negative before and positive after x = -3.
For x = -1 we have a turning tangent since the sign of the curvature changes.

x	y
-4	21
0	1
1	-4
2	3



Double-click and edit

Exercise. Do other examples of graph drawing from p. 6 and 7.

5 TWO VARIABLES

Question. How can we predict the behaviour of a two-variable graph?
Answer. By identifying its turning points.
A total T containing two variable numbers x and y will form a surface when drawn in a three dimensional coordinate system. Restricting one variable creates partial curves with slopes found by partial differentiation.

Example. T = x^2 + y^2 – 2*x + 3
Slope in the i x-direction Tx’ = ∂T/∂x = ∂/∂x (x^2 + y^2 – 2*x + 3) = 2*x – 2 (y is constant)
Slope in the i y-direction Ty’ = ∂T/∂y = ∂/∂y (x^2 + y^2 – 2*x + 3) = 2*y (x is constant)
The surface is locally horizontal where Tx’ = 0 og Ty’ = 0, i.e. in (x,y) = (1,0)
A surface has level-curves with a constant T-number e.g. T = 6:

Level-curve = ?	x^2 + y^2 – 2*x + 3 = T
T = 6	x^2– 2*x + 1 + y^2 = 6 – 3 + 1
	(x-1)^2 + (y-0)^2 = 4 = 2^2

So the level-curve T = 6 is a circle with centre (x,y) = (1,0) and radius r = 2.

T = a*x^2 + b*x + c*y^2 + d*y + e

a	b	c	d	e
1	-2	1	0	3

	T	y -3	-2	-1	0	1	2	3
x	-3	27	22	19	18	19	22	27
	-2	20	15	12	11	12	15	20
	-1	15	10	7	6	7	10	15
	0	12	7	4	3	4	7	12
	1	11	6	3	2	3	6	11
	2	12	7	4	3	4	7	12
	3	15	10	7	6	7	10	15

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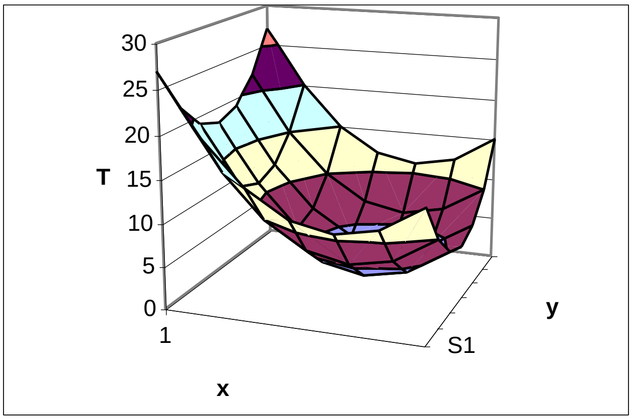


Exercise. Make other examples of graph drawing.

6 CHANGE EQUATIONS

Question. How can we solve the general change equation?
Answer. By adding the change ΔT to the initial number To to produce the terminal number T.
A change equation tells how the change can be calculated. A change equation is solved by calculating the terminal number from the initial number and the change. It can be solved by calculating techniques (e.g. integration) or by manual addition (numerical integration). Examples of growth equations:

Type of change	*Initial number*	*Change equation*	*Terminal number*
Constant change	To = b	ΔT = a	T = b + a*x
Constant change-percentage	To = b	ΔT = r%*T	T = b * a^x, a = 1+r
Constant change and change-percentage	To = 0	ΔT = r%*T + a	T/a = R/r, 1+R = (1+r)^n
Variable predictable change	To = b	dT = f*dx	T = b + ∫ f*dx
Variable unpredictable change	-	ΔT = ?	T = Tave ± 2*ΔTave

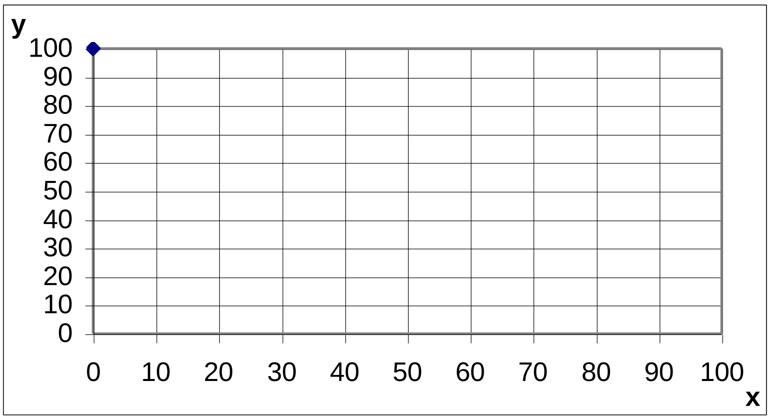
Example1. Numerical integration:

Stop/Start: 0/1 *0*

	x	y
Initial number	0	100
Change	1	-5
Terminal number	**0**	**100**

Double-click and edit

Guide
1. Enter 0 in B1, press F9
2. Enter initial number in C3
3. Enter change equation in C4
4. Enter 1 in B1, press F9
5. Repeat pressing F9



Exercise. Do other examples of solving change equations.

Exemple2. Predicting planetary orbits in EXCEL. *Double click and edit.*

ChangeEquations

d vx =	ax*dt
d vy =	ay*dt
d x =	vx*dt
d y =	vy*dt

LevelEquations

ax =	-x/r^3
ay =	-y/r^4
vx =	vx + d vx
vy =	vy + d vy
x =	x + d x
y =	y + d y
r =	V(x^2+y^2)
a =	1/r^2

Guide

1. Choose the field DEL and delete by pressing Delete
2. Choose the field Table
3. Pull down in the black spot in the lower right corner, one or two steps at a time

SUN	0	0
dt:	0,1	

ax	ay	dvx	vx	dvy	vy	dx	x	dy	y	r
			0		1,10		1		0	1,00
-1,00	0,00	-0,10	-0,10	0,00	1,10	-0,01	0,99	0,11	0,11	1,00
-1,00	-0,11	-0,10	-0,20	-0,01	1,09	-0,02	0,97	0,11	0,22	0,99
										0,99
										1,00
										1,00
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										1,04
										1,06
										1,07
										1,09
										1,10
										1,12
										1,14
										1,16
										1,18
										1,20
										1,22
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										1,32
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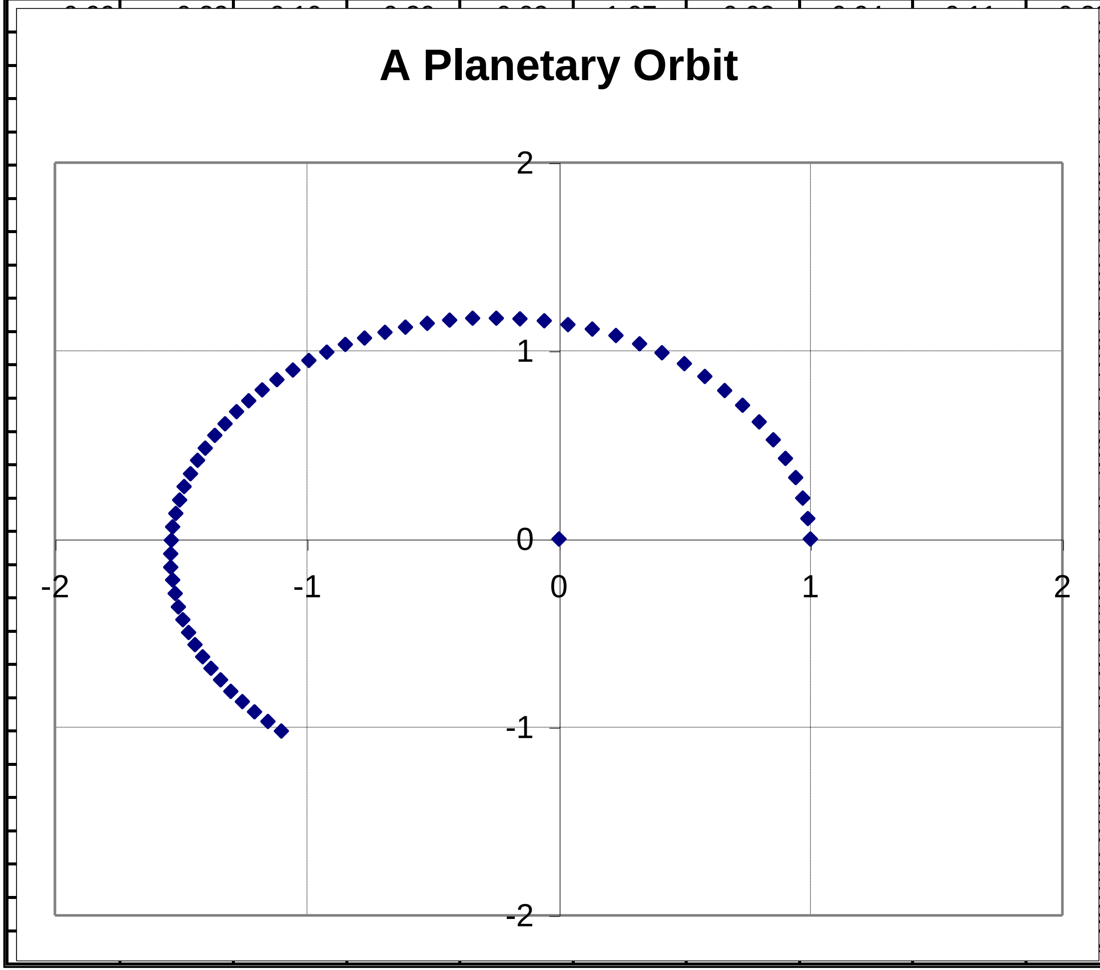
										1,40
										1,42
										1,43



Double-click and edit

Exercise1 *Answers*

		x = 3 y = ?	y = 0 x = ?		y '	y ' = 0	y "	∫ y dx	Turning points		Tangent i x=2	∫ y dx from 1 to 2	Signs	Factorise
1	y = x^2 -6x+5	-4	1	5	2x-6	3	2	0,33x^3-3x^2+5x+k	3	-4	y = -2x+1	1,67	+ - +	(x-1)(x-5)
2	y = x^2 -3x+2	2	1	2	2x-3	1,5	2	0,33x^3-1,5x^2+2x+k	1,5	-0,25	y = +1x-2	0,17	+ - +	(x-2)(x-1)
3	y = 2x^2 -10x+12	0	2	3	4x-10	2,5	4	0,67x^3-5x^2+12x+k	2,5	-0,5	y = -2x+4	-1,67	+ - +	2(x-3)(x-2)
4	y = 2x^2 -6x-8	-8	-1	4	4x-6	1,5	4	0,67x^3-3x^2-8x+k	1,5	-12,5	y = +2x-16	12,33	+ - +	2(x-4)(x+1)
5	y = 3x^2 -18x+15	-12	1	5	6x-18	3	6	1,00x^3-9x^2+15x+k	3	-12	y = -6x+3	5,00	+ - +	3(x-5)(x-1)
6	y = 3x^2 -24x+36	-9	2	6	6x-24	4	6	1,00x^3-12x^2+36x+k	4	-12	y = -12x+24	-7,00	+ - +	3(x-6)(x-2)
7	y = 4x^2 -40x+84	0	3	7	8x-40	5	8	1,33x^3-20x^2+84x+k	5	-16	y = -24x+68	-33,33	+ - +	4(x-7)(x-3)
8	y = 4x^2 -40x+64	-20	2	8	8x-40	5	8	1,33x^3-20x^2+64x+k	5	-36	y = -24x+48	-13,33	+ - +	4(x-8)(x-2)
9	y = -4x^2 -12x-8	-80	-1	-2	-8x-12	-1,5	-8	-1,33x^3-6x^2-8x+k	-1,5	1	y = -28x+8	35,33	- + -	-4(x+1)(x+2)
10	y = -4x^2 -4x+8	-40	1	-2	-8x-4	-0,5	-8	-1,33x^3-2x^2+8x+k	-0,5	9	y = -20x+24	7,33	- + -	-4(x+2)(x-1)
11	y = -3x^2 -6x+9	-36	1	-3	-6x-6	-1	-6	-1,00x^3-3x^2+9x+k	-1	12	y = -18x+21	7,00	- + -	-3(x+3)(x-1)
12	y = -3x^2 -6x+24	-21	2	-4	-6x-6	-1	-6	-1,00x^3-3x^2+24x+k	-1	27	y = -18x+36	-8,00	- + -	-3(x+4)(x-2)
13	y = -2x^2 -4x+30	0	3	-5	-4x-4	-1	-4	-0,67x^3-2x^2+30x+k	-1	32	y = -12x+38	-19,33	- + -	-2(x+5)(x-3)
14	y = 2x^2 +8x-24	18	-6	2	4x+8	-2	4	0,67x^34x^2-24x+k	-2	-32	y = +16x-32	7,33	+ - +	2(x+6)(x-2)
15	y = 3x^2 +18x-21	60	-7	1	6x+18	-3	6	1,00x^39x^2-21x+k	-3	-48	y = +30x-33	-13,00	+ - +	3(x+7)(x-1)
16	y = x^2 +6x-16	11	-8	2	2x+6	-3	2	0,33x^33x^2-16x+k	-3	-25	y = +10x-20	4,67	+ - +	(x+8)(x-2)

Exercise 2 *Answers*

		x=4 y=?	Zeros			Signs	Turning points		Factorising	Tangent in x=2	Differentiated	Integrated
1	y = x^3 -2x^2-5x+6	18	1	-2	3	- + - +	-0,79	2,12	(x-1)(x+2)(x-3)	y=-4-1(x-2)	3x^2-4x-5	0,25x^4-0,67x^3-2,50x^2+6x+k
							8,21	-4,06				
2	y = x^3 -6x^2+11x-6	6	1	2	3	- + - +	1,42	2,58	(x-1)(x-2)(x-3)	y=-1(x-2)	3x^2-12x+11	0,25x^4-2,00x^3+5,50x^2-6x+k
							0,38	-0,38				
3	y = 2x^3 -8x^2-22x+60	-28	2	-3	5	- + - +	-1,00	3,67	2(x-2)(x+3)(x-5)	y=-30(x-2)	6x^2-16x-22	0,5x^4-2,67x^3-11,00x^2+60x+k
							72,00	-29,63				
4	y = 2x^3 -20x^2+62x-60	-4	2	3	5	- + - +	2,45	4,22	2(x-2)(x-3)(x-5)	y=6(x-2)	6x^2-40x+62	0,5x^4-6,67x^3+31,00x^2-60x+k
							1,26	-4,23				
5	y = 3x^3 -18x^2-57x+252	-72	3	-4	7	- + - +	-1,21	5,21	3(x-3)(x+4)(x-7)	y=90-93(x-2)	9x^2-36x-57	0,75x^4-6,00x^3-28,50x^2+252x+k
							289,30	-109,30				
6	y = 3x^3 -42x^2+183x-252	0	3	4	7	- + - +	3,46	5,87	3(x-3)(x-4)(x-7)	y=-30+51(x-2)	9x^2-84x+183	0,75x^4-14,00x^3+91,50x^2-252x+k
							2,64	-18,19				
7	y = 4x^3 -32x^2-116x+720	0	4	-5	9	- + - +	-1,43	6,76	4(x-4)(x+5)(x-9)	y=392-196(x-2)	12x^2-64x-116	1x^4-10,67x^3-58,00x^2+720x+k
							808,75	-290,82				
8	y = 4x^3 -72x^2+404x-720	0	4	5	9	- + - +	4,47	7,53	4(x-4)(x-5)(x-9)	y=-168+164(x-2)	12x^2-144x+404	1x^4-24,00x^3+202,00x^2-720x+k
							4,51	-52,51				
9	y = -4x^3 +24x^2+76x-336	96	-4	3	7	+ - + -	5,21	-1,21	-4(x+4)(x-3)(x-7)	y=-120+124(x-2)	-12x^2+48x+76	-1x^4+8,00x^3+38,00x^2-336x+k
							145,74	-385,74				
10	y = -4x^3 +148x+336	672	-4	-3	7	+ - + -	3,51	-3,51	-4(x+4)(x+3)(x-7)	y=600+100(x-2)	-12x^20x+148	-1x^40,00x^3+74,00x^2+336x+k
							682,51	-10,51				
11	y = -3x^3 +12x^2+33x-90	42	-3	2	5	+ - + -	3,67	-1,00	-3(x+3)(x-2)(x-5)	y=45(x-2)	-9x^2+24x+33	-0,75x^4+4,00x^3+16,50x^2-90x+k
							44,44	-108,00				
12	y = -3x^3 +57x+90	126	-3	-2	5	+ - + -	2,52	-2,52	-3(x+3)(x+2)(x-5)	y=180+21(x-2)	-9x^20x+57	-0,75x^40,00x^3+28,50x^2+90x+k
							185,63	-5,63				
13	y = -2x^3 +4x^2+10x-12	-36	-2	1	3	+ - + -	2,12	-0,79	-2(x+2)(x-1)(x-3)	y=8+2(x-2)	-6x^2+8x+10	-0,5x^4+1,33x^3+5,00x^2-12x+k
							8,12	-16,42				
14	y = -2x^3 +14x+12	-60	-2	-1	3	+ - + -	1,53	-1,53	-2(x+2)(x+1)(x-3)	y=24-10(x-2)	-6x^20x+14	-0,5x^40,00x^3+7,00x^2+12x+k
							26,26	-2,26				
15	y = -x^3 +3x+2	-50	-1	-1	2	+ - + -	1,00	-1,00	-(x+1)(x+1)(x-2)	y=-9(x-2)	-3x^20x+3	-0,25x^40,00x^3+1,50x^2+2x+k
							4,00	0,00				
16	y = -x^3 +3x^2+1x-4	-16	-1	2	2	+ - + -	2,15	-0,15	-(x+1)(x-2)(x-2)	y=2+1(x-2)	-3x^2+6x+1	-0,25x^4+1,00x^3+0,50x^2-4x+k

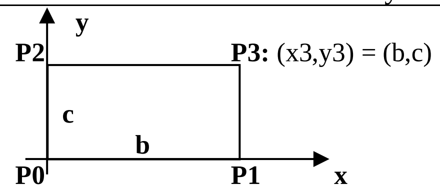
COUNT&ADD IN SPACE

Question	Answer
How to predict the position of points and lines?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14)
How to use the new calculation technology?	Computers can calculate set of numbers (vectors) and set of vectors (matrices)

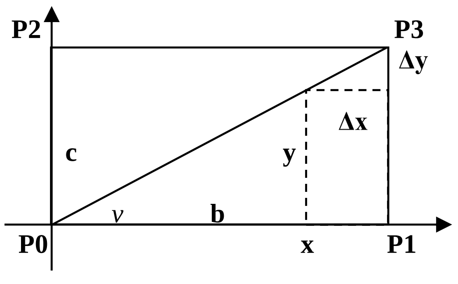
1 COUNTING POSITION

Question. How can we count the position of a point and a line?
Answer. By using a Cartesian coordinate-system assigning numbers to vertical and horizontal distances:
Example. A stack has 4 corners. The lower left corner is chosen as the centre form which steps are counted both in the horizontal direction (x-axis) and in the vertical direction (y-axis). The numbers are called the 1. and 2. coordinates coordinating points and numbers. The two axis are called a Cartesian coordinate system.

Points. In a c*b stack the 4 corner points have the coordinates
P0(x0,y0) = (0,0) is called the origin
P1(x1,y1) = (b,0) where b is the right/left-number
P2(x2,y2) = (0,c) where c is the up/down-number
P3(x3,y3) = (b,c)



Lines. The four side lines have the equations
P0P1: y=0, P2P3: y=c, P0P2: x=0, P1P3: x=b
The two diagonals have the equations:
P0P3: Δy = Δy/Δx * Δx
y-0 = (c-0)/(b-0) * (x-0)
y = c/b * x
or y = tan v * x
where v is the altitude angle of the diagonal.
P2P1: Δy = Δy/Δx*Δx = (0-c)/(b-0)*Δx = -c/b*Δx
or y-c = -c/b*(x-0), or y = -c/b*x + c



Distances. Horizontal and vertical distances are the differences between the coordinates: |P0P1|= |b-0| = b & |P0P2|= |0-c| = c where |b| means the numerical value of b: |±3| = 3.
The length of the diagonal is predicted by Pythagoras:
|P0P3|^2 = |P0P1|^2 + |P1P2|^2 = (x1-x0)^2 + (y3-y1)^2
= (b-0)^2 + (c-0)^2 = b^2 + c^2

The distance from P3 to the diagonal P2P1 is predicted by inserting the P3's coordinates in the distance-formula:
Distance = |a1x+b1y+c|/√(a1^2+b1^2)

The equation for a line y = m*x+b

Point&Point: (5,3) og (7,9)

y = ?	Δy = Δy/Δx * Δx
Δy = y – 3 Δx = x – 5 $\frac{\Delta y}{\Delta x} = \frac{9-3}{7-5} = \frac{6}{2} = 3$	y – 3 = 3*(x - 5) y = 3*x –15 + 3 **y = 3*x –12**

Point&Slope: (5,3) with the slope 2

y = ?	Δy = Δy/Δx * Δx
Δy = y – 3 Δx = x – 5 $\frac{\Delta y}{\Delta x} = 2$	y – 3 = 2*(x - 5) y = 2*x –10 + 3 **y = 2*x –7**

Exercise. Describe the corners, side lines and diagonals in a 3*5 stack.

2 PREDICTING INTERSECTION POINTS

Question. How can we predict the intersection point between two lines?
Answer. By inserting one line equation in the other, or by reversing vector calculation as in 4.
Intersection points. The two diagonals have the intersection point S(x,y):

(x,y) = ?	y = y
y = c/b*x y = -c/b*x+c	c/b*x = -c/b*x + c 2*c/b*x = c **x = b/2** **y =** c/b*b/2 = **c/2**
Check:	*c/b*b/2 = -c/b*b/2 + c* *c/2 = -c/2 + c* *c/2 = c/2*

(x,y)=?	y = y
y = a1x+b1 y = a2x+b2 D = a1b2-a2b1 $= \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$ D: Determinant a1 = c/b, b1 = 0 a2 = -c/b, b2 = c	a1x+b1 = a2x+b2 x(a1-a2) = b2-b1 x = (b2-b1)/(a1-a2) y = a1x+b1 = (a1*(b2-b1)+ b1*(a1-a2))/(a1-a2) = (a1*b2-a1*b1+ b1*a1-b1*a2)/(a1-a2) = (a1b2 - a2b1)/(a1-a2) = D/(a1-a2) x = (c-0)/(c/b+c/b) = c/(2*c/b) = c/2/c*b = b/2 y = (c/b*c-(-c/b)*0)/(c/b+c/b) = c/2

Exercise. Predict the intersection point of the diagonals in a 3*5 stack. And in a parallelogram.

3 VECTOR AND MATRIX

Question. How can we perform multiple simultaneous calculations?
Answer. By using number sets (vectors) and vector sets (matrices).
Number sets (vectors) are used to describe position in a coordinate system, and to describe goods and prices.
Example. Two goods weighing 15 and 35 kg are priced at 4$/kg and 6$/kg. The total value then is
T = 15*4 + 35*6. This can be described by two vectors, vertical or horizontal:

kg	$/kg	$
$\begin{pmatrix}15\\35\end{pmatrix}$	$\begin{pmatrix}4\\6\end{pmatrix}$	$\begin{pmatrix}15\\35\end{pmatrix} * \begin{pmatrix}4\\6\end{pmatrix} = 15*4 + 35*6 = 270$
$\begin{pmatrix}15 & 35\end{pmatrix}$	$\begin{pmatrix}4 & 6\end{pmatrix}$	$\begin{pmatrix}15 & 35\end{pmatrix} * \begin{pmatrix}4 & 6\end{pmatrix} = 15*4 + 35*6 = 270$

Vectors can be added	Vectors can be multiplied (a scalar product):
$\begin{pmatrix}15\\35\\10\\12\end{pmatrix} + \begin{pmatrix}40\\10\\5\\30\end{pmatrix} = \begin{pmatrix}15+40\\35+10\\10+5\\12+30\end{pmatrix} = \begin{pmatrix}55\\45\\15\\42\end{pmatrix}$	$\begin{pmatrix}15\\35\\10\\12\end{pmatrix} * \begin{pmatrix}40\\10\\5\\30\end{pmatrix} = 15*40 + 35*10 + 10*5 + 12*30 = 1360$

Exercise. Describe blending 5 sorts of tea by vectors.

4 REVERSING VECTOR CALCULATIONS

Question. How can we reverse vector calculations?
Answer. By using determinants.

Two vectors equations are united to	one matrix equation
$\begin{pmatrix}7 & 5\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = 7*x+5*y = 29$ & $\begin{pmatrix}8 & 3\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = 8*x+3*y = 25$	$\begin{pmatrix}7 & 5\\8 & 3\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}7*x+5*y\\8*x+3*y\end{pmatrix} = \begin{pmatrix}29\\25\end{pmatrix}$

A general matrix equation looks like this $\begin{pmatrix}a1 & a2\\b1 & b2\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a1*x+a2*y\\b1*x+b2*y\end{pmatrix} = \begin{pmatrix}c1\\c2\end{pmatrix}$. It can be solved by introducing the determinant of a matrix: Determinant $\begin{pmatrix}a1 & a2\\b1 & b2\end{pmatrix} = \begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix} = a1*b2 – a2*b1$

a1*x + a2*y = c1 (I)		b1*x + b2*y = c2 (II)
a1*x = c1 - a2*y a1*b1*x = c1*b1 - a2*b1*y →	a1*c2 – a1*b2*y = c1*b1 - a2*b1*y a1*c2 - b1*c1 = a1*b2*y - a2*b1*y a1*c2 - b1*c1 = (a1*b2 - a2*b1)*y $\frac{a1*c2 - b1*c1}{a1*b2 - a2*b1} = y$ $\frac{\begin{vmatrix}a1 & c1\\b1 & c2\end{vmatrix}}{\begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix}} = y$	b1*x = c2 - b2*y a1*b1*x = a1*c2 – a1*b2*y ←
a2*y = c1 - a1*x a2*b2*y = c1*b2 - a1*b2*x →	c1*b2 - a1*b2*x = a2*c2 - a2*b1*x c1*b2 - a2*c2 = a1*b2*x - a2*b1*x c1*b2 - a2*c2 = (a1*b2 - a2*b1)*x $\frac{c1*b2 - a2*c2}{a1*b2 - a2*b1} = x$ $\frac{\begin{vmatrix}c1 & a2\\c2 & b2\end{vmatrix}}{\begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix}} = x$	b2*y = c2 - b1*x a2*b2*y = a2*c2 – a2*b1*x ←

Example. Three different goods or spatial positioning involves 3x3 matrices leading to solving 3 equations with 3 unknowns by letting Excel calculate the determinants:

$\begin{pmatrix}3 & 5 & 6\\2 & 4 & 3\\6 & 4 & 2\end{pmatrix} * \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3*x+5*y+6*z\\2*x+4*y+3*z\\6*x+4*y+2*z\end{pmatrix} = \begin{pmatrix}52\\31\\42\end{pmatrix}$ or as 3 equations:	3*x + 5*y + 6*z = 52 2*x + 4*y + 3*z = 31 6*x + 4*y + 2*z = 42

$$x = \frac{\begin{vmatrix} 52 & 5 & 6 \\ 31 & 4 & 3 \\ 42 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-152}{-38} = 4 \qquad y = \frac{\begin{vmatrix} 3 & 52 & 6 \\ 2 & 31 & 3 \\ 6 & 42 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-76}{-38} = 2 \qquad z = \frac{\begin{vmatrix} 3 & 5 & 52 \\ 2 & 4 & 31 \\ 6 & 4 & 42 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-190}{-38} = 5$$

DETERMINANTS							
2x2 Matrix	7	29		3x3 Matrix	3	5	6
-57	8	25		**-38**	2	4	3
DoubleClick&Edit					6	4	2

Exercise. Solve the equation system

5 DISPLACEMENT AND ROTATION WITH VECTORS AND MATRICES

Question. How can we use vectors and matrices in geometry?
Answer. To predict displacements and rotations in the plane and in the space.

Example1. A displacement vector $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ displaces the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ going from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ to a vector going from $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

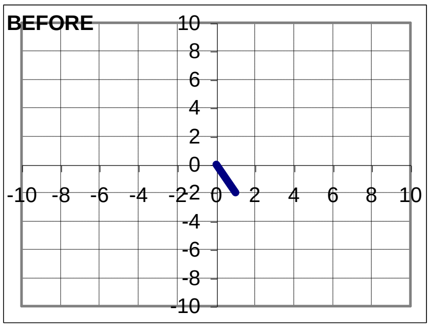
DISPLACEMENT

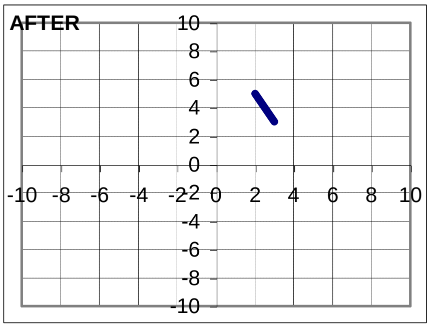
Δx	2
Δy	5

Enter Δx & Δy. Enter point2 before

		Point1	Point2
Before	x	***0***	***1***
	y	***0***	***-2***
After	x	***2***	***3***
	y	***5***	***3***

DoubleClick&Edit





Example2. A rotation matrix $\begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}$ rotates a vector v degrees around its starting point.

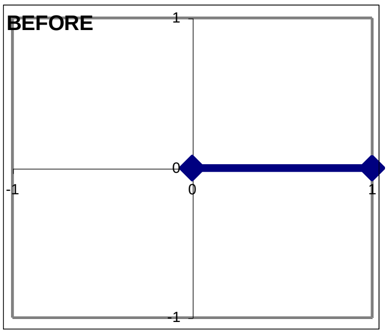
The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is rotated 63 degrees to the vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 63 & -\sin 63 \\ \sin 63 & \cos 63 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 63 \\ \sin 63 \end{pmatrix} = \begin{pmatrix} 0.454 \\ 0.891 \end{pmatrix}$

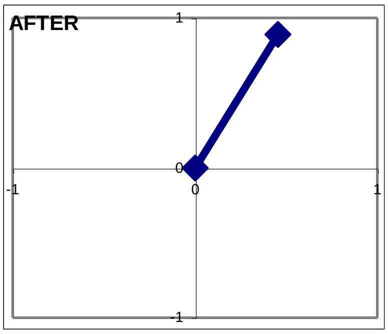
ROTATION *Enter Angle*

Angle	RotationMatrix	
63	**0,454**	**-0,891**
	0,891	**0,454**

DoubleClick&Edit

		Point1	Point2
Before	x	**0**	**1**
	y	**0**	**0**
After	x	**0**	**0,454**
	y	**0**	**0,891**





Exercise. Try other examples of displacements and rotation by editing the Excel-windows.

6 TRIANGLES WITH VECTORS AND MATRICES

Question. How can a triangle be predicted by vectors and matrices? **Answer**. By using the triangular vector formulas.
Example. A triangle has the vertices A(2,1), B(6,3) and C (4,6). Predict the sides, the angles and the area.
Method1. Frame&Cut. The triangle is framed in a rectangle, from which 3 right triangles are cut.
Method2. The triangle vectors are AB= (6-2, 3-1) = (4,2), BC= (4-6, 6-3) = (-2,3) and AC = (4-2, 6-1) = (2,5).
The side AB: IABI = √(AB*AB) = √(4,2)*(4,2) = √(4*4 + 2*2) = √20, IACI = √29 and IBCI = √13
The angle A: AB*AC=IABI*IACI*cosA, so cosA=AB*AC/IABI/IACI=(4*2+2*5)/√20/√29=0.747, A = 41.6
The area = ½*IÂB*ACI = ½*determinant(AB, AC) = ½*IABxACI = ½*I(-2,4)*(2,5)I = ½* I-4+20I = 8
Here ÂB is the AB's cross-vector, and x is the cross-product between the two vectors.
Exercise. A triangle has the vertices A(4,2), B(7,-1) and C (6,5). Predict the sides, the angles and the area.

7 PREDICTING CONIC SECTIONS

Question. How can we predict the shape of conic sections? **Answer**. By second degree equations. Rotated around the y-axis a line with altitude angle v forms a vertical cone cutting out different conic sections from a plane depending on the angle of intersection u.
The shapes are predicted by the formula y^2 = 2*x – (1-e^2)*x^2, where e is called the eccentricity:
circle (u = 0, e = 0); ellipse (0<u<v, 0<e<1); parabola (u = v, e = 1); hyperbola (u = 90, e>1).
Parallel displacement. The coordinate system K is displaced x1 to the right and y1 up to K'. Thus the origin in K' has the coordinates (x,y) = (xo,yo) in K, and (x',y') = (0,0) in K'. Thus the coordinates are related by: x' = x-xo and y' = y-yo.

	Centre in (0,0)	Centre in (xo,yo)
Circle	A circle consists of the points P(x,y) having a constant distance r to a given centre C = P1(0,0): x^2 + y^2 = r^2 (= ǀPCǀ^2). The tangent passing through P(xo,yo) is predicted by x*xo + y*yo = r^2.	In K' a circle with centre in (x',y') = (0,0) and radius r is predicted by x'^2 + y'^2 = r^2. In K the circle is predicted by (x-x1)^2+ (y-y1)^2 = r^2.
Ellipse	An ellipse consists of the points P(x,y) having a constant distance-sum to two given centres B1 and B2 (the foci). An ellipse has a horizontal major axis b and a vertical minor axis c: (x/b)^2 + (y/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by x*xo/(b^2) + y*yo/(c^2) = 1.	In K' an ellipse with centre in (x',y') = (0,0) and major and minor axis b and c is predicted by (x'/b)^2 + (y'/c)^2 = 1. In K the ellipse is predicted by ((x-x1)/b)^2 + ((y-y1)/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by (x-x1)*xo/(b^2) + (y-y1)*yo/(c^2) = 1.
Hyperbola	A hyperbola consists of the points P(x,y) having a constant distance-difference to two given centres B1 and B2 (the foci). An hyperbola has a horizontal major axis b and a vertical minor axis c: (x/b)^2 - (y/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by x*xo/(b^2) - y*yo/(c^2) = 1. When a hyperbola is turned 45 degrees, its equation becomes y = k/x.	In K' an hyperbola with centre in (x',y') = (0,0) and major and minor axis b and c is predicted by (x'/b)^2 - (y'/c)^2 = 1. In K the hyperbola is predicted by ((x-x1)/b)^2 - ((y-y1)/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by (x-x1)*xo/(b^2) - (y-y1)*yo/(c^2) = 1.
Parabola	A parabola consists of the points P(x,y) having the same distance to a given centre, the focus, and a given line, the directrix. The parameter p is twice the distance between the focus and the line. A vertical parabola is predicted by y = a*x^2 having the parameter p = 1/a. The tangent passing through P(xo,yo) is predicted by y+yo = 2*a*xo*x.	In K' an parabola with vertex in (x',y') = (0,0) and parameter p = 1/a is predicted by y'= a*x'^2. In K the ellipse is predicted by y-y1=a*(x-x1)^2. This can be transformed to: y = y+a*(x^2+x1^2 -2*x*x1) = a*x^2 + (-2*a*x1)*x + (y1+a*x1^2) = a*x^2 + b*x + c (#)

*(#) Since -2*a*xo= b, xo= -b/(2*a). Since yo+a*xo^2= c, yo= c-a*xo^2= c-a*b^2/(4*a^2)= -(b^2-4*a*c)/4*a = -D/4*a.*
In K the parabola vertex has the coordinates (xo,yo) = (-b/(2*a), -D/(4*a)), where the discriminant D = b^2-4*a*c.
Thus the intersection points between a parabola and the x-axis is xo±Δx, where a*Δx^2=D/(4*a) giving x = (-b±√D)/(2*a).
Example. What is the intersection points between x^2-14*x+y^2+6*y+33=0 and x-2*y-8=0?
x^2-14*x+y^2+6*y+33=0 gives x^2-2*7x+7^2+y^2+2*3y+3^2=-33+7^2+3^2 or (x-7)^2+(y+3)^2 = 25 = 5^2 i.e. a circle with centre (7,-3) and radius 5. x-2*y-8=0 gives x-8=2*y or ½*x-4=y i.e. a line with slope ½ and y-intercept –4.
Inserting x=2*y+8 is in x^2-14*x+y^2+6*y+33=0 gives (2y+8)^2-14(2y+8)+y^2+6y+33=0 or 5y^2+10y-15=0 having the solutions y=(-10±√(10^2-4*5*(-15)))/(2*5) = 1 & -3 giving x = 10 & 2. Intersection points: (x,y) = (10,1) & (2,-3).
Exercise. Predict & construct the intersections points between a conic section and a conic section or a line.

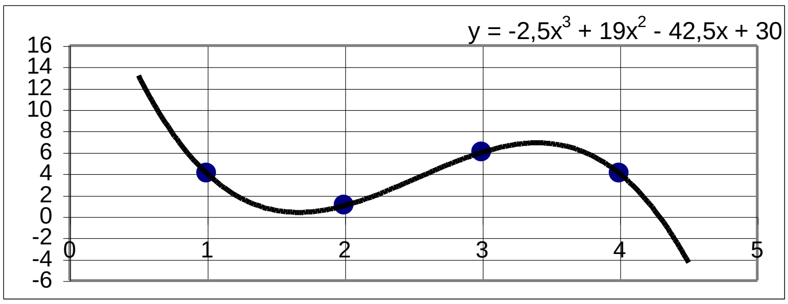
8 FITTNG CURVES

Question. How can we fit curves to points? **Answer**. By Excel trend lines or solving equations.
2 points. Through 2 points pass infinitely many 2^{nd} degree polynomials (parabolas) but only 1 line.
3 points. Through 3 points pass infinitely many 3^{rd} degree polynomials but only 1 2^{nd} degree polynomial.
4 points. Through 4 points pass infinitely many 4^{th} degree but only 1 3^{rd} degree polynomial y = a*x^3+b*x^2 +c*x+d. d is the initial level, c is the initial slope, b is the initial curvature and a is the counter-curvature.
a, b, c and d can be predicted by solving 4 equations with med 4 unknown or by the Excel trend line.

4 POINTS

	x	y
Point1	1	4
Point2	2	1
Point3	3	6
Point4	4	4

DoubleClick&Edit



Exercise. Try other examples of curve fitting by editing the Excel-window.

10. The ‘KomMod Report’, a Counter-report to the Ministry’s Competence Report

Allan Tarp, 2002, translated into English in 2017.

The KomMod report provides an alternative response to KOM-project terms of reference, in the expectation that the Science Board of education and the Ministry of education want to respect a common democratic IDC-tradition with Information and Debate between alternatives before a Choice is made. The report replies to the following 12 questions relating to mathematics education:

(a) What is the society's requirements for the education?

(b) To what extent is there a need to renew the existing education?

(c) How can education consider the new student type?

(d) What content can contemporary mathematics education have?

(e) How can the future of education be organized?

(f) How to secure progression and consistency in education?

(g) What impact will a modified education have for teacher training?

(h) Which competences and qualifications can be acquired at the various stages of the education?

(i) How can competences and qualifications be measured?

(j) How can future teaching materials look like?

(k) How to secure a continuous development of the education?

(l) How can Denmark exchange educational experience with other countries?

Ad a. Our democratic society needs citizens and specialists to have a common number-language to communicate about quantities and calculations. Society needs mathematics as a human right, both as a discursive qualification and as silent competence.

Ad b. There is a need to renew the current mathematics education to solve its three main problems: 1. There is a widespread number-languages illiteracy, where many citizens are reluctant to use the number-language. 2. There are major transition issues between primary, secondary, and tertiary education. 3. There is a decreasing enrolment to math-based education in science, technology, and economy, as well as a large shortage of new secondary school teachers in mathematics.

Ad c. Future mathematics teaching should respect today's democratic, anti-authoritarian youth and its requirements on meaning and authenticity. This can be achieved if the subject respects its historical roots and re-humanizes itself by presenting abstractions as abstractions and not as examples, i.e. as abstractions from examples (a function is a name for a formula with variable numbers), and not as examples of even more abstract abstractions (a function is an example of a set-relation). In short, the subject should portray itself as mathe-matics, recognizing its outside roots from which it has grown bottom up through abstractions. And the subject must say goodbye to the current ‘meta-matics’ and its belief that it has meta-physical roots and has grown top down as examples. Finally, the subject should respect the fact that people learn differently. Children learn by touching the world, i.e. by building competences. Young people learn by listening to the world, i.e. by building narratives and skills from the learning question "tell me something I don't know about something I know" (gossip-learning).

Ad d. Mathematics must respect its history as grown through abstractions, and thus also its construction as a number-language grammar, which can only be introduced after the number-language has been developed. The number-language has grown out of the meeting with quantity in time (repetition) and in space (many-ness). This meeting constructed numbers to describe the total, either through counting in pieces, bundles, bundles of bundles, bundles of bundles of bundles etc. Or faster by means of calculations to unite and divide unit-numbers (3$) and per-numbers (3$/day, 3%): Plus and minus unite and divide in variable unit-numbers (3 + 5 = ?, 3 + ? = 8). Multiplication and division unite and divide in constant unit-numbers (3 * 5 = ?, 3 * ? = 15). Potency and root &

logarithm unite and divide in constant per numbers (3 times 5% = ?%, 3 times ?% = 20%, ? times 5% = 20%). Integration and differentiation unite and divide variable per-numbers (5 seconds at 2 m/s growing evenly to 4 m/s =? m, 5 seconds of 2 m/s growing to 4 m/s = 18 m?). In short, the subject must respect the fact that geometry has grown out of what the word means in Greek, earth measurement; and respect that algebra has grown out of what the word means in Arabic, reunion, i.e. uniting and dividing constant and variable unit-numbers and per-numbers. Geometry and algebra must therefore respect their historical roots in an agricultural culture with two main questions: "How to share the Earth, and what it produces?" The number-language has several typical applications: Geometry deals with forms and shapes. Formulas deal with number levels. Growth deals with predictable change. Statistics/probability deals with change that is not pre-dictable but post-dictable. It is important to clean teaching of 'killer-Mathematics' (i.e. mathematics, that does not occur outside of the classroom, and that can only be used for one thing, killing students' interest). Addition should only occur within the parentheses, which ensures that the units are equal (T = 2 * 3 + 5 * 3 = (2 + 5) * 3 = 7 * 3 = 21). Fractions should only act together with their totals (1/2 of 2 plus 2/3 of 3 = 3/5 of 5). Equations should be solved by reversed calculation. Since the set concept cannot be well-defined it should be removed, and functions be postponed until it pops up historically after differential calculus.

Ad e. Future mathematics lessons can be organized in two main areas: Child math and youth math from respectively grade 1-7 and 8-12. Meeting the roots of mathematics roots, Many in time and space, will develop the learner's two core competences: to count and to add.

Ad f. Progression and consistency in teaching can be ensured by letting the child's math grow out of the local examples of Many, and of agricultural examples from rural and urban areas, and by letting the youth's math grow out of industrial culture and its global diversity. As well as by the child primarily working with unit-numbers, and young people primarily with per-numbers.

Ad g. By dividing education into the child's mathematics and the youth's mathematics, it will also be natural to divide teacher education in primary school teacher and secondary school teacher, as in the rest of the world approximately. This means that all future teacher-training takes place at a university. In the end, this will coincide with the division of the school into a primary school and secondary school that will take place within the next decade in connection with the high school collapse due to increased teacher retirement and decreasing enrollment of new teachers in mathematics and natural science.

Ad h. By meeting Many in time and space, the child develops competences in uniting and dividing constant and variable unit-numbers. In the countryside, bundling and re-bundling leads to multiplication and division. In the city, stacking and re-stacking leads to addition and subtraction. Calculating repetition and diversity develops the skills of young people to unite and divide constant and variable per-numbers. Totaling interest rates leads to power, root, and logarithm. Totaling distances leads to integral and differential calculus.

Ad i. Competences are tacit knowledge and can therefore be neither described nor measured but will evolve automatically through the meeting with meaningful and authentic situations and grow from the many concrete experiences with Many in time and space, bundling and stacking, uniting and splitting, unit-numbers and per-numbers. Qualifications is measured as now through three types of tasks: Routine tasks, text tasks and projects.

Ad j. Future teaching materials should be short and concise so that time could be dedicated for student learning through self-activity. The material should respect that students have two brains, a reptile's brain for routines and a human brain for conceptual understanding. There should therefore be training tasks with responses, so learners can progress at their own pace and do as many exercises as wanted. As well as textbooks telling how mathematics has grown from practice through layers of abstractions, and accepting different names so concept may be named both bottom-up and top-down, as e.g. growth by adding and linear function etc.

Ad k. A continuous development of education can be ensured by continuously relating mathematics to its roots and not to the current political correctness.

Ad l. Exchange of experience with foreign countries can be done through establishing a Danish development research, in which practitioners can combine being researcher at a university with being attached to a teacher team at a school. This will avoid the current barren 'ghost research' performed by researchers without experience background in teaching practice. Development research should be difference-research (Cinderella-research) using practice based and sociological imagination to discover and try out hidden alternatives.

The Difference between the KOM- and KOMMOD Reports

In mathematics education, the two main questions are: 'How do concepts enter into the world and into the student's head - from the outside or from the inside?' These questions give rise to different answers. Secondary school structuralism says 'outside-outside': Concepts exist in the meta-physical world, they are discovered by researchers and mediated by teachers. Primary school constructivism says 'outside-inside': Concepts exist in the meta-physical world, but are discovered through experimentation, in which each student constructs their own knowledge and abilities (schemata and competences), both being silent and only to be observed through use. Post-structuralism says the 'inside-outside': Concepts are created through invention and social construction, and should be presented as such. Apprenticeship says 'inside-inside': Concepts are constructed by the apprentice during the participation in the master's practice. Worldwide, two knowledge wars rage, a math-war between structuralism and constructivism, and a science-war between structuralism and post-structuralism. Instead of acknowledging this diversity, the report is trying to conceal it by taking over the core constructivist concept, competence, but giving it a structuralist content (insight-based action-readiness). The French philosopher Foucault has shown how new words create new clients: 'Qualification' creates the unqualified, and 'competent' creates the incompetent. But where the unqualified can cure themselves by qualifying themselves, the incompetent cannot cure themselves by 'competencing' themselves, and are thus left to be cured by others, the competence-competent. Adoption and modification of the word competence can therefore be interpreted as a structuralist attempt to win the math-war by a coup, instead of using it to a fruitful dialogue with equal partners.

First structuralism tried to solve the math-crisis through the wording 'responsibility for your own learning'. Students took this seriously and turned their back to 'meta-matics' with its meaningless self-reference (a function is an example of a set-relation: bublibub is an example of bablibab).

Now instead the teachers are disciplined and incapacitated by constructing them as incompetent, with a consequent need for competence development through massive in-service training. Omitting the competence 'experimenting' shows that the report only respects science as an end-product, and neither the process nor its roots in the outside world. Neither does it respect the way in which young people and especially children acquire knowledge through self-activity and learning.

Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical professionalism. This makes it a report on 'catholic mathematics' with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a 'protestant mathematics' that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add; and emphasizes that linguistic competence precedes grammatical competence. Meaning that also with quantitative competence, the number-language comes before its grammar, mathematics; and as with the word-language, grammar remains a silent competence for most.

Will the math-war end with a KOM-coup? Or will it be settled through a democratic negotiation between opposing views? The choice is yours, and the KomMod report gives you an opportunity to validate the arguments, not from above from political correctness, but from below from the historic roots of mathematics. Best of luck.

SET-based 'MetaMatics', or Many-based ManyMatics: Learning by Meeting the Sentence or by meeting its Subject

Class 1-2

SETS are united: addition
2 + 3 = 5
47 + 85 = 135
82-65 = 17
PROBLEM:
Addition is a false abstraction: 2 m + 3 cm = 203 cm
2 weeks + 3 days = 17 days
2 C + 3 D = 23 D
3 stones = stone + stone + stone

Country: Bundle & ReBundle
Multiplication is true abstraction:
3 stones = 3 times stone = 3·stone
2·3·days = 6·days
2·m·3·cm = 6·m·cm = 600 cm ^ 2
Bundling and ReBundling:
Total = 6 1s =? 2s
Response: 6·1 = 6 = (6/2)·2 = 3·2
ReBundling-rule: T = (T/b)·b
6/2: Counted in 2s
6·2: Counting 2s
To find the total, count or calculate: ReBundling (division)
Multiplication rebundles in tens:
T = 8·3 = 24 = 2·ten & 4·1
= 2·D & 4·1
Multiplication is division!
Max-height 3:
T = 8 3s = overload
T = 8·3 = 2·3 ^ 2 & 2·3
Unbundled can also bundled in parts, for example in 5s:
T = 8·3 = (24/5)·5 = 4·5 & 4·1
= 4·5 & (4/5)·5 = (4 4/5)·5

Class 3-4

SETS are repeated: multiplication
2·3 = 6
7·85 = 595
372/7 = 53 1/7

City: Stacks & ReStack
T = 653 + 289 =?
653 = 6·C & 5·D & 3·1
279 = 2·C & 7·D & 9·1
T = 8·C & 13·D & 12·1
T = 8·C & (13 + 1)·D & (12-10)·1
T = (8 + 1)·C & (14-10)·D & 2·1
T = 9·C & 4·D & 2·1 = **942**
<u>ReStack rule: T = (T – b) + b</u>
T = 654-278 =?
653 = 6·C & 5·D & 4·1
278 = 2·C & 7·D & 8·1
T = 4·C & -2·D & -4·1
= (4-1)·(C) & (-2 + 10)·D & -4·1
= 3·(C) & (8-1)·(D) & (-4 + 10)·1
= 3·C & 7·D & 6·1 = 376

T = 7·653 =?
T = 7·(6·C & 5·D & 3·1)
= 42·C & 35·D & 21·1
= 42·(C) & (35+2)·(D) & (21-20) ·1
= (42 + 3)·(C) & (37-30)·D & 1·1
= 45·C & 7·D & 1·1 = 4571

T = 653/7 =?
T = 6/7·C & 5/5·D & 3/7
= 65/7·D & 3/7
= (65-2)/7·(D) & (20 + 3)/7
= 9·D & 23/7
= 9·D & 3 2/7 = 93 2/7
(double book-keeping)

Class 5-6

SETS are divided: fractions
$\frac{1}{2}+\frac{2}{3}=?$
$\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$
PROBLEM:
$\frac{1}{2}+\frac{2}{3}=\frac{1+2}{2+3}=\frac{3}{5}$ if
1 coke of 2 bottles plus 2 cokes of 3 bottles is (1 + 2) cokes of (2 + 3) bottles.

City: weighted average
$T=\frac{1}{2}\cdot 2+\frac{2}{3}\cdot 3=3=\frac{3}{5}\cdot 5$, or
$T=\frac{1}{2}\cdot 4+\frac{2}{3}\cdot 3=4=\frac{4}{7}\cdot 7$
So there are many different answers to the question
$\frac{1}{2}+\frac{2}{3}=?$
But NEVER more than 1!
Trade calculation
5 kg cost 60 \$, 3 kg cost ? \$

ReBundle \$	ReBundle kg
\$ = (\$/kg)·kg \$ = (60/5)·3 \$ = 36	3 kg = (3/5)·5kg 3 kg = (3/5)·60\$ 3 kg = 36\$

Percentages part 1
• 8 has 2, so 100 has ?
100 = (100/8)·8 has (100/8)·2 = 25
• 100 has 25, so 8 has ?
8 = (8/100)·100 has (8/100)·25 = 2
• 100 has 25,so ? has 2
2 = (2/25)·25 had by (2/25)·100 = 8

Class 6-7

Solution-**SETS**: open statements (equations)

2 + 3·x = 8 (2+3·x)-2 = 8-2 (3·x + 2)-2 = 6 3·x + (2-2) = 6 3·x + 0 = 6	3·x = 6 (3·x)/3 = 6/3 (x·3)/3 = 2 x·(3/3) = 2 x·1 = 2

L = {x ∈ R | 2+3·x = 8} = {2}
PROBLEM:
The weight-metaphor hides the count process, and creates many error possibilities as e.g.
If 2 + 3·x = 8, then 5·x = 8

Castle & Monastery: Coding 2 + (3·5) = 17 → 2 + (3·x) = T
DeCoding (solving an equation):
ReStacking 8 in two stacks:
2 + (3·x) = 8 = (8-2) + 2
3·x = 8-2 = 6
ReBundling from 1s to 3s:
3·x = 6 = (6/3)·3
x = 6/3 = 2
Forward- & back calculations :
To opposite side with opp. sign

<u>Forward</u>		<u>Back</u>
2 + 3·x	=	8
+ 2	↑↓	*-2*
3·x	=	8-2 = 6
·3	↑↓	*/3*
x	=	6/3 = 2

Percentage part 2
• 25% of 8 is ?
0.25·8 = x
• 25% of ? is 2
0.25·x = 2, så x = 2/0.25 = 8
• ? % of 8 is 2
x·8 = 2, so x = 2/8 = 0.25 = 25%

Class 8-9

SETS are connected: functions
Function: an example of a many-one set-relation
E.g. f (x) = 2 + 3·x
A function's value and graph
PROBLEM:
The function came after calculus!
A syntax error to confuse the language and meta-language: the function's value corresponds to the verb's tie.

City: Trade and Tax
Per-numbers: Tax, custom, exchange and interest rates, profit, loss, bonds, assurance.
Adding per-numbers:
3 kg at 4\$/kg + 5 kg at 6\$/kg gives 8 kg at ? \$/kg
Geometry: area and volume of plane and spatial forms. Right-angled triangles: Pythagoras, sine, cosine & tangent.
Linear funct.: growth by adding:
T = b + a + a + a + … = b + a·n
A function is a name for a calculation with variable numbers, such as. T = 2 + 3 x. (Euler 1748)
Calculations give fixed and functions give variable number.
The change of a function can be shown in tables or on curves.
The Inn: Redistribution by games
Winn on pools, lotto, roulette.
Statistics counts number of wins.
Risk = Consequence·propability.

Class 10	Class 11	Class 12
Set theory Function theory: Domains & values. Algebraic functions: Polynomials and polynomial fractions. First- & second-degree polynomials. Trigonometry. Analytical geometry.	Function theory: reverse and composite function. Non-algebraic functions: trigonometric functions. Logarithm- & exponential functions as homomorphisms: f (x*y) = f (x) # f (y) Stochastic functions. Core calculus.	Vector spaces. Main calculus. Simple differential equations.
The Renaissance: Constant per-numbers Numbers as many-bundles (polynomials): T = 2345 = 2·B ^ 3 + 3·B ^ 2 + 4·B + 5·1 Reversed calculations with powers: B ^ 4 = 81 4 ^ n = 1024 B = 4 √ 81 n = log1024/log4 Interest rates: Single r, total R, compound RR (1 + r)^n – 1 = R = n·r + RR Change with constant per-number and percentage: x: + 1 → T: + a$ linear change T = b + a·x x: + 1 → T: + r% exponential T = b·(1+r)^x x: + 1% → T: + r% power change T = b·x ^ r x: + 1 → T: + r% + a$ savings T = a·R/r Change with unpredictable (random) variation Δ T =? T = MID ± 2·SPR Adding percentages by their areas (integration): 300$ at 4% and 500$ at 6% is 800$ at ? %. Change percentage: T = a·b: ΔT/T ≈ Δa/a + Δb/b T = a/b: ΔT/T ≈ Δa/a – Δb/b Trigonometry: SIN & COS: short sides in percent of the long. TAN: the one short side in percent of the other.	**Industry: Variable per-numbers** Coordinate geometry: Geometry & algebra, always together, never apart. Curve fitting with polynomials: T = A + B·x + C·x^2 + D·x^3 (or y = A + B·x + C·x^2) A: level, B: rise, C: curvature, D: counter-curvature Variable, predictable change: Differential calculus: dT = (dT/dx)·dx = T'· dx The non-linear is locally linear: (1+r)^n ≈ 1 + n·r (= 1 + n·r + RR: with a small interest, the compound-interest can be neglected) T = x^n: dT/T = n·dx/x, dT/dx = n·T/x = n·x^(n-1) Optimization tasks in engineering and economics. Integral calculus: Δ T = T2 – T1 = ∫ dT = ∫ f·dx, Total change = terminal – start = the sum of single changes, regardless of their number or size. Integration is done by rewriting to change form: Since 6·x^2 + 8·x = d/dx (2·x^3 + 4·x^2) = d/dx(T) then ∫(6·x^2 + 8·x) dx = ∫ d(2·x^3 + 4·x^2) = ∫ dT = ΔT = T2 – T1 Accumulation tasks in engineering and economics.	**Major works in the Quantitative Literature:** Geometry, Trade, Economics, Physics, Biology. The three genres for quantitative literature: - *Fact or since-then calculations* quantifies the quantifiable, and calculates the calculable: since the price is 4$/kg, then the cost of 6 kg is 6·4$ = 24$. - *Fiction or if-then calculations* quantifies the quantifiable, and calculate the incalculable: if my income is 4m$/year, then 6 years of income will be 6·4m$ = 24 million $. - *Fiddle or so-what calculations* quantify the non-quantifiable: If the consequence 'broken leg' C is taken to be 2 million $, and if the probability p is taken to 30%, then the risk R will be R = C·p = 2m$·0.3 = 0.6 million $. The three courses of action: fact models are controlled especially for the units; fiction models are supplemented with alternative scenarios; fiddle models are referred to a qualitative treatment. Change equations solved by numerical integration. Functions of two variables. Differentiation and integration. Optimization and accumulation. Vectors used in trade and in the movement on a surface and in space.

11. Word Problems

Unit-number tasks

Type1.1 Numbers

Two numbers have the sum 72, and one is twice as large as the other. What are the numbers?

Text	*Numbers*	*ANSWER*	*Equation*
Number1	x = ?	*24*	x + y = 72
Number2	y = 2*x	*48*	x + 2*x = 72 3*x = 72 x = 72/3 = 24

Type1.2 Money

A pays a bill of 210$ with three types of coins: 1s, 2s and 5s. There are 4 times as many 1s as 2s, and 20 fewer 2s than 5s. How many coins of each type were used?

Text	*Numbers*	*ANSWER*	*Equation*
5s	x = ?	*30*	x*5 + (x-20)*2 + 4*(x-20)*1 = 210
2s 1s	x-20 4*(x-20)	*10* *40*	5*x + 2*x – 40 + 4*x – 80 = 210 11*x = 210 + 120 x = 330/11 x = 30

Type1.3 Age

A is 4 times as old as B. 5 years ago, A was 7 times as old as B. How old are A and B now?

Text	*Numbers*	*ANSWER*	*Equation*
B's age now	x = ?	*10*	7*(x-5) = 4*x – 5
A's age now B's age then A's age then	4*x x - 5 4*x - 5	*40*	7*x – 35= 4*x – 5 7*x – 4*x = – 5 + 35 3*x = 30 x = 30/3 x = 10

Type1.4 Geometry

A rectangle has a circumference of 224 meters. The length is 4 meters shorter than 3 times the width. What is length and width?

Text	*Numbers*	*ANSWER*	*Equation*
Width	x = ? meters	*29*	2*x + 2*(3*x-4) = 224
Length	3*x-4 meters	*83*	2*x + 6*x – 8 = 224 8*x = 224 + 8 x = 232/8 x = 29

Type1.5 Lever

A, B and C settle on a seesaw, B and C on the same side. They weigh 100kg, 80kg and 40kg respectively. A and B both sit 3 meters from the focal point. Where must C sit for equilibrium?

Text	*Numbers*	*ANSWER*	*Equation*
C's meter-tal	x = ?	*1.5*	100*3 = 80*3 + 40*x
A's contribution B's contribution C's contribution	100*3 80*3 40*x		300 = 240 + 40*x 300 – 240 = 40*x 60/40 = x 1.5 = x

Tasks.

1. Two numbers have the sum 48, and one is twice as large as the other. What numbers are they?
2. Two numbers have the sum 48, and one is three times as large as the other. What numbers are they?
3. A pays a bill of 290 kr. with three types of coins: 1ere, 2ere and 5ere. There are 5 times as many 1s as 2s, and 10 fewer 2s than 5s. How many coins of each type were used?
4. A pays a bill of 200 kr. with three types of coins: 1ere, 2ere and 5ere. There are 3 times as many 1s as 2s, and 20 more 2s than 5s. How many coins of each type were used?
5. A is 5 times as old as B. 4 years ago, A was 6 times as old as B. How old are A and B now?
6. A is 8 times as old as B. 5 years ago, A was 9 times as old as B. How old are A and B now?
7. The circumference of a rectangle is 128 meters. The length is 4 meters longer than 5 times the width. What is length and width?
8. The perimeter of a rectangle is 110 meters. The length is 5 meters shorter than 4 times the width. What is length and width?
9. A, B and C settle on a seesaw, B and C on the same side. They weigh 120kg, 60kg and 50kg respectively. A and B both sit 4 meters from the focal point. Where must C sit for equilibrium?
10. A, B and C settle on a seesaw, B and C on the same side. They weigh 90kg, 70kg and 20kg respectively. A and B both sit 2 meters from the focal point. Where must C sit for equilibrium?

Per-number tasks

In per-number tasks, they must always be converted to unit-numbers before the equation can be established.

Travel

Train1 runs from A to B at a speed of 40 km/h. Two hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*
Hours		x = ?	*4*	40*(x+2) = 60*x
Speed1	40 km/h			40*x + 80 = 60*x
Speed2	60 km/h			80 = 60*x – 40*x = 20*x
Km-number1		40*(x+2) km	*240*	80/20 = x
Km-number2		60*x km	*240*	4 = x

Train1 runs from A to B at a speed of 40 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 300 km?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*
Hours		x = ?	*4*	40*x + 60*x = 300
Speed1	40 km/h			100*x = 300*x
Speed2	60 km/h			x = 300/100
Km-number1		40*x km	*120*	x = 3
Km-number2		60*x km	*180*	

The same distance takes 3 hours upstream, and 2 hours downstream. What is the speed of the motorboat when the speed of the current is 5 km/h?

Text	*Per-num.*	*Unit numbers*	*ANSWER*	*Equation*
Speed	x = ? km/h		*25*	km = km/h*h = (x-5)*3 = (x+5)*2
Speed upstream	x – 5 km/h		*20*	3*x-15 = 2*x+10
Speed downstream	x + 5 km/h		*30*	3*x-2*x = 10+15
Hours		3 hours		x = 25

Mixture

? Liter 40% alcohol + 3 liters 20% alcohol gives ? liter 32% alcohol

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*
Liters		x = ? liters	*4.5*	0.4*x + 0.2*3 = 0.32*(x+3)
Liters-number3		x+3 liters	*7.5*	0.4*x + 0.6 = 0.32*x + 0.96
Alcohol1	40%	0.4*x liters		0.4*x - 0.32*x = 0.96- 0.6
Alcohol2	20%	0.2*3 liters		0.08*x = 0.36
Alcohol3	32%	0.32*(x+3)	liters	x = 0.36/0.08 = 4.5

Finance

A invests a gain of 400,000$ in the following way: Some is set at a rate of return at 3% p.a., the rest is put into 8% bonds. How much did he invest in each when the annual dividend is $20,000?

Text	*Per-num*	*Unit numbers*	*ANSWER*	*Equation*	
Bank in a thousand		x = ? $	*240*	3%*x + 8%*(400-x)	= 20
Bonds in thousands		x+3 $	*160*	0.03*x + 32 – 0.08*x	= 20
Interest rate in bank	3%			32 - 20	= 0.08*x – 0.03*x
Interest rate on bonds	8%			12	= 0.05*x
The Bank's contribution		3%*x $		12/0.05	= x
Bonds' contribution		8%*(400-x) $		240	= x

Work

A can dig a trench in 4 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*
Time		x = ? hours	*12/7*	¼*x + 1/3*x = 1
A's speed	1/4 trench/h			(¼ + 1/3)*x = 1
B's speed	1/3 trench/h			7/12*x = 1
A contributes		¼*x		x = 12/7
B contributes		1/3*x		

Tasks.

1. Train1 runs from A to B at a speed of 50 km/h. Three hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?
2. Train1 runs from A to B at a speed of 50 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 400 km?
3. Same distance 4 hours countercurrent, and 3 hours downstream. What is the speed of the motorboat when the speed of the current is 6 km/h?
4. A can dig a trench in 5 hours. B can dig the same trench in 4 hours. How long does it take to dig it together?
5. A can dig a trench in 6 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Mechanics

M1. A ball falls from the top of a skyscraper (air resistance is disregarded). After 0 seconds, the ball is at an altitude of 300 meters. After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 0 meters. What is the impact velocity?.

Height after 5 sec: *Time:* *Speed:*

s = ? meter	s = ½*g*t^2		t = ? sek.	s = ½*g*t^2		v = ? m/s	v = g*t
t = 5 sek. g = 9.8 m/s^2 Height = ?	s = ½*9.8*5^2 s = 123.7 meter H = 300-123.7 H = 177.3 m		s = 300 m g = 9.8 m/s^2	2*s/g = t^2 √(2*s/g) = t √(2*300/9.8g) = t 7.82 seconds = t		t = 7.82 sek. g = 9.8 m/s^2	v = 9.8*7.82 v = 76.6 m/s

M2. A ball is shot vertically up at an initial velocity of 30 m/s (air resistance is disregarded). After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 40 meters. After? seconds is the ball at maximum height?

Height after 5 sec. *Time to 40 m:*

s = ? meter	s = ½*g*t^2+vo*t		t = ? sek.	s = ½*g*t^2+vo*t
t = 5 sek. g = -9.8 m/s^2 vo = 30 m/s	s = -½*9.8*5^2+30*5 s = 27.5 meter		s = 40 m. g = -9.8 m/s^2 vo = 30 m/s	40 = -4.9*t^2+30*t 4.9*t^2-30*t+40 = 0 t = 1.96 and 4.16 seconds

Rising time until speed = 0 *Rising height:*

t = ? sek.	v = g*t+vo		s = ? meter	s = ½*g*t^2+vo*t
v = 0 m/s g = -9.8 m/s^2 vo = 30 m/s	(v-vo)/g = t (0-30)/(-9.8) = t 3.1 seconds = t		t = 3.1 sek. g = -9.8 m/s^2 vo = 30 m/s	s = -½*9.8*3.1^2+30*3.1 s = 45.9 meter

The height part of the task can also be counted as a task in the conversion of energy from kinetic to potential energy.

Rising height	h = ? meters	Ep = Ek
Rising time Acceleration Initial speed Kinetic energy Potential energy	t = 3.1 sek. g = -9.8 m/s^2 vo = 30 m/s Ek = 1/2*m*v^2 Ep = m*g*h	m*g*h = 1/2*m*v^2 h = 1/2*v^2/g h = 1/2*30^2/9.82 h = 45.8 meters

M3. A 100 kg person performs a Bounty jump from a bridge (air resistance is disregarded). It is 220 meters down. The feet are fixed in a rope of 120 meters, which is fixed in a spring with spring constant k = 100 N/m, corresponding to 10 kg being able to extend the spring 1 m. How far does the person get down? What if the person weighed 150 kg?

Spring dislocation	x = ? metre	Ef = Eb
Fall distance Acceleration Kinetic energy Potential energy Spring energy	d = 120+x g = -9.8 m/s^2 Ek = 1/2*m*v^2 Ep = m*g*h If = 1/2*k*x^2	1/2*k*x^2 = m*g*h x^2 = 2*m*g*h/k x =√(2*m*g*h/k) x = √(2*100*9.82*120/100) x = 48.5 d = 120+48.5 = 168.5 meters

M4. A person swings in a swing (air resistance is disregarded). The swing set is 4 m high and cord length is 3 m. What is the oscillation time? In the extreme position, the fluctuation is 50 degrees. What is the maximum speed? How far is the jump if the takeoff is in the bottom position?

Swing time	T = ? seconds	T = 2*π* √(l/g)
Cord length Acceleration	l = 3 m g = -9.8 m/s^2	T = 2*π* √(3/9.82) T = 3.47 seconds

Rise height at 50 degree oscillation *Maximum speed at 0 degree oscillation:*

s = ? meters	s = l – l*body v		v = ? m/sec.	Ek = Ep
l = 3 meters v = 50 degrees	s = 3 – 3*cos50 s = 1.07 meters		h = 1.07 m g = 9.8 m/s^2	1/2*m*v^2 = m*g*h v^2 = 2*g*h v =√(2*g*h) v = √(2*9.82*1.07) v = 4.58 meters/seconds

Drop time at 0 degree oscillation *Jump length at 0 degree oscillation:*

t = ? seconds	s = ½*g*t^2		s = ? meter	s = v*t
s = 4-3 = 1 meters g = 9.8 m/s^2	2*s/g = t^2 √(2*s/g) = t √(2*1/9.82) = t 0.45 seconds = t		v = 4.58 m/s t = 0.45 s	s = 4.58*0.45 s = 2.06 meters

The Economic Flow Diagram

The basic economic flow consists of two sectors, production and private households. We have a number of needs that we meet by producing goods and services for others. In return, we receive an income that we can use to cover our own needs. This creates the basic economic flow consisting of the two money flows: Production creates an income A (wage), which is used for consumption C (food, clothing, etc.), which in turn leads to a new production, which in turn creates new consumption, etc. If income and consumption are in balance, the economic flow is stable. However, there is a drain and a source in the flow: Savings B and investments D. Savings are money that is not spent on consumption. Investment is money used to buy goods that cannot be consumed, e.g. buildings and machinery, etc.

The flow is stable if savings and investments are in balance. If savings are greater than investment, the flow will shrink, resulting in mass unemployment. This was the case after the First World War, when Germany was forced to send money to France as war reparations without France being obliged to buy German goods for the money. This caused the English economist J. M. Keynes to withdraw from the peace negotiations.

And this was the case in the United States during the Great Depression of the 1930s, where investment in stocks fell dramatically after the Great Wall Street crash of 1929, and where savings increased in order to repay the large loans taken out to participate in speculation on the stock market.

An economic flow with a state sector.

Keynes showed how a third public sector can balance a two-sector circuit. The public sector pulls taxes E out of the loop and uses this money to pump money back into the flow through transfer income H to the unemployed, public consumption F (more public employees, etc.) and public investment G (more roads, etc.). The public authorities may borrow loans, which will be repaid when the flow is back in balance.

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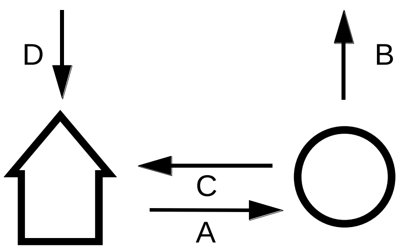

production households
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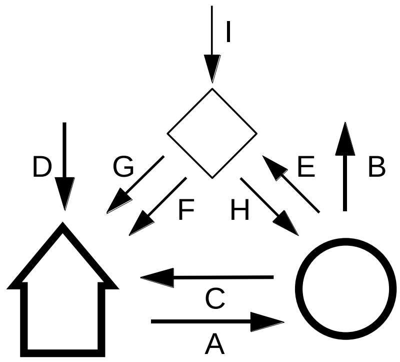

production state households
</td>
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<td>
A model of the basic economic flow could contain 4 equations:
First trip:
1 Initial consumption Co = 100
2 Initial savings Bo = 20
3 The investment is assumed to be a constant percentage of consumption Do = d*Co
4 Income is consumption plus investment Ao = Co+Do
Next trip:
1 Consumption is assumed to be a constant percentage of income C1 = c*Ao
2 Savings are the income that is not consumed B1 = Ao–C1
etc.
2 and 4 are fact equations, 1 and 3 are fiction equations. That is, the model as such is a fiction that should be supplemented with alternative models and scenarios.
For example. Could proportionality equations 1 and 3 be replaced by linearity equations:
1 C1 = c*Ao -> C1 = c*Ao + K
4 D = d*C -> D = d*C + L
Finally, an intervention can be made that changes the investment rate d from d to d+i
D = d*C -> D = (d+i)*C
This justifies the introduction of a state sector in the flow.

In both cases, the systems of equations can be solved on an Excel spreadsheet:
</td>
<td>
A model for this 3-sector economic flow with 9 equations:
First trip:
1 Initial consumption Co = 100
2 Initial savings Bo = 20
3 Initial investment Do = 20
4 Income is consumption plus investment Ao = Co+Do
5 Initial transfers Ho = 4
Next trip:
1 Taxes are assumed to be a constant percentage of income and transfers E1 = e*(Ao+Ho)
2 Private consumption is assumed to be a constant percentage of availability C1 = c*(Ao+Ho–E1)
3 Savings are the disposable amount that is not consumed: B1 = Ao+Ho–E1–C1
4 Government consump. assumed to be constant F1= constant
5 Public investment assumed to be a constant percentage of the investment gap G1 = g*(B1–Do)
6 Private investment is assumed to be a constant percentage of consumption D1 = d*(C1+F1)
7 The next income is that produced for consumption and investment A1 = Co+Do+Fo+Go
8 Transfers are assumed to be a constant percentage of the employment gap H1 = h*(Ao-A1)
9 Borrowing is the difference between taxes and government expenditure I1 = E1-F1-G1-H1
10 Debt is the summed up borrowing
3, 7 and 9 are fact equations, the rest are fiction equations. That is, the model as such is a fiction that should be supplemented by alternative models and scenarios.
</td>
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</table>

Simulation of the economic cycle with and without a public sector

Change the rates in cells C5-C10
Don't edit the equations

Rates

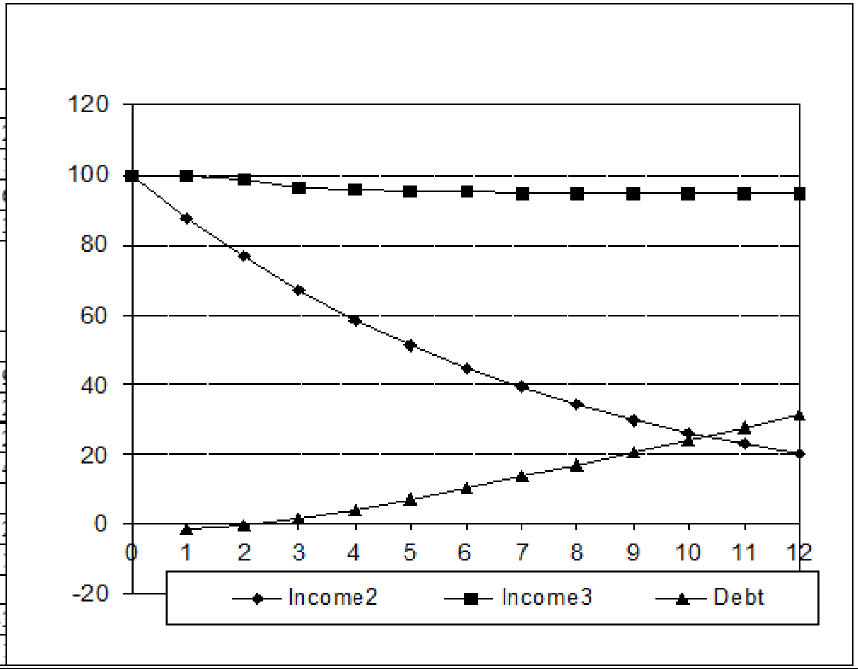
Time	n	0	1	2	3	4	5	6	7	8	9	10	11	12
Private consumption	c	**70%**	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%
PrivateInvestment	d	**25%**	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%
Tax	e	**30%**	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%
Public investment	g	**50%**	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Income transfer	h	**50%**	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Public consumption	F	**20**	20	20	20	20	20	20	20	20	20	20	20	20

2sector model

Time	*n*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Consumption	C	80	70	61,25	53,59	46,89	41,03	35,9	31,42
Savings	B	20	30	26,25	22,97	20,1	17,59	15,39	13,46
Investment	D	20	17,5	15,31	13,4	11,72	10,26	8,976	7,854
Income2	A	100	87,5	76,56	66,99	58,62	51,29	44,88	39,27

3sector model

Time	*n*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Income3	A	100	100	98,7	96,25	95,85	95,1	94,98	94,75
Transfers	H	4	0	0,65	1,875	2,074	2,449	2,51	2,625
Tax	E		31,2	30	29,81	29,44	29,38	29,27	29,25
Private consumption	C	80	50,96	49	48,68	48,08	47,98	47,8	47,77
Public consumption	F		20	20	20	20	20	20	20
Savings	B	20	21,84	21	20,86	20,61	20,56	20,49	20,47
PrivateInvestment	D	20	17,74	17,25	17,17	17,02	17	16,95	16,94
PublicInvestment	G		10	10	10	10	10	10	10
Borrowing	I		1,2	-0,65	-2,07	-2,637	-3,071	-3,245	-3,378
Debt			-1,2	-0,55	1,52	4,157	7,228	10,47	13,85



Meeting many in a STEM context

OECD (2015b) says: "In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth." STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e., tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force.

In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e., from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about 22.4*1000 liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times.

But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, $p*V = n*R*T$, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used.

The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out. The leaving steam is visible on e.g., steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes.

In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per 'carrier' (charge-unit). Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law 'Volt = Resistance*Ampere', i.e., 220 = Resistance*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120-Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a recounting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g., 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

How to construct a road up a steep hill side

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2. In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2. In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Jumping from a swing

When I jump from a swing I have the maximum speed at the bottom point. But here the angle with horizontal begins increasing from zero. But now the speed decreases. So, a what point should I jump to obtain a maximum length?

● SECTION II, REFLECTING on the New Paradigm

12. A short History of Mathematics

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means 'earth measuring' in Greek. Algebra means 'reuniting' in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces? Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus trade was only possible with those highlanders that had silver in their mountains. The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number system and algebra. The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2.

Pythagoras generalized his findings by claiming: All is numbers. This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don't know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation. The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g; and the distance s and the time t are connected by the formula: s = ½*g*t^2. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middle men. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine. The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon.

But how does the moon move? The church said 'among the stars'. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth. But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance. Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system, but could not prove it without sending up new planets.

Newton, however, could validate his theory by different examples of falling and swinging bodies. Newton's discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

13. What is Math - and why Learn it?

"What is math - and why learn it?" Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, 'mathematics' means 'knowledge'. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms, and numbers.

Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite.

With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

"The table is green" and "The total is 3 4s" or "T = 3*4".

Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles.

Operations are icons also:

Counting 8 in **2s** can be predicted by <u>division</u>, iconized by a broom pushing away **2s**:

8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by <u>multiplication</u>, iconized by a lift pushing up the **2s**:

8 = 4x2.

Looking for unbundled can be predicted by <u>subtraction</u>, iconized by a rope pulling away the 4 **2s**:

8 – 4x2.

Uniting bundles and singles is predicted by <u>addition</u>, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a 'recount-formula':	**T = (T/B)*B**	
saying 'From T, T/B times, B can be pushed away'.	9/2	4.some
Recounting 9 in **2s**, the calculator predicts the result	9 – 4x2	1
9 = 4B1= 4.1 **2s** = 4 **2s** + 1		

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get $T = 3*B^2 + 4*B + 5*1$.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula.

Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8? An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3. We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8. When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8.
- So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8.
- So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8.
- So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to <u>opposite side</u> with <u>opposite sign</u>'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = 8/5	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges:

To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x+d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial

- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking '3kg at 5$ per kg gives what?', the answer can be predicted by T = 3*5 = 15$.
- Asking '10 years at 5% per year gives what?', the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.
- Asking 'If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?', the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking 'Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?', the answer is 'yes and no'.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers.

Algebra **unites**/ *splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** $T – b = a$	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T = a^b** $\sqrt[b]{T} = a$ $\quad$ $log_a(T) = b$

Love, your uncle Allan.

14. Fifty Years of Research without Improving Mathematics Education, why?

An academic essay written after the CERME 10 congress, February 2017

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies organised by the Organisation for Economic Co-operation and Development, OECD, showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report ‘Improving Schools in Sweden, an OECD Perspective’: “PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.”

Researchers in mathematics education meet in different fora. On a world basis, the International Congress on Mathematical Education, ICME, meets each four years. And on a European basis, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is “Only one: to improve, mathematics education should ask, not what to do, but what to do differently.” Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it ‘paralogy’ inventing dissension to the reigning consensus. Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory that is able to produce new researchers and to feed a growing research industry, but unable to reach its goal, to improve mathematics education.

To elaborate, mathematics education research is sterile because its three words are not well defined.

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation’s need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for new discoveries. So, seeing education as an institutional help to children and youngsters master outside phenomena leads to the question: What outside phenomena roots mathematics?

The Outside Roots of Mathematics

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, thus providing the holes in the head with our two basic needs, food for the body and information for the brain.

The sounds developed into languages. In fact, we have two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many totally?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, which we abbreviate to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’.

And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

And since we master outside phenomena through actions, learning the word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. You cannot learn how to math, since math is not an action word, it is a label as is grammar. Thus, mathematics may be seen as the grammar of the number-language.

Using the phrasing ‘the number-language is an application of mathematics’ implies that then ‘of course mathematics must be taught and learned before it can be applied’. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the ‘New Math’ that turned mathematics upside down to become a ‘meta-matics’ presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as ‘many-matics’, a natural science about Many.

Thus, Euler defined a function as a common name for calculations with unspecified numbers, in contrast to calculations without that could be calculated right away without awaiting numbers to be specified. Defining all concepts as examples of the mother concept set, New Math turned a function into an example of a set-product where first-component identity implies second-component identity, which learners heard as ‘bublibub is an example of bablibab’.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear. This means that what research has studied is problems coming from teaching how to math. So, one alternative presents itself immediately: Forget about New Math and, once again, teach mathematics as rooted in numbers and reckoning and reuniting totals and measuring earth.

Re-rooting mathematics resonates with its historic origin as a common label chosen by the Pythagoreans for their four knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

As to New Math, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox ‘This sentence is false’, being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn’t and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type-theory in which fractions could not be numbers if defined by numbers as done by New Math defining fractions as equivalence classes in a set of number-pairs. Insisting that fractions are numbers, New Math invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word ‘apple’ that cannot. By mixing things and their names, New Math and its meta-matics ceases to be a language about the real world. Still, it has entered universities worldwide as the only true version of mathematics.

So, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

Rethinking Mathematics from Below

To improve it we must rethink mathematics. To rethink we seek guidance by one of the greatest thinkers of the 20th century, Heidegger, being very influential within existentialist thinking and French skeptical post-structural thinking.

Heidegger holds that to exist fully means to establish an authentic relationship to the things around us. To allow a thing to open its ‘Wesen’ and escape its gossip-prison created by reigning essence-claims we must use constant questioning. So, returning to the fundamental goal of education, preparing humans for what is outside, we must keep on asking to the Wesen of the root of the number language, the physical fact Many, and allow Many to escape from its New Math gossip, ‘Gerede’.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can describe meeting Many in theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent.

2. Using a cup for the bundles we discover that a total can be ‘cup-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘cup-writing’ to report cup-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. We discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s.

On a folding ruler, distances are counted in tens. Here one centimeter is a bundle of ten millimeters, and ten centimeters gives a bundle of one decimeter. If the length of a hand is counted to 6 strokes after 1.7 tens, we write the length as T = 1.76 tens centimeters = 17.6 centimeters leaving the 6 unbundled millimeters outside.

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that cup-writing offers an alternative way to perform and write down operations:

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92 ; and T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336 ; and T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. To count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. Showing

7/3 = 2.more, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, dividing and subtracting a multiple, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

5. Wanting to describe the three parts of a counting process, bundling, and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. A ‘recount formula’ T = (T/B)*B saying that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and a ‘restack formula’ T = (T–B)+B saying that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Thus, we discover the nature of formulas: formulas predict.

6. Wanting to recount a total in a new unit, we discover that again a calculator can predict the result by bundling and stacking and dragging away the stack:

T = 4 5s = ? 6s. First (4*5)/6 = 3.more. Then (4*5) – (3*6) = 2. Finally, T = 4 5s = 3.2 6s

Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

7. Wanting to recount a total in tens, we discover that a calculator can predict the result directly by multiplication. Only, the calculator leaves out the unit and misplaces the decimal point:

T = 3 7s = ? tens. Answer: T = 21 = 2.1 tens

Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit:

T = 3 tens = ? 7s. First 30/7 = 4.more. Then 30 – (4*7) = 2. Finally T = 30 = 4.2 7s

Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged

8. Using the letter u for an unknown number, we can rewrite recounting from tens, e.g. 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u. Officially this is called to solve an equation, so here we discover a natural way to do so: Move a number to the opposite side with the opposite sign. The equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, thus predicted by the restack-formula as 8 = (8–2)+2. Thus, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.

9. Once counted, totals can be added. But we discover that addition is not well defined. With two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition can be reversed. Thus, the equation above restacking 8 by moving 2, 8 = u+2, can also be read as reversed addition: u is the number that added to 2 gives 8,

which is precisely the formal definition of u = 8–2. So, we discover that subtraction is reversed addition. And again we see that the equation u+2 = 8 is solved by u = 8–2, i.e. by moving to the opposite side with the opposite sign. Likewise, the equation recounting 8 in 2s, 8 = u*2, can be read as reversed multiplication: u is the number that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2? So, we discover that division is reversed multiplication. And again we see that the equation u*2 = 8 is solved by u = 8/2, i.e. by moving to the opposite side with the opposite sign. Also, we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root and logarithm as u = 3√20 and u = log3(20). So, again we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition, we can ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation. By observing that this is reversing multiplication and addition we discover that differentiation is reversed integration.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So, 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra's four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

How School Teaches Mathematics

Before addressing how school guides children on their way to mastering Many let us look at the number-language children bring to school. Asking a three-year old child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, that is not four, that is two twos!"

So, children come to school with two-dimensional 'block-numbers' all carrying a unit, corresponding to LEGO-bricks that stack as 1, 2, 3 or more 4s. Thus, by combining geometry and algebra in their shapes and knobs, they are an excellent basis for connecting the starting point, children's block-numbers, with the final goal, the Arabic numbers also being a collection of blocks of 1s, tens, ten-tens etc.

To emphasize that we count by bundling and stacking, the school could tell children that eleven and twelve is a special ‘Viking-way’ to say ten-1 and ten-2. Then they probably would count ‘2ten9, 3ten, 3ten1’ instead of saying ‘ten-and-twenty’ and risk being diagnosed with dyscalculia. In Danish, eleven and twelve mean ‘one left’ and ‘two left’, implying that the ten-bundle has been counted already. And, except from some French additions because of the Norman conquest, English is basically Anglish, a dialect from Harboøre on the Danish west coast where the ships left for Angland.

Now let us see how school prepare children and youngsters to meet Many by offering them what is called mathematics education. Again, we use the form of theses.

1. School could respect the origin of the word mathematics as a mere name for algebra and geometry both grounded in the physical fact Many and created to go hand in hand. Instead, school teaches mathematics as a self-referring ‘meta-matics’ defining concepts as examples of abstractions, and not as abstractions from examples. Likewise, school teaches algebra and geometry separately.

2. School could respect that a digit is an icon containing as many sticks as it represents. Instead, school presents numbers as symbols like letters. Seldom it tells why ten does not have an icon or why ten is written as 10; and seldom it tells why ten1 and ten2 is called eleven and twelve.

3. School could follow the word-language and use full sentences ‘The total is 3 4s or T = 3 4s or T = 3*4’. Instead, by only saying ‘3’, school removes both the subject and the unit from number-language sentence, thus indicating that what children should learn is not a number-language, but a one-dimensional number system claimed to be useful later when meeting life’s two-dimensional numbers.

4. School could develop the two-dimensional block-numbers children bring to school and are supposed to leave school with. Instead, school teaches its one-dimensional line-numbers as names for the points along a number line, using a place-value system. Seldom numbers are written out as we say them with the unit ones, ten, ten-tens, etc. Seldom a three-digit number is taught as a short way to report three countings: of ones, of bundles, and of bundles of bundles. Seldom tens is called bundles; seldom hundreds is called ten-tens or bundles of bundles.

5. School could respect that a number is a horizontal union of vertical blocks of 1s, bundles, bundles of bundles etc., and that counting-on means going up one step in the 1-block until we reach the bundle level where a bundle of 1s is transformed into 1 extra bundle making the bundle block go up 1 while the 1-block falls back to zero; and school could respect that a natural number is a decimal number with a unit. Instead, school represents numbers by a horizontal number-line, where counting-on means moving one step to the right and where a natural number is presented without unit and with a misplaced decimal point.

6. School could respect that totals must be counted and sometimes recounted in a different unit before being added. Instead, without first teaching counting, school teaches addition from the beginning regardless of units, thus transforming addition to mere counting-on. Seldom school teaches real on-top and next-to addition respecting the units.

7. School could respect that also operations are icons showing the three basic counting activities: division as bundling, multiplication as stacking the bundles, and subtraction as removing the stack to look for unbundled singles; and school could respect the natural order of operations: division before multiplication before subtraction before addition. Instead, school reverses this order without respecting that addition has two meanings, on-top and next-to, or that division has two meanings, counted in and split between.

8. School could respect that 3*8 means 3 8s that may or may not be recounted in tens. Instead, school insists the 3*8 IS 24 and asks children to learn the multiplication tables by heart. Seldom the geometrical understanding is included showing that recounting in tens means the stack increases its width and therefore must decrease its height to leave the total unchanged.

9. School could respect that basic calculations become understandable by recounting a total in the same unit to create or remove over- or underloads. Instead, school does not allow over- and underloads and insists on using specific algorithms with a carry-technique.

10. School could respect that proportionality is just another word for per-numbers coming from double-counting, and that per-numbers are operators that need a number to become a number. Instead, school renames per-numbers to fractions, percentages and decimal numbers and teach them as numbers that can be added without considering the unit, and teaches proportionality as an example of a linear function, which isn't linear since the b in y = a*x+b makes it an affine function instead.

11. School could respect that equations are just another name for reversed calculation rooted in recounting tens in icons and solved by moving to the opposite side with the opposite sign. Instead, school teaches equations as statements expressing equivalence between two different number-names to be solved by performing the same operation to both sides aiming at using the laws of abstract algebra to neutralize the numbers next to the unknown.

12. School could respect that integrating means adding non-constant per-numbers to be taught in primary school as next-to addition of block-numbers, and in middle school as mixture tasks; and respect that reversed integration is called differentiation made relevant since adding many differences boils down to one single difference between the end- and start-number. Instead school neglects primary and middle school calculus; and it teaches differentiation before integration, that is reduced to finding an antiderivative to the formula to be integrated. Seldom continuity and differentiability are introduced as formal names for local constancy and local linearity. Seldom the units are included to make clear that per-numbers are integrated, and that differentiation cerates per-numbers.

How School Could Teach Mathematics

Seeing the goal of mathematics education as preparing students for meeting Many, doing so in a Heideggerian gossip-free space offers many differences to be tried out and studied. Again, we use a list form.

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to teach cup-counting, recounting in the same and in a different unit, calculator prediction, on-top and next-to addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with cup-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with cup-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with cup-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with cup-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 33-5B6+50 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to

have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ A hymn can be created: “Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to ‘pre-dict’ a number, instead statistics can ‘post-dict’ its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5^{th} root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise, or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'CupCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in cup-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format.

The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, with at least two alternative meanings for all three words, at least 2*2*2 i.e. 8 different forms of mathematics education research exist. The past 50 years has shown the little use of the present form applying theory to study meta-matics taught in compulsory multi-year classes or lines. So, one alternative presents itself directly as an alternative for future studies: to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many, and to teach it in self-chosen half-year block at the secondary and tertiary level; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining and trying to understand the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so, stop teaching essence and start experiencing existence.

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15. Postmodern Enlightenment, Schools, and Learning

Is it Meeting the Concrete or Meeting the Abstract that Educates? An essay from 2000.

The Basis of Enlightenment, Bildung

A crisis in educational institutions makes the enlightenment discussion reappear: What is the purpose of an educational institution? What is educating, building, and forming humans? In short what is enlightenment or 'bildung'? Wishing not to convince but to inspire by pointing at contingency and otherwiseness this paper discusses late feudal, postfeudal, late modern and postmodern enlightenment. What has changed, and what has not? And what is the basis of enlightenment?

Matter, energy and information flow through nature. Through the holes in the head humans import their share. Humans are built from food for the body, routines for the reptile brain and stories for the human brain. But is it meeting the concrete or is it meeting the abstract that is building? Should enlightenment stories be told in the native or in a foreign language? And should enlightenment stories have concrete or abstract subjects? The feudal storyhouse, the church, talked about an abstract subject in a foreign language, Latin.

The postfeudal storyhouse, the school, talked about concrete subjects in foreign languages, Greek and Latin. Modernity introduces the native language but falls back to abstract subjects. Disclosing contingency in local institutions the introduction of the postmodern storyhouse, the global TV, in local cultures entails both globalisation and individualisation in the form of self-enlightening humans asking for stories about concrete subjects told in the native language. Modernity liberated the enlightenment story from the foreign language. It is the postmodern challenge to liberate it from foreign subjects. A source of inspiration could be the concrete enlightenment ideals of the postfeudal enlightenment, but this time told in the native language. In short, the postmodern globalisation is asking for a postmodern enlightenment.

The Enlightenment of the Church

The social order of feudality was a double feudalisation: Humans had two masters, a concrete and an abstract, a physical and a metaphysical, the King and the Creator. The story of the feudal storyhouse, the church, had an abstract subject: The Creator has created humans in his own picture and will help them to recreation after the Fall. In the Catholic Church the meeting is a triple meeting, two abstract and one concrete: a foreign language, Latin; a story about an abstract subject, the Creator; and a meeting with statues and pictures. The reformation of Luther substitutes the foreign language with the native, but takes away the concrete meeting, and the enlightenment story still has an abstract metaphysical subject only accessible through preaching and praying.

The Enlightenment of the Classical High School

The 'reformation' of Newton introduces a concrete physical subject into the enlightenment story. It is the will of not the metaphysical but the physical that is fulfilled: The apple is falling, not because the Creator wants it to, but because the earth is pulling it down. Physical forces are the creators and changers of motion. And these forces can be understood by human rationality, they can be quantified, calculated, and predicted. This quantification of nature changes the master/servant relationship: humans can become the masters of nature by listening to the enlightenment stories told in the enlightenment institution and storyhouse of modernity, the school (originally meaning the place for pause), making the enlightenment stories accessible through teaching and learning.

Newton's story takes away the metaphysical part of the double feudalisation and makes the way for the Enlightenment, modernity, industrialisation and democracy. As to the physical part, the king, Kant, and Humbolt argue for a new humanistic enlightenment. Kant asks "Was ist Aufklärung (what is enlightenment)?" and answers:

> *Enlightenment is man's emergence from his self-incurred immaturity. Immaturity* is the inability to use one's own understanding without the guidance of another. This immaturity is *self-incurred* if its cause is not lack of understanding, but lack of resolution and courage to use it without the guidance of another. The motto of enlightenment is therefore: *Sapere aude!* Have courage to use your own understanding! For enlightenment of this kind, all that is needed is *freedom*. And the freedom in question is the most innocuous form of all freedom to make public use of one's reason in all matters (Kant in Cahoone 1996 p. 51, 53)

Kant thus replaces the master/servant relationship with a political public space populated by free, equal, and empowered humans are their own masters. Humbolt, the creator of the modern university and the modern high school, agrees to this and sees enlightenment taking place through the meeting with the free political humans of antiquity, the old Romans, and Greeks:

> The true end of Man is the highest and most harmonious development of his powers to a complete and consistent whole. Freedom is the first and indispensable condition which the possibility of such a development presupposes; but there is besides another essential - intimately connected with freedom, it is true - a variety of situations. ...The highest ideal, therefore, of the co-existence of human beings, seems to me to consist in a union in which each strives to develop himself from his own inmost nature, and for his own sake.... And is it not exactly this which so inexpressibly captivates us in contemplating the age of Greece and Rome... (Humboldt 1969 p. 16, 19)

In this way the vision of the enlightenment story of the postfeudal high school is created. Enlightenment takes place through stories with concrete subjects, stories about the exemplars: nature in motion and the free humans of antiquity in speech, thinking and action. A beautiful dream that inherits two things from the Catholic Church: Its practice of arranging people in rows and columns pointing the nose in the same direction towards the grand narrative and the narrator. And its belief that the story must be told in a foreign language, the number & calculation language in the case of nature, and Greek and Latin in the case of antiquity. These three languages also have abstract creators, the concrete the students must meet the abstract, the grammars of meta-languages mathematics and grammar. So before meeting the concrete the students must meet the abstract, the grammars of these foreign languages. A very unsuccessful meeting for many students. The dream of an enlightening school turns into the nightmare of the black school. The classical high school soon produces two deviations, a modern language, and a science branch.

The Enlightenment of Modern Languages

The enlightenment ideal of the modern languages is the languages themselves and their literature. It is not stories about exemplary humans but exemplary stories about humans that educates. Now it is the form and not the content of the story that matters. Also, modern languages are presented as formed by an abstract grammar, and exemplary literature is presented as formed by abstract principles for exemplary language application. The enlightenment ideal of modern languages thus remains abstract and produces in the middle of the 20th century a deviation, a democratic enlightenment ideal.

On Democratic Enlightenment

The feudal double feudalisation can be removed in two ways, by escape or by revolution. With its freedom under Good" principle the US removes the double feudalisation with a stroke of the pen. On this background the American John Dewey formulates in the beginning of the 20th century a pragmatism linking democracy and education together (Dewey 1916). Germany had to go through revolutions and wars. After the First World War the new German democracy was sabotaged by the treaty of Versailles putting a drainage tube into the German economy (Keynes 1920). After the Second World War the physical part of the double feudalisation seemed finally removed, so after a period of rebuilding the enlightenment dream of empowerment now finally could be implemented. Modernity had become high modernity. The Frankfurter school formulates a democratic critical enlightenment ideal, where Adorno puts as the first claim to a modern school, that it prevents a new Auschwitz (Adorno 1988). As a road to democracy Habermas proposes the power free dialogue where the most convincing argument wins (Habermas 1996). In the Danish enlightenment debate the Frankfurter school also inspires to a concept of critical enlightenment:

> Besides specific knowledge enlightenment also contains a criterion for the application of this knowledge accepting responsibility for how, when and for what the knowledge is applied (Nielsen 1973 p. 40-41).

The critical enlightenment ideal is not an alternative but an extension of the traditional ideal thus importing its abstract enlightenment ideal. Later the critical enlightenment ideal is problematised by the emergence of postmodernity and postmodernism. The power free dialogue of Habermas could also be considered a power game between competing phrasings and discourses trying to feudalise and clientify the other (Foucault 1972). And Bauman points out how three central aspects of modernity: authorisation, routinisation and dehumanising can hide amorality in a web of transfer of morality, technical rationality and covering phrasings. Auschwitz might not be an abnormality but a normality of modernity (Bauman 1989).

On Scientific Enlightenment

The postfeudal science tried to practice a concrete enlightenment ideal: scientific knowledge is created through laboratory meetings with concrete examples of nature in motion and the following quantification. In high modernity it was still possible to tell Newton's discovery as a concrete story about motion: forces change motion, and the change can be quantified and calculated. Per second the force adds a certain amount of momentum, and per meter the force adds a certain amount of energy.

In late modernity, called risk society by Ulrik Beck, science can no longer legitimise itself by referring to its successes. It now also produces problems and risk: atomic bombs, pollution etc. As a reaction science turns from being a taboo-breaker to a taboo-creator (Beck 1992). Science adopts a voluntary self-feudalisation beginning to present itself as enlightened and formed by and coming from, not social practices but universal principles and concepts. In the stories of physics the concrete word ‘motion’ is substituted by abstract concepts as conservation principles, energy and momentum. Likewise, the number language is now rephrased to mathematics substituting the word calculation with the abstract concept of a function. With this change of focus from the concrete to the abstract, from motion and calculations to energy and functions social constructivism (Järvinen et al. 1998) is substituted with new Platonism. Through its self-feudalisation late modern science resembles late feudality in its abstract enlightenment ideal telling stories in a foreign language about abstract subjects. This self-feudalisation is frozen by the late modern phrasing. A phrasing as “mathematics is the grammar of the number language” would entail a bottom-up practice “of course language must be learned before its grammar”. But the late modern phrasing “the number language is applied mathematics” entails the opposite top-down practice “of course mathematics has to be learned before it can be applied”.

The Late Modern Re-Feudalisation

The late modern enlightenment institution thus shows many signs of a re-feudalisation (Habermas 1975). Late modernity partly took away the foreign language but reintroduced the abstract subjects into the enlightenment stories. And the abstract is now yet more abstract: The storyteller of late feudality, the preacher, was obliged to retell the words of the bible as a concrete everyday story. The storyteller of late modernity, the teacher, will often just echo the textbook and repeat it once more when students ask for an explanation (Tarp 1998 b).

The Postmodern

The postmodern can mean many different things (Bertens 1995). Rudely we can say that postmodernity is about the social, and postmodernism about the individual. Modernity used electrons for transporting energy. Postmodernity uses electrons for transporting information, through computers, internet, and satellite TV. The new storyhouse of postmodernity, the global multichannel TV tells stories about contingency and alternatives to local traditions. Postmodernism means accepting contingency, as opposed to modernism which is trying to hide contingency.

> Modernity, or desperately seeking structure. The kind of society that, retrospectively, came to be called modern emerged out of the discovery that human order is vulnerable, contingent and devoid of reliable foundations. That discovery was shocking. The response to the shock was a dream and an effort to make order solid, obligatory, and reliably founded. This response problematized contingency as an enemy and order as a task....Modernity was a long march to prison. It never arrived there, albeit not for the lack of trying (Bauman 1992).

Through traditions modernity succeeded in encapsulating and institutionalising contingency to the political top of the social iceberg. The rest, institutions and culture, or social systems and structures is imbedded and frozen in humans as tacit routinised practical consciousness (Giddens 1984). Disclosing contingency in local institutions and cultures the introduction of global TV in local cultures bring about both globalisation, i.e. local de-traditionalisation (Giddens in Beck et al. 1994), and individualisation, meaning humans can no longer obtain an identity by just echoing the tradition, humans have to build a self-identy and a biographical self-story by asking for authenticity and meaning (Giddens 1991). The post-traditional postmodern individual has become a self-enlightener.

Postmodern Self-Enlightenment

The postmodern self-enlightener is asking for stories that can be linked to the existing self-story, i.e. sentences with concrete subjects: “Tell me something I don't know about something I know”. The concrete is what we share ‘being’ with (Dasein, Heidegger 1926), and what provides the foundational names of our existing self-story. Postmodern learning resembles Ausubel's authentic verbal learning, but talks about the existing self-story where Ausubel talks about the existing cognitive structure (Ausubel 1968).

Post-feudality saw an increasing exodus of self-savers away from the feudal story-house, the church, wanting to make them clients of its Creator. Likewise, the decreasing interest for natural science (Jensen et al. 1998) is indicating that postmodernity will see a similar exodus of self-enlighteners away from the modern story-house, the school, wanting to make them clients of its Discourse. Postmodern self-enlighteners have problems with the late modern school and its upwards referring sentences with unknown abstract subjects presenting abstract concepts as examples of more abstract concepts: “Bablibab is an example of Bublibub” or, “a function is an example of a relation between two sets assigning to each element in one set one and only one element in the other set”. Referring upwards such sentences cannot be rephrased (I only have one mother) thus creating ‘echo-teaching’, ‘echo-learning’, and ‘echo-resistance’ (Tarp 1998 b). Self-enlighteners are looking for downwards referring sentences “Bablibab is a name for a teaspoon” or, “a function is a name for a calculation with a variable quantity” (Euler 1748). In short sentences with known concrete subjects presenting something abstract as a name for something less abstract and thus linkable at some level to the learners existing self-story.

Postmodern "Reducation"

Modernity sees the hiding of contingency as a means against chaos and institutionalises this hiding in the form of education in schools and research at universities. The power and knowledge monopolies of the modern state melt together in the modern school, where knowledge is reproduced as symbolic violence: “All pedagogic action is, objectively, symbolic violence insofar as it is the imposition of a cultural arbitrary by an arbitrary power (Bourdieu 1977)”.

Postmodernity sees the hiding of contingency as a means to clientification and feudalisation. Nietzsche is among the first to point out, that there are many perspectives on the world (Robinson 1999). Foucault carries Nietzsche's argument further by showing how modern emancipated humans are controlled by the master of modernity, the ruling phrasings and discourses (Foucault 1972). Phrasing is freezing and rephrasing is defreezing. To the postmodern the road to declientification and defeudalisation goes through disclosure of contingency. Only by constantly looking for alternative phrasings and redescriptions (Rorty 1989) can humans avoid the blindness of the ruling discourse and enjoy their 'being' completely.

The postmodern sees the enlightenment institution as an institution for re-education. In postmodernity education becomes 'reducation': The institution has to choose between reduction and re-education. The modern institution reduces humans to clients and consumers of its discourses, its ideal being the peasants living next to and defending the source of truth against all the scepticism of the world. The postmodern institution offers instead a possibility for declientification and defeudalisation through constant re-education, its ideal being the nomads travelling between different 'neotribes' (Bauman 1992) redescribing themselves through new perspectives, phrasings, and vocabularies. To reproduce humans must consume, but humans need not accept being reduced to passive consumers of goods and discourses. Consumption can also be used for active reproduction of counterculture using human imagination to see contingency (Rorty 1989).

On Postmodern Automatic Learning

Humans are built form food for the body, routines for the reptile brain and stories for the human brain. Modernity's constructivists pointed out that students cannot be filled with concepts; they construct their own versions. Learning is a chaotic process that can be guided by attractors (Doll 1993). Using bio-power (Foucault 1976) to control all levels of human knowledge modern attractors are metaphysical concepts as momentum and functions. In reverse postmodern attractors could be the authentic social practices from which the school subjects arose and which they could be telling about (see below). Authenticity is precisely what the postmodern self-enlighteners are asking for. Authentic practice can provide both a tacit learning of routines, and stories with concrete subjects. If furthermore the school could make its tales fairy it could offer stories, the global TV can't. And it would rediscover automatic assessment free learning, a way of learning witnessed by the existence and self-reproduction of fairytales in a non-written prefeudal culture - but alas long forgotten.

On Modern Full- and Half-Research and Postmodern Counter-Research

Modern research and postmodern counter-research meet on the border between nature and culture, between what is given and what could be different, between necessity and contingency. The dispute is about questions as: Is contingency hidden necessity - or is necessity hidden contingency? Is truth discovered or invented? Will a description describe or construct what is described? Do concepts make us conceive or conceited? Modern research chases hidden necessity asking "what is the nature of this?" wanting to produce convincing knowledge claims of the form "A is B". In reverse postmodern counter-research chases hidden contingency asking "what is the culture of this?" wanting to produce inspiring knowledge suggestions of the form "A could also be B".

The postmodern is sceptical towards the modern belief in a final vocabularium being able to mirror and represent the world in a convincing way (Rorty 1989). The postmodern might accept that nature can talk in numbers through number meters but does not accept that culture can talk in words since no word meter exists. A quantification of nature can be true by necessity, but all wordings of culture could be different. All phrasings are contingent except for this meta-phrasing. So modern research come as full-research assigning numbers to nature, and half-research assigning words to culture. Postmodern counter-research is also half-research only talking about other sides of the coin. Phrasing can be used to clientify, and phrasing can be used to rephrase and declientify. As to premodern ethnography the missionary wants to help the field to salvation through teaching. In modern ethnography the researcher wants to help the field to research based innovation through teaching. The postmodern ethnographer (Denzin 1997) wants to help himself to become declientified through learning about contingency from the field. Especially the postmodern gather/hunter culture could learn a lot from the prefeudal dittos.

On Postmodern Gather/Hunter Culture II

With its opposite attitude towards contingency modernism tries to demonise postmodernism to keep humans frozen in a modernity discourse. Postmodernism helps you getting out of this discourse but being a meta-discourse it does not offer a competing discourse. One alternative discourse could be a ‘Gather/Hunter II discourse’.

In gather/hunter I cultures nature is producing and humans are consuming. In agriculture and industry humans take over production. In postmodernity artificial humans, the robots, are doing 80% of the jobs reducing most humans to gathering food in the supermarket and hunting stories on the global TV.

The fundamental difference between the first and the second gather/hunter culture is that nature was owned and controlled by nobody, whereas the robots are controlled by those who know their language, mathematics, and owned by stockholders using mathematics as stochastic differential equations to optimise their profit on the stock market. The postmodern 20% job reality can develop in two ways: Work to 20% of the people - or work to 20% of the time, all depending on who becomes the victorious of two mathematics discourses. The ‘no math - no job’ discourse (Tarp 2000a) could develop a US-like scenario: Real high salary jobs to few and phony low paid jobs to many blaming themselves for not having learned mathematics and accepting that those who know how the stock market develops by knowing stochastic differential equations also have the right to a bigger share.

The ‘math - a human right’ discourse (Tarp 2000) could develop a solidarity scenario where humans share working time and income, and use the spare time to be active and creative reproducers of culture inspired by the humans of gather/hunter cultures I. In this discourse the phrasing "stochastic differential equations" can be rephrased. Stochastic means unpredictable and differential equations mean change calculations or change predictions. So ‘stochastic differential equations’ could be rephrased to ‘predicting the unpredictable’ e.g. a logical impossibility unless one would like to pretend to be a god. Similar scenarios are described e.g. by Bauman (1998) and Rorty (1989).

Postmodern School Preparing Humans for Gather/ Hunter Culture II

With robots providing the food for the body humans can concentrate on providing routines for the reptile brain and stories for the human brain. The institutionalised school can help humans getting access to the different routines, rituals, and stories of their own and of other cultures: Myths, fables, fairytales, songs, dances, games, plays, musical instruments, drawing techniques etc. And the postmodern school could see a challenge in telling the subject of the modern school as a postmodern ‘fairy.tale’ to provide solidary automatic learning.

From a postmodern mother ‘fairy.tale’ the school subjects grow as specialised stories. Stories about nature and its flows of matter, motion and information controlled by forces and cells. Stories about cultures created through human endeavor to control these flows with technology gradually constructing the artificial human, the robot. Language stories about how humans invent four languages, a word language and a number language to talk about the world in statements and calculations. And two meta-languages, grammar, and mathematics, to talk about the word language and the number language. And ethical stories pointing out the difference between numbering and wording the world: nature can talk in numbers, but culture cannot talk in words, making all phrasings contingent and setting ethical demands for the exchange of stories in the interpersonal conversation room and in public space.

A Postmodern Mother ‘Fairy.Tale’ about Nature, Culture and Humans

“Once upon a time there was three forces continuously pumping matter and motion through the universe: the strong, the medium and the weak. First the strong force crushed the old universe in a big bang liberating light matter and the medium force. Then the strong force pulled matter together in stars, which the medium force tried to crush in small bangs. This liberated light, which pumped motion from the stars into the space.

Finally, the strong force crushed the stars in medium bangs liberating heavy matter and the weak force. And creating black holes where the strong force was so strong that it put a brake to the expansion of the universe and ended up swallowing all matter and other black holes creating one big black hole so heavy that it was crushed by its own strong force in a new big bang”.

The strong force unites heavy matter into planets, where the weak force unites atoms into molecules with big distance between the atoms thus neutralising the strong force. As long as the universe is expanding, the light of the stars is weak. When the universe will start to retract, the light will blow the planets apart.

On planet earth the strong light, the lightening, splits the strong nitrogen-nitrogen compounds of the air thus adding strength to the widespread network of carbon-nitrogen molecules from which life is build. The weak light from the sun pumps ordered motion to the earth, while the void sucks unordered motion away. The resulting flow of motion makes nature's matter move in cycles. An air-cycle creating winds. A water-cycle creating rain and rivers. And an organic/inorganic cycle pumping motion to the three life forms, black, green, and grey cells.

The black cells survive in oxygen free places in stomachs and on the bottom of lakes only able to take oxygen in small amounts from organic carbon-structures thus producing gas. The green cells use the weak light to remove the oxygen from the inorganic carbon dioxide molecules thus producing both organic matter cells to release storing motion and the needed by the grey the motion again. Green cells form cell communities, plants, unable to move for the food and the light. oxygen

Grey cells form animals able to move for the food in form of green cells or other grey cells thus needing to collect and process information by senses and brains to decide which way to move. Animals come in three kinds.

The reptiles have a reptile brain for routines. The mammals having live offspring in need of initial care have developed an additional mammal brain for feelings. Humans have developed human fingers to grasp the food, and a human brain to grasp the world in words and sentences. Humans thus can share and store not only food but also stories, e.g. stories about how to increase productivity by transforming nature to culture.

The agriculture invents an artificial human hand, a tool, enabling humans to transform the wood to a field for growing crops. The industrial culture invents an artificial human muscle, a crops. motor. By integrating tools and motors to machines humans can transform nature's raw material to material goods. The information culture invents an artificial human reptile brain, a computer. By integrating the artificial hand, muscle and brain to an artificial human, a robot, humans are freed from routine work.

Human production and exchange of goods has developed a number-language besides the word-language, to quantify the world and calculate totals. Agriculture totals crops and herds by adding. Trade totals stocks and costs by multiplying. Rich traders able to lend out money as bankers total interest percentages by raising to power.

And finally industrial culture calculates the total change-effect of forces through integrating: by adding a certain amount of momentum per second and energy per meter a force changes the meter-per-second number, which again changes the meter-number.

Postmodern Mathematics - Stories About Quantification & Calculation

Modern mathematics presents its two main areas, algebra, and geometry, as examples of abstract universals: the concepts of sets, relations, and functions. Postmodern mathematics lets algebra and geometry grow out of the social practices they are naming. Geometry means earth measuring in Greek. Geometry grows out of dividing and measuring earth posing questions like “How can we divide this piece of earth?” and “How much will each get?”.

Numbers grow out of different bundling practices revealing their cultural dependence: 74 = seven bundles at ten pieces per bundle plus four single pieces (UK standard) = half-four bundles twenty pieces per bundle plus four single pieces (DK standard). Algebra means reunite in Arabic. The four uniting techniques grow out of uniting variable and constant unit numbers and per numbers posing the question "How much in total?".

Addition and multiplication unite variable and constant unit numbers. Integral and power unite variable and constant per-numbers. In reverse the reverse calculations split the total: Minus and division split the total into variable and constant unit numbers. Differentiation and root/logarithm split the total into variable and constant per numbers (Tarp 1998 a).

	Questions		lead to	Equations		
unit-numbers	the Total of 3 \$ and	5 \$	is ? \$	T =	3 + 5	= ?
	the Total of 3 \$ and	? \$	is 21 \$	T =	3 + x	= 21
					x	= 21-3
unit-numbers	the Total of 3 \$	5 times	is ? \$	T =	3 * 5	= ?
	the Total of 3 \$	? times	is 21\$	T =	3 * x	= 21
					x	= 21/3
per-numbers	the Total of 3 %	5 times	is ? %	T =	103% ^ 5	= ?
	the Total of 3 %	? times	is 21%	T =	103% ^ x	= 121%
					x	= log 1.21/log 1.03
	the Total of ? %	5 times	is 21%	T =	x^5	= 121%
					x	= $\sqrt[5]{1.21}$
per-numbers	the Total of 3 m/s increasing to 4 m/s over 5 sec.		is ? m	T =	$\int_0^5 (3 + \frac{4-3}{5} x)\, dx = ?$	

Figure 32. The Algebra Square from a different view

Postmodern Physics Stories About Controlling Motion

Modern physics will present a bouncing ball as an example of an abstract universal, the metaphysical law of conservation of energy: The energy is changing between potential and kinetic energy to end up as internal energy. Postmodern physics could present a bouncing ball through a metaphor of forces pumping motion: The gravity pumps motion into and out of the ball when the ball is moving in the same or the opposite direction of the force. Constant collisions pump motion from the ball into micro-balls, molecules, from which the motion is pumped into new molecules through new collisions. The motion is thus transferred and spread to more particles. The energy is conserved but the entropy has increased.

Of special interest is the pumping of motion from nature into the machines of industry through artificial windmills. Two water chambers, a hot and a cold, are connected by tubes to from a water cycle. The burning of coal produces fast carbon dioxide molecules pumping their motion into the water molecules of the hot chamber through collisions. This transforms the water molecules into fast steam molecules being sucked into the cold chamber where collisions pump the motion into the cooling water. This transforms the steam back to water, which is then pumped back into the hot chamber by an electrical pump.

From the cooling water the motion is pumped into the ocean or into cold rooms (district heating). The cooling water is then pumped back into the cold chamber by an electrical pump. On its way to the cooling chamber the steam molecules are pumping motion into a rotor through collisions. The rotor pumps motion into electrons by means of the weak electromagnetic force. Later this force pumps motion from electrons into machines. Only 1/3 of the motion ends ordered electronic motion, the rest becomes up as unordered molecule motion.

Postmodern Language - Stories About Rephrasing

Modern language subjects consider the meeting with good phrasings as educating. Postmodern language subjects consider the meeting with alternative phrasings as educating. Since all phrasings are contingent the important thing to discuss is not, what makes a phrasing good, but how it can be rephrased or deconstructed (Derrida 1991). Modernity's clientification of humans in ruling phrasings and discourses legitimises a postmodern rephrasing subject able to assist individuals in declientification through alternative phrasings. The basis of public democracy is the respect for other phrasing in texts and the other's phrasing in the conversation room. The attention towards hidden power games in the conversation room can be sharpened by the metaphor names child/adult/ parent of transactional analysis (Berne 1964). Democracy is founded in studies of and practice in ethics of conversation: you don't fight to become the narrator but give away this position to the other by inviting the other into the conversation room as the narrator and by practising active listening.

Conclusion

The modern enlightenment discussion is frozen in a freedom discourse: Are humans free? Can humans freely master their freedom? The late feudal and late modern answer is no: Humans need to be taught about the metaphysical laws they serve. The feudal reformation of Luther freed the story for its foreign language but not for its abstract subject. Modernity tried to realise the potentials of human freedom, but late modernity falls back to late feudal forms with abstract subjects. The postmodern recognises clientification and contingency as conditions of life. To reproduce humans will always be consumers of food, routines, and stories. Humans can however free themselves from being reduced to passive consumers and become active reproducers of social routines and stories. So, by not covering but uncovering contingency and otherwiseness the postmodern enlightenment institution can help postmodern humans in the transformation from unempowered client to empowered self-enlightener.

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16. Can Postmodern Thinking Contribute to Mathematics Education Research

An auto-interview from year 2000

Abstract

Q: I have three questions: What is the postmodern? Is there a postmodern research paradigm? To what field of mathematics education research can postmodern research contribute, and what are examples of postmodern studies? Please be brief, we can take the details later.

R: OK. 1. The postmodern means goodbye to the modern, both as postmodernism on the discursive level and as postmodernity on the practice level. Postmodernism is poststructuralism rejecting the belief that the world has a structure to be represented and echoed by language, and warning against becoming enslaved and clientified by echo-phrasings and echo-discourses. Postmodernity is post-industry now also using electrons to carry information, to machines and to global TV's, thus creating automation and globalisation. And individualisation since the individual can no longer get an identity through echoing the local tradition, but has to construct a self-identity through self-stories. In short the postmodern means a transition from an echo-society to a dilemma-society. 2. Modern research echoing the structure of the world becomes problematic in a world without structure. Postmodern research is a declienfying counter research producing counterexamples to modern structure claims and echo-discourses by rephrasing echo-phrasings. 3. Postmodern counter research is working within the sociological part of the social turn in mathematics education. In mathematics education 'mathematics' and 'education' are echo-phrasings that can be rephrased as 'grammar of the number language' and 'cure'. This rephrasing exposes the contemporary mathematics education tradition as treating clients in need of a number language with a grammar cure; and it provides a counter cure for, not the client, but the cure: language before grammar; a counter cure that might make mathematics a human right.

Q: By the way, why are you interviewing yourself?

R: In my studies I have been inspired by Foucault, who is often easier to understand in interviews than in his writing (Foucault 1972). This is where I got the idea to use an interview to tell this story.

Postmodernism and Poststructuralism

Q: So there are different sides to the postmodern?

R: Yes. In his book about the idea of the postmodern, Hans Bertens distinguishes between postmodernism and postmodernity. About postmodernism he writes

> Postmodernism, then, means and has meant different things to different people at different conceptual levels... If there is a common denominator to all these postmodernisms, it is that of a crisis in representations: a deeply felt loss of faith in our ability to represent the real, in the widest sense (Bertens 1995 p.10).

Later he distinguishes between two moments of postmodernism, a deconstructionist derived from Derrida and a poststructural derived from Foucault:

> Like the earlier deconstructionist postmodernism, the later poststructural postmodernism assumes a reality of textuality and signs, of representations that do not represent. Here however, the emphasis is on the workings of power, and the constitution of the subject. From the perspective of this postmodernism, knowledge, which had once seemed neutral and objective to the positivists and politically emancipatory to the left, is inevitably bound up with power and thus suspect. ... This postmodernism interrogates the power that is inherent in the discourses that surround us - and that is continually reproduced by them - and interrogates the institutions that support those discourses and are, in turn, supported by them. It attempts to expose the politics that are in work in representations and to undo institutionalised hierarchies, and it works against

> the hegemony of any single discursive system - which would inevitably victimize other discourses - in its advocacy of difference, pluriformity, and multiplicity (Bertens 1995 p.7).

To sum up 'under erasure' (Derrida 1991) as Derrida puts it to acknowledge the eventuality of all phrasings, postmodernism means poststructuralism, i.e. a goodbye to the conviction that the world has a structure that can be represented or echoed by language and discursive knowledge; discovered through research, transmitted through education making student knowledge an echo of scientific knowledge, and finally applied in industries. Phrasings are not representing, but constructing what they phrase. All phrasings could be otherwise. Unaware of the hidden eventuality of phrasings the emancipated free human being becomes enslaved and clientified by ruling echo-phrasings. Also phrasing is praising, but this time the Lord is not living in castles or churches, but in schools and universities, where the hidden subjection of the apparent free and rational modern human being takes place through echo-phrasings.

Q: You don't mention nihilism. Often postmodernism is considered just negative nihilism and rejected because of that.

R: Well, the key idea of poststructuralism is that things cannot name themselves. And of course structuralism has good reasons to demonise poststructuralism by phrasing it with a negative word. As structuralism itself was demonised by what it replaced, the feudal order.

Q: So structuralism is basically postfeudalism?

R: Yes. The premodern feudal world was governed by the will of two masters, a physical and a metaphysical; and the ways of the metaphysical Lord were many, His will was unpredictable. All we could do was to go to church, believe and pray. Or go to a monastery and study the Holy Scripture. Then Newton discovered, that the will that makes things fall can be quantified and calculated, hence predicted. Calculation thus freed us from the arbitrary laws of the Lord, we had become our own master, since we could make the calculable laws serve us at will. Thus structuralism and enlightenment and democracy was born: The modern free world would be governed by the will of a parliament representing the will of the people, and by the will of metaphysical laws, that were predictable and representable in language. All we had to do was to go to school, learn and calculate. Or go to a university and study the research scriptures, and study the laws through experiments and observations. The physical world had a structure echoed from metaphysical laws. These laws could be discovered by researchers, taught by teachers and applied by professionals. Hence a modern society should invest in, not impressive baroque monasteries and churches as in the catholic southern Europe, but in universities, schools and industry as in the protestant northern Europe. At the universities the natural sciences searches for the quantitative laws determining particle behaviour in physical force fields. Likewise human and social sciences search for the qualitative laws determining human behaviour in social force fields.

Q: And post-structuralism rejects these beliefs?

R: Yes, in a way the structuralism/poststructuralism debate is a repetition of the former Plato/Aristotle and realist/nominalist debates. To poststructuralism there is no metaphysical structure to be echoed in the world, and there is no physical structure to be echoed in language. Any description constructs what it describes. Words are not echoes from above, words are names from below. The structure of the world is not a natural consequence of metaphysical laws, but a cultural consequence of the way we have chosen to phrase the world, of ruling discourses. A branch of poststructuralism, the SSK, Sociology of Scientific Knowledge, even argues that also natural sciences are social constructions (Pickering 1995). We can only ask nature through measuring instruments, and since measuring instruments are human constructs, the answers of nature will also be human constructs. Another name for the structuralism/poststructuralism debate is 'the science war' with social constructionism on the one side writing off the enlightenment project, and the paradigm of complexity on the other trying to save it through system theory, chaos theory etc.

Q: Foucault has been rather influential to postmodernism?

R: Yes. In his inaugural speech 'Orders of discourse' Foucault says:

> The fundamental notions now imposed upon us are no longer those of consciousness and continuity (with their correlative problems of liberty and causality), nor are they those of sign and structure. They are notions, rather, of event and of series, with the group of notions linked to these (Foucault 1970, p. 23).

This is how Foucault says goodbye to the core concepts of modernism: the concepts of liberty, causality and structure, and replaces them with the core concept of postmodernism: eventuality, possibility, potentiality, otherwiseness, alternatives. Or contingency as Rorty phrases it, or re-enchantment as Bauman phrases it (Rorty 1989, Bauman 1992).

Postmodernity and Postindustry

Q: And then there is postmodernity

R: Yes. Postmodernity is about technological and social changes. In his book 'Runaway world' Giddens talks about globalisation:

> Globalisation also influences everyday life as much as it does events happening on a world scale. ... [Globalisation] contributes to the stress and strains affecting traditional ways of life and cultures in most regions of the world. ... Fundamentalism originates from a world of crumbling traditions (Giddens 1999).

Instead of postmodernity Giddens uses other terms as radicalised, late, high, reflexive, post-traditional modernity. In his book 'Modernity and Self-Identity' he writes

> In the post-traditional order of modernity ... self-identity becomes a reflexively organised endeavour. The reflexive project of the self, which consists in the sustaining of coherent, yet continuously revised, biographical narrative, takes place in the context of multiple choice as filtered through abstract systems (Giddens 1991).

In short we can say that postmodernity means post-industry. In modernity electrons carry energy nationwide to machines producing echoes of their products in high numbers at low costs. In postmodernity electrons also carry information to machines, and to global TV's. Routine echo-knowledge is moved from human brains to artificial brains, computers, becoming integrated with machines to robots taking over the echo-production. Global TV's in local cultures produce dilemmas by echoing global counter-answers to local echo-answers. This globalisation creates individualisation: with many answers individuals can no longer obtain identity by echoing the answer, individuals now have to construct their own self-identity through biographical self-stories build upon authenticity and meaning. The postmodern individual has become a self-story builder and a self-educator.

Q: So post-modernity is post-industry, which probably also is post-something?

R: The history of technology can briefly be thought of as the gradual construction of the artificial human. First the construction of an artificial hand, a tool, enabled a transition from gather/hunter culture to agriculture. Then the construction of an artificial muscle, a motor, and the combination of a tool and a motor to a machine enabled a transition to the industrial culture, to modernity. And now the construction of an artificial brain, a computer, and the combination of a machine and a computer to a robot has made possible a transition to a postmodern information culture. This postmodern culture resembles very much the former gather/hunter culture: Then nature was producing and humans just had to gather and hunt the food. Now robots, clones and construction molecules made by the new self reproducing GNR-technologies, i.e. gene-, nano- and robot-technology, will take over production. Again most humans just have to gather the food in the supermarket and hunt for stories on the global TV. So one way of naming the postmodern culture will be 'gather/hunter culture II' (Tarp 2000 b).

Q: Does it matter how you name it? A name is just a label.

R: In postmodern thinking a name constructs what it names. Hence we witness a fierce battle of definition. As long as the modern can demonise the postmodern its institutions can survive. But if the term postmodern becomes accepted, all the institutions of modernity have to defend and re-legitimise themselves. In the SSK terminology they then stop being closures, i.e. accepted naturalized solutions, and become instead possible and potential solutions that have to fight with other potential solutions to become the new closure. And just calling the new order 'post' isn't enough, we have to find a new name in order to think and talk about it, make decisions and institute institutions.

Q: What would be gained by naming postmodernity the new gather/hunter culture.

R: Then we wouldn't have to build a new culture from the bottom; we could learn and become inspired by former and present gather/hunter cultures. Instead of phrasing these cultures undeveloped and primitive we can phrase them exemplars to be learned from and to get inspiration from as to how life can be organised so individuals become actively producers instead of passive consumers of culture.

Q: To sum up, what is the effect of the postmodern to research and education?

R: Well, the university and the school are modern institutions that succeeded the premodern institutions of the monastery and the church. All are studying or teaching metaphysical laws, in the premodern case unpredictable laws understood through belief, in the modern case predictable laws understood through knowledge. Postmodernism cancels the metaphysical world, which of course seriously questions the relevance of research and education in a postmodern world. Can these modern institutions survive a transition to a postmodern era, or do they in a desperate 'apres nous le deluge' situation try to keep the illusion of modernity alive as long as possible?

Also the students have changed. In the premodern will-society the church wanted to save the lost individual by offering faith to believe in. In the modern echo-society the school wanted to cure the uneducated individual by offering knowledge to be echoed. The postmodern dilemma-society has created the individualised self-story builder shopping among numerous stories, rejecting those who do not tell them something new about what they already now, thus rejecting modern education from above.

Postmodern Counter Research

Q: So does research have any meaning in a postmodern world? Is there a postmodern research paradigm? Please start with a summary.

R: In a postmodern world research has meaning as counter research studying how things could also be otherwise. Postmodern counter research means goodbye to modern research trying to echo the structure of the world in true phrasings. The world does not phrase itself, we do, so all phrasings are eventual, contingent, and could be otherwise. Counter research produce counter examples to ruling discourses and to structure claims, which by claiming a status of truth automatically become echo-statements. Postmodern counter research is a 're-search' re-searching for re-phrasings, using both discovery and imagination. Discovery is used to search ruling discourses for echo-phrasings. And imagination is used to rephrase these echo-phrasings to unhide hidden alternative counter-discourses that might change our convictions, institutions and routines. Thus transforming rituals to routines. In short, postmodern counter research is guarding the borderline between necessity and eventuality by unhiding hidden eventuality in claimed necessity.

Q: So postmodern research is based upon poststructuralism?

R: Yes. Again let us turn to Foucault. In his inaugural speech Foucault talks about four methodological principles: a principle of reversal, a principle of discontinuity, a principle of specificity and a principle of exteriority. As to the principle of specificity he says:

> The principle of specificity declares that a particular discourse cannot be resolved by a prior system of significations; that we should not imagine that the world present us with a legible face, leaving us merely to decipher it; it does not work hand in glove with what we already know;

> there is no prediscursive fate disposing the world in our favour. We must conceive discourse as a violence that we do to things, or, at all events, as a practice we impose upon them; it is in this course that the events of discourse find the principle of their regularity (Foucault 1970 p. 22).

This is Foucault's version of the post-structuralism statement. The world has no structure to be echoed, so whenever we are talking about the structure of the world, we are talking about something we impose upon the world. Thus one kind of a postmodern methodology is the one Foucault described in his inaugural speech and carried out in his studies.

Q: So basically Foucault's thinking is the foundation of postmodern research?

R: Not exactly. In my own work I have tried to make a compromise between structuralism and poststructuralism by formulating what I call a postmodern counter research paradigm. Research is supposed to produce true knowledge claims about the world. But when we are talking about truth and describing the world we are faced with three levels: the world, the language describing the world and the meta-language describing the language. Between the two upper levels we meet the 'meta-dilemma': Is the meta-language describing the language - or is the language an application or echo of the meta-language? This dilemma is parallel to the theory-dilemma: Is theory describing the world - or is the world an echo of theory? And between the two lower levels we meet what I call the 'pencil-dilemma'.

Q: The pencil-dilemma?

R: Place the thing we call a pencil between a ruler and a dictionary, both enabling us to lift up things from the thing-level to the language-level by assigning numbers and words to them. The pencil-dilemma says that the pencil can point to its own length, but not to its own name. Hence there is a fundamental difference between number-statements and word-statements: Number-statements about a thing as 'its length is 25 cm' can be verified by asking the thing itself, word-statements as 'this is a pencil' cannot. Number-statements follow from necessity; word-statements follow from eventuality, from a choice that could be otherwise.

Q: Just a moment, a cup is a cup. I mean nobody will deny, that this thing is a cup?

R: Correct. But the fact that nobody would deny it doesn't make it a cup. 'Cup' is not a necessary quality by the thing; it is a description of its use in the current culture. People from other cultures or other times might name the thing differently. In 200 years the thing we call a cup might be called e.g. a stick-container, again depending of the use. Also 'cup' can be seen as an excellent social inclusion and exclusion technique, nobody denies it is a cup, for if they did they would be excluded. The point is that the thing cannot name itself in the same way as it can number itself e.g. by being put on a balance. Also we have different ways to decide if number- and word statements are true. Number-statement are decide through measuring, word-statements are decided in court, i.e. through a choice between two different phrasings of the same thing.

Q: But both words and numbers are constructed by humans?

R: Of course humans have to construct their own interface towards the world. But once the ruler and the dictionary have been constructed there is still a necessity in the thing that enables it to point to its own length. Modern science discovered the five necessities of nature that physics is build upon: mass, charge, extension in space, extension in time, and multiplicity. These necessities exist, but of course how we name them is dependent on social constructions in local cultures. The pencil-dilemma points to the fact, that there exists a borderline between necessity and eventuality, contingency, i.e. a borderline between numbering nature and naming culture.

Q: Didn't Foucault consider numbers as eventual?

R: Yes, both Foucault's and Nietzsche's scepticism towards truth also include numbering. As a matter of fact neither structuralism nor poststructuralism accepts the borderline between necessity and eventuality. Structuralism believes all is necessity and sees apparent eventuality as hidden necessity to be uncovered through research. And poststructuralism believes all is eventuality and sees apparent necessity as hidden eventuality to be uncovered through narratives. The pencil-dilemma opens up for a moderate postmodernism paying tribute to both sides by taking on the task as a guardian of the borderline between necessity and eventuality by pointing to hidden eventuality in phrasings and by accepting the numbering of nature, where the truth can be controlled by asking the numbered itself through measuring instruments. To use a metaphor from mathematics: Within numbers there is a borderline between constants and variables to be guarded against parameters who hide a variable nature beneath a constant appearance.

Q: But we cannot live in a world full of variation. It would be chaotic and without order.

R: Correct, we need to have what the SSK calls closures and we use our political democratic institutions to install such closures all the time. Also different cultures shows different form of closures. But still it is important to distinguish between what might be changed and what might not, between political laws and physical laws. Inspired by the concept of entropy from physics measuring the degree of order, we could talk about different degrees of eventuality. And inspired by Giddens' terms system and structure (Giddens 1984) we can divide the social in two layers: culture and society. Nature would then have no eventuality, culture would have a low, society a medium and individual lives a high degree of eventuality.

Q: So phrasing might be problematic, but we are phrasing all the time.

R: Of course we are phrasing all the time, but we should be aware of when we phrase eventuality as necessity. The ability of phrasing is what separates humans from animals. Animals are bound by rituals, but we humans can free ourselves from rituals by using phrasing and installing to generate an RRR-cycle: Ritual-Reflection-Routine. Rituals can be lifted from the practice level to the reflection level by a phrasing. Once phrased we can think, discuss, form convictions, make decisions and take action by returning to the practice level to install and institute rational institutions and routines - but these routines might become new rituals, freezing us until we rephrase them again. Phrasing is freeing, echo-phrasing is freezing, and rephrasing is refreeing. Routine is hot, ritual is not. Postmodern counter research is aiming at re-freeing through re-phrasing, at transforming rituals into routines, both at a practice and at a discursive level.

Q: But how can you tell routine from ritual?

R: A routine is generated and justified by a discourse rationalising why the routine is installed. If the generating and justifying discourse has degenerated into an echo-discourse, then the generated routine has become a ritual. As e.g. if mathematics education is constantly justified by a "mathematics is, and mathematics is applied, hence mathematics must be taught" discourse.

Q: But still I have problems distinguishing between modern research and postmodern counter research.

R: The difference is that modern human and social science produce word statements and general claims. Postmodern counter research produces counter examples, and a counter claim is en existence claim and not a general claim. So postmodern counter research has to provide proof of existence and not proof of generality. And a proof of existence is indirectly a number statement, for as e.g. Frege says, a confirmation of existence is a negation of zero-occurrence. So postmodern counter research includes first a numbering, then a rephrasing. First the institutionalised and ruling traditional practise is searched for rituals by searching its justifying discourse for echo-words and excluded words, i.e. words that are never varied or absent, word with zero variation or presence. Once these zero-occurrences have been identified through discovery, through modern research, then imagination takes over to rephrase the echo-phrasing thus creating a counter discourse, that might make us change the way we see the world and might change our convictions, institutions and routines. But this rephrasing is just one example of a rephrasing serving to prove the existence of hidden eventuality, and an example might always be otherwise. So the inherent eventuality of an example protects postmodern counter research from its own demand to respect the eventuality of phrasings. And the example serves two purposes, to convince and to inspire: it serves to convince about the existence of a counter example and it inspires to look for other examples. In this way the postmodern counter research unites two essential values, the convincing power of modern research and the inspiring power of postmodern narratives.

Q: Do you say that quantitative research is OK?

R: Yes and no. Numbering nature is OK if it is based upon the five necessities of nature. Numbering culture is not OK since it is based upon a previous phrasing as e.g. ‘abnormal student’. The eventuality of such a phrasing will be transferred to the corresponding numbering.

Q: But you say that qualitative research is problematic?

R: Yes and no. Theory based qualitative research will always contain hidden eventuality, partly for poststructural reasons, partly because questionnaires and interview lists basically are interrogation methods only allowing the informants to answer to what they have been asked. However empiric based grounded research is OK as long as it only produces new words and not new theory. I.e. as long as it is ‘grounded naming’ and not ‘grounded theory’; new names are examples of existence-statements, and they can inspire to new rephrasings. It is only with new sentences one can ask if it is fact or unintended fiction. And grounded naming is used in postmodern ethnography.

Q: Postmodern ethnography?

R: Ethnography means graphing the ethno, i.e. phrasing the other. The premodern ethnographer was the missionary send out to save the “Prodigal” natives by preaching to them. The modern ethnographer is the researcher send out to develop the “under developed” natives by teaching about them, telling innovation agents how they can be phrased. The postmodern ethnographer is the postmodern researcher dropping all claims that his phrasing of the other can be true. Instead you want to learn about hidden eventuality in yourself from the other, and you can do so by exchanging gifts with the other. Instead of conquering the conversation room to preach or teach, the others are invited in to receive a gift: the prestigious role of the narrators, by being asked and allowed to tell about themselves. If the other returns the gift, the two self-stories can produce a new third mutual story and perhaps a new practice in the case of postmodern action research. So using postmodern ethnography both the learner and the other will leave the conversation room with two stories about hidden eventuality: the story of the other and another story.

Q: Could you give an example?

R: In many cases the students are considered the raw material of the educational process. The processing of the students is studied, and the students are observed as if they were particles in a force field. Occasionally students are interrogated through questionnaires or interviews. In short, students are what Foucault calls excluded from the discourse. But students are human beings who are reflecting about their experiences and able to tell stories about them. So to them it is a present to be asked ‘Tell me about your learning life with mathematics’. In the case of Ruth she tells about the teacher entering the room, opening the textbook and starting to teach. But what he says is precisely what is in the book. At home the students find the book difficult to understand, so next day they ask the teacher to explain the book, and once more he repeats the book. I coin the name ‘echo-teaching’ from this story. I handed out Ruth’s story to other students, many of whom recognised the echo-teaching. From their reaction I coined two other words. ‘Echo-rejection’ takes place when students say ‘I don’t want to learn what I can’t understand. If the teacher will not explain, I will not learn.’ ‘Echo-learning’ takes place when the student echo the teacher or the textbook, e.g. by saying ‘I want good marks for my future career so I cheated the system by learning by heart.’ Instead of ‘echo’ I could have chosen names as copy, parrot, clone etc. Thus there is a fair chance that ‘echo-teaching’ will develop into en echo-word. By choosing ‘echo’ ‘echo-teaching’ connects to the ‘echo-society’ discourse, which reveals a hidden understanding of the exodus-problem in mathematics, where more students reject mathematics based educations within science and technology (Jensen et al 1998): a postmodern dilemma-society rejects the echo-answers of a modern echo-society. So while modern students might like modern mathematics, postmodern students reject must reject it.

Another example was Africa, where I was invited to assist in an action research programme. However the wish for change only existed at a rector level, but not at department level ad hence not at teacher and student level. Staff development meetings were planned but not realised. So I changed strategy from modern action research to postmodern ethnography using gift exchange: Instead of waiting for a meeting in the seminar room or at my office, I went to the teachers room and said ‘tell & show me what YOU do’. This gift was then returned by asking me what I did. I then told them postmodern mathematics as a hidden alternative to modern mathematics, and I was asked to implement it in the classroom. In this way the students were also exposed to the hidden eventuality. In the following discussion the terms ‘Top-Down’ and ‘Bottom-Up’ mathematics arose substituting modern and postmodern mathematics. Thus both parts left the conversation room with two new stories, the story of the other and another story.

Q: In your summary you said that postmodern counter research produces counter examples to structure claims, which by claiming a status of truth automatically become echo-statements. But doesn’t this counter example become a new structure claim itself that becomes a new echo-statement?. Isn’t there a contradiction here?

R: Yes. It seems as if we have a problem of self-reference and infinite regress here, another example of Russell’s paradox ‘This sentence is false’. It is a main argument against the postmodern that it cancels itself. If all phrasings are eventual, so is yours, so how can you claim to be telling truths? And against Loytard’s statement ‘Simplifying to the extreme, I define *postmodern* as incredulity toward metanarratives’: Doesn’t his incredulity also apply to his own metanarrative?

Q: Precisely! So how can postmodern statements claim to be true? Isn’t here that postmodernism becomes nihilism?

R: Apparently. But we forget that Russell solved his paradox himself in his type theory: Self-description leads to nonsense; a level can only be described from a metalevel. So a statement about a statement is a meta-statement and therefore not affected by the eventuality of statements. Or in the case of Lyotard: A narrative about a metanarrative is a meta-metanarrative and hence not affected by the incredulity toward metanarratives. Or the poststructuralist version: Al phrasings are contingent - except for this metaphrasing. From this perspective postmodern counter research becomes a meta-research, not studying the world, but studying truth descriptions about the world, assuming the role of the naughty boy in Hans Christian Andersen’s fairy tale about the emperor’s new clothes commenting ‘but he is not wearing anything’.

Q: But isn’t it destructive just to be critical?

R: Yes and no. Postmodern counter research has a destructive critical part and a constructive emancipating part. And even the destructive part becomes constructive if it is used to destruct a destructive practise or cure within e.g. education, as presenting mathematics so difficult that it excludes people when it has the potential to do the opposite.

Q: Is postmodern counter research just another word for critical research?

Critical mathematics is critical towards the application of mathematics. Postmodern mathematics is critical towards the echo-phrasing of mathematics. Critical mathematics is accepting mathematics as it is, but believes humans become empowered by being critical and reflective to how mathematics is applied in society. To postmodern mathematics the current mathematics discourse is just another ruling echo-discourse that hides its hidden eventuality. Postmodern thinking believes that humans become empowered by rephrasing existing echo-discourses. In this case by rephrasing mathematics, from the ruling top-down application discourse presenting mathematics as examples of more abstract concepts, to a bottom-up number language discourse presenting mathematics as tales about the social practices that created mathematics: how to divide the earth and what it produces.

Postmodern Counter Research in Mathematics Education

Q: How is postmodern research relevant within education?

R: In modern thinking institutions are created to serve a public need for e.g. education. Foucault questions this, suggesting that institutions are a result of the disappointment of the enlightenment discovering that the normality of nature was not paralleled by a normality of culture. This unbalance however could be cured by modern institutions created to cure the phrased or diagnosed abnormality. The institution even delivers a circular self reference proof for the correctness of the diagnose: Abnormal are those who are not cured by the institution's cure against abnormality. Thus the institution school arises from the diagnose 'uneducated' to be cured through education. Which unfortunately is not successful in all cases. A problem the school blames on a bad funding. Instead inspired by Foucault postmodern counter research suggests that it is the cure and not the client, that has a problem: Cure the cure, not the client.

Q: Within mathematics education, in what field is postmodern research working?

R: Postmodern counter research is working within the sociological part of the social turn in mathematics education research (Lerman in Boaler 2000). Postmodern re-search will first search for rituals on the practise level by identifying echo-phrasings in the ritual's justifying discourses. The practices can be studied in the classroom by observing or by listening to students and teachers. The justifying discourses can be studied in textbooks and curriculum descriptions. Then imagination and inspiration is used to rephrase these echo-phrasings to unhide hidden eventuality. Then curriculum-architecture is used to design alternative micro- or macro-curricula, which implementation finally is reported. There is a widening gap between theory and practice within mathematics education (Niss 2000). This gap could be narrowed by practice based research methods avoiding the 'London Syndrome': 'move all national universities to London, since local knowledge doesn't matter anyhow'. Postmodern counter research accepts the principle of situated knowledge (Lave et al 1992), thus having curriculum architecture as a core ingredient: If knowledge is local, then the locals should be allowed to develop local knowledge by bringing micro- or macro curriculum design into the classroom. Curriculum architecture offers to teachers and students a creative alternative to just being textbook echoes, and a possibility to perform postmodern counter research thus closing the theory-practise gap. Such research reports will not make claims convincing about necessity, but suggestions inspiring to look for other examples of hidden eventuality.

Q: Within mathematics education, who is postmodern research discussing with?

R: At the moment not very many for two reasons: First of all he postmodern research community is limited. At the first Mathematics Education and Society conference in Nottingham Steve Lerman and Anna Tsatsaroni in their plenary lecture about the role of sociology handed out an education matrix. One of the columns called 'Disciplines legitimising the pedagogic mode' had four rows: Cognitive and developmental psychology; Sociology (e.g. ethnomaths) and Critical social theory; psychology-behaviourism; and postmodernist poststructuralist theories. While the matrix was filled out in all cells belonging to the first three rows, the cells of the fourth row were practically all empty.

And second of all, to postmodern research talking with the classroom is more important that talking with other theorists, since producing counter examples in the classroom through curriculum architecture is more important than extending ruling discourses hiding eventuality. So a visiting postmodern researcher is more interested in easy classroom accessibility than in well equipped libraries.

Q: Tell me about your own research.

R: I call my working theme 'mathematics as a human right'. As a part of this I have focused on the exodus problem or enrolment problem in mathematics, where more students reject mathematics based education within science and technology (Jensen et al 98). I ask: Can there be hidden reasons behind this exodus phenomena? And here I find postmodern thinking useful suggesting we are blindfolded by ruling echo-discourses. Hence I have set up a research symphony consisting of four movements or phases asking: What is postmodern research? How do postmodern students react to modern mathematics? What is a postmodern mathematics curriculum? How do postmodern students react to postmodern mathematics? In brief the answers are: Postmodern counter research is a research producing counter examples to ruling rituals by searching for and rephrasing echo-words in the ritual's justifying discourses. Postmodern students ask 'tell me something I don't know, about something I know', but modern Top-Down mathematics by phrasing abstract concepts as examples or echoes of more abstract concepts, tells the students something they don't know about something they don't know, which makes some students reject and other students echo modern mathematics. A postmodern mathematics curriculum will phrase mathematics Bottom-Up as quantitative calculation stories about two fundamental social practices: dividing the earth and what it produces. Postmodern mathematics transforms many dropouts to dropins asking for more mathematics until they meet Top-Down mathematics.

So the conclusion of this study is: A possible hidden factor behind the exodus-problem in mathematics is its ruling top-down ritual installed by a discourse having mathematics and application as echo-phrasings. This echo-phrasing is freezing the educational system, forcing it to install mathematics education from above resulting in echo-teaching and echo-learning, and preventing it to see the hidden alternative, mathematics from below. This creates no problem to modern students getting their identity through echoing; but it creates problems to postmodern students building self-identity through self-stories. Thus the exodus problem might be an 'echodus'-phenomena: The echo-tradition of modern mathematics forces postmodern students to exodus. It is however possible to install an alternative bottom-up practise making mathematics a human right, but its corresponding justifying 'mathematics is easy' discourse has to fight the ruling 'mathematics is difficult' discourse. Which parallels the classical situation of the reformation, where Luther and later Kierkegaard pointed out, that it is possible to be saved without the assistance of the saving institution: It is possible to develop a number language without the assistance of an institutionalised Top-Down mathematics cure. This conclusion makes me very sad remembering all the unhappiness that came out of the wars between Catholics and Protestants.

Q: But is it possible to make mathematics a human right. It is generally accepted, that mathematics is difficult.

R: Mathematics is not difficult by nature; mathematics has been made difficult by being phrased from above. Phrased form below my research shows that mathematics becomes easy. And I think it is important to make mathematics a human right. If not mathematics could be used to exclude the unfitted from the future labour market where the 'math-based' GNR technologies makes most jobs superfluous, except of course the 'math-based' ones. Making mathematics difficult creates a 'natural' selection technique, where the excluded only have themselves to thank for not working hard enough with the 'difficult' mathematics. Also it is problematic to say that these technologies are math-based. It is the number-language and not its grammar, mathematics, is applied in these new technologies.

Q: This study sounds as traditional modern research.

R: Traditional research adds to the zone of necessity. This study adds to the zone of eventuality. It demonstrates the existence of a counter example to an apparent necessary top down ritual and to an apparent unavoidable exodus problem. It does not offer a new truth claim, a new statement of necessity about the client that can be used to install a new cure of the client. It offers a counter claim, a statement of hidden eventuality in the justifying discourse of the ruling cure. It does not prescribe a new cure; it only gives an example of how the cure could also be otherwise. It is a meta-cure curing the cure and not the client.

Q: To what kind of questions within mathematics education can postmodern research contribute?

R: With its ability to unhide hidden eventuality postmodern counter research reminds very much about architecture. So one field of application is to help to establish a new education and profession called 'curriculum-architecture'. Traditional curricula are filled with echo-phrasings, that could be rephrased. It was calculation that freed humans from feudality by changing the metaphysical laws from unpredictable to predictable thus making modern industrialised society possible. Hence we could assume, that calculation was a core subject in the curriculum of modern education. However it is not, instead mathematics is. At the curriculum level 'mathematics' is used as self referring echo-word in most goal statements: The goal of mathematics education is mathematics; mathematics is taught so that students can learn mathematics. There is no mentioning of outside needs or goals calling for mathematics as a means. Terms like 'multiplicity', 'number-language', 'quantity', 'calculation', 'predictability, 'change', 'per-numbers' etc. are absent. A list of mathematical qualities might be listed as positive values: its ability to further logical thinking, its widespread application in modern society, its role as a tool in further education etc. In short the goal-discourse seems to say: "Mathematics is, and mathematics is applied, hence mathematics is taught". The feudal origin of this statement can be seen be substituting 'mathematics' with 'God'. This mathematics&application-discourse automatically installs education from above together with 'echo-teaching' and 'echo-learning'.

Q: How does a mathematics&application-discourse automatically install education from above?

R: The mathematics&application-discourse says: 'Modern society builds heavily upon application of mathematics, hence mathematics education must be instituted at all levels'. If we ask: 'What should be learned first, mathematics or application of mathematics?' the answer automatically becomes: 'Of course mathematics must be learned before it can be applied! Without mathematics we have nothing to apply!?'. Within a mathematics&application-discourse you cannot say 'mathematics should be learned after its applications' without being considered mad and thus excluded form the discourse. This discourse is typical for modern structuralist thinking. Up there are metaphysical mathematical laws that are echoed down here. These laws have to be found and taught before they can be applied, creating a knowledge flow from the metaphysical down through universities and schools to applications in the industry.

Q: But mathematics is mathematics, how can mathematics be rephrased?

R: I once asked a mathematics professor what this 'mathematics' is. His answer was: 'Mathematics is what mathematicians do'. I.e. a self referring echo-answer without meaning installing mathematics as a ritual. An alternative to the structuralist top-down thinking would be a poststructuralist bottom-up thinking starting out from the zone of necessity. It is culturally given that we describe the world in words and numbers using a word-language and a number-language. And that we describe our languages in metalanguages or grammars. So one possible rephrasing of mathematics could be: Mathematics is a grammar of our number-language. Based upon the fact that 'multiplicity is, hence a number language and its grammar should be developed' this rephrasing enables us to formulate a non self referring goal statement as e.g.: 'The goal of mathematics education is to develop the students number-language enabling them to describe multiplicity in quantities and calculations'. This multiplicity-discourse installs education from below giving mathematics meaning and authenticity, which is what the postmodern self-story builders and learners are demanding.

In this rephrased grammar-discourse the question 'What should be learned first, mathematics or application of mathematics?' is rephrased to 'What should be learned first, the grammar of the language or the language?' Again totally convinced most people would say: 'Of course the language must be learned before the grammar of the language! Without a language, a grammar of a language has nothing to talk about!?'. The problem here of course is that now we are totally convinced about the opposite. This proves the postmodern point, that our beliefs and actions depend on the ruling phrasing and discourse, and can be changed through a rephrasing. Thus rephrasing mathematics to the grammar of the number-language frees us to think and to act otherwise.

Q: In what way?

R: We could compare the grammar of the number-language with the grammar of the word-language. The grammar of the word-language is successfully applied within the mother tongue by all, it is installed as a human right, but it is tacit knowledge developed through extensive use of the word-language. Very few are able to discursively justify why they are forming the sentences as they do. And if we make the rules discursive through teaching, many would drop out. Inspired by this analogy we can now ask: Maybe the grammar of the number-language, mathematics, can only become a human right if is developed as tacit knowledge through extensive use of the number-language? Meaning if we want mathematics to become a human right, maybe we should not teach it. In any case we should not teach it before its applications but through its applications. And since mathematics is applied to the number language and not the world, we should teach the number-language before its grammar. It is however impossible because we are frozen in a discourse, where the words as number-language, reckoning (regning in Danish, rechnung in German) and calculations are absent.

Q: But still mathematics is increasingly applied in modern society?

R: Is grammar increasingly applied in today's society? A grammar is applied to a language, which is applied to the world. It is the number-language, which is being increasingly applied, not its grammar.

Q: But isn't there a new discourse coming up talking about competences?

R: Yes, a new ruling goal-discourse seems to be developing. Mogens Niss (1999) says that in order to prevent 'syllabilitis', the blindly following of content lists, we should instead describe mathematics in competence-terms. He then lists a series of mathematical competences all based on insight: Mathematical thinking, mathematical reasoning, etc. But again 'mathematics' is still used as a self-referring echo-word. And secondly, by being build upon insight 'competence' tries to unite two complementary words, qualifications and competence. Until now the biological fact that humans have both a human and a reptile brain has been reflected in language through the words 'qualifications' and 'competences'. Giddens calls it discursive and practical consciousness (Giddens 1984). Qualifications are discursive knowledge and insights resulting from discursive teaching. Competences are practical, tacit know-how. I can qualify my students, but I can't 'competence' them. To introduce a new word 'competence' by reusing the name 'competence' but giving it the meaning of 'qualification' seems deeply problematic. It becomes a meaningless echo-word impossible to rephrase. Is this an act of despair by an institution whose justification is fainting?

Q: So much about the goal discourse. How about the content list of mathematics.

R: In the content list we also find an echo-tradition. Mathematics is presented from above derived from the mother concept set: number sets, expressions, equations, functions, and calculus. And the operation order is always addition, subtraction, multiplication, division etc. A rephrasing of this would unhide a 'mathematics from below' content list containing e.g.: multiplicity, quantities and qualities, number-language and word-language, differentiating degrees of many by number-names, counting by ones, counting bundles, stacking, rebundling, multiplication as division, ...

Q: Just a moment. What do you mean with multiplication as division? And how about addition?

R: Multiplicity is one of nature's five necessities. Number are names for different degrees of multiplicity being bundled in ten-bundles, although France and Denmark partly used twenty-bundles. A stack of four three-bundles can be rebundled or divided into six two-bundles or into one ten-bundle and two ones. In this way multiplication becomes rebundling into ten-bundles: Asking what is 7*4, is asking what happens, if seven four-bundles are rebundled or divided into ten-bundles; and the answer is two ten-bundles and eight ones: 7*4 = 28 = 2*10+8. If we want to rebundle in other bundle sizes than ten, we can no longer multiply, then we have to divide: 28 = ?·8, ? = 28/8 = 3 rest 4, so 28 = 3*8 + 4.

As to addition it is deeply problematic since addition is a false abstraction, in opposition to multiplication, which is a true abstraction, meaning it is true whenever you meet it. 7*4 is always 28 since multiplication is just describing a rebundling. With addition it seems a little difficult to claim that 2+3 = 5, when in most cases it is not: 2m+3cm=203cm, 2w+3d=17d, 23 is 23 and not 5 etc. It only has meaning to add if the units are the same: 2·m+3·cm = 2·100·cm+3·cm = (200+3)·cm = 203·cm. I.e. addition is only meaningful inside a parenthesis, that ensures that the units are the same. So multiplication and addition belong to different sides of the borderline between necessity and eventuality.

Adding fractions suffers from the same problem as adding numbers without units. According to the principle of a common denominator 2/3+4/5 = 22/15. Adding numerators and denominators 2/3+4/5 = 6/8 is considered a meaningless mistake. However 2 cokes out of 3 cans and 4 cokes out of 5 cans total 6 cokes out of 8 cans, and not 22 cokes out of 15 cans. Now the meaningless becomes meaningful and vice versa.

So from this viewpoint multiplication is more fundamental than addition, which of course calls for a whole new curriculum design. A curriculum where the starting point is not a metaphysical concept set contaminated with syntax errors, but the social practices of bundling, stacking and totalling through counting and calculating.

Q: What do you mean with the syntax errors of set theory?

R: Geometry was successful to set up a system of axioms from which geometry could be derived. Algebra also tried, but the resulting set-theory is mixing elements and sets in its axioms thus committing the syntax error, that comes from mixing language and metalanguage: “This sentence is false”, “The verb had to much to drink” etc. Both Russell and Gödel have shown, that mathematics can’t prove itself. Still modern mathematics builds upon set theory making mathematics almost fundamentalism in Giddens’ sense: “Fundamentalism may be understood as an assertion of formulaic truth without regard to consequences (Giddens in Beck 1994).”

In the textbooks we also find echo-definitions, echo-names and killer-equations. The definition of a function is an echo-statement: “A function is an example of a relation between two sets that assigns to each element in one set one and only one element in the other set.” This phrasing is an echo of the university’s definition and is echoed in other textbooks nationally and internationally. But using Foucault’s genealogy going back in time to when the name was born, we find another phrasing when Euler says, that a function i a name for a calculation with a variable number, or more precisely: “A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities (Euler 1748).” The modern function concept also contains syntax errors. f(x)=x+3 means: Let f(x) denote the calculation x+3 with x as a variable number. But then f(x)=7 means: Let f(x) denote the calculation 7 with x as a variable number. And f(2)=5 means: Let f(2) denote the calculation 5 with 2 as a variable number. Both cases are syntax errors violating Russell’s type theory by mixing language and metalanguage. Calculations belong to the language level, functions to the metalanguage level. Calculations are calculating numbers, function are labelling calculations. Historically ‘function’ is born just after calculus. And with good reason. Before calculus calculations had basically been used to predict rebundling results thus only operating on numbers. Calculus is used to predict change: Can we calculate the change of a calculation? Calculus thus operates on calculations with changing quantities, and this is precisely how Euler defined a function.

By using the term ‘example of’ the modern definition of a function is relating an abstract concept upwards to something more abstract thus presenting mathematics from above as examples or echoes of the mother concept set. Such definition can only be echoed ant not be rephrased, since you can only have one mother. The meaning structure of the modern definition is ‘bublibub is an example of bablibab’, which can only be learned by heart as an echo. In short, coming from above the modern mathematics is forced to become echo-statements forcing teachers to practice echo-teaching and forcing students to learn by echo-learning. This is no problem in a modern echo-society where students get their identity by echoing. But it is a problem in a postmodern society, where students are self-story builders asking for stories that can be related to their existing story.

On the contrary by using the term ‘name for’ the Euler definition is relating an abstract concept downwards to something less abstract, thus presenting mathematics from below, allowing indefinite rephrasings by relating it to other examples of calculations with variable numbers; allowing teachers to practice example-teaching showing how mathematics is constructed through abstraction;

and allowing students to extend their self-stories. The meaning structure of the Euler definition is ‘bublibub is a name for a calculation’ making the ideal type of a postmodern learner say: ‘I know what a calculation is. I didnot know that a calculation could be called a function. But I know now since you have told me something I don’t know about something I know. Now tell me something I don’t know about functions’.

Q: But in most textbooks the function concept is introduced after many practical examples. In this way the concept seems to be abstracted from below!

R: It seems so, yes. But the examples are all means, sugar around the pill: the definition, which is phrased from above. All examples can be otherwise, but the definition can’t. The examples are chosen because they are examples or applications of the concept to be taught. If the examples were the goal, we have already seen, that e.g. Euler made a different abstraction than the modern one. But of course we can say that both definitions are naming something. In the case of Euler we name a socially constructed classification of concrete objects, calculations; a classification made socially important by calculus. The other definition is naming an abstract structure: a relation between two sets. So the first definition is a nominalist one naming concrete objects, and the second a structuralist one naming an abstract structure, that after a rephrasing appears to be a triviality: Of course a calculation can only give one result!

Q: You also mentioned echo-names and killer-equations.

R: Echo-names from above occur all over mathematics just waiting to be rephrased from below. Equations can be rephrased to calculation stories. Functions can be rephrased to calculations with variable quantities. Linear and exponential functions can be rephrased to change by adding and multiplying. Differential equations can be rephrased to change calculation stories. Differentiable can be rephrased to locally linear. Continuous can be rephrased to locally constant. Limits can be rephrased to about-numbers. Numbers can be divided into unit-numbers and per-numbers. Unit-numbers are e.g. \$, m, s, kg etc. Per-numbers are e.g. \$/m, m/s or m/100m = %.

And equations can be divided into killer-equations and practice equations. Killer-equations are equations you never meet outside the classroom, and whose only application is to kill the interest of the students. Killer-equations come from above where an equation is an example of an equality relation between two expressions allowing e.g. arbitrary polynomial fractions to be equalled. Rephrased killer-equation become practice-equations allowing only equation to arise as calculation stories from social practices as e.g. 2x+4=10 arising from the question “2 kg @ ?\$/kg including a 4\$ fee total 10\$”.

In the word-language we always use full sentences to evaluate the truth of a sentence. Instead of “green” we say e.g. “This table is green”. For the same reason also the number-language should use full sentences from day one, saying “T = 3·5” instead of just “3·5” thus specifying both what is being calculated and the calculation. Standard formulations from first year mathematics as “3+5” is a third order abstraction being abstracted from reality, from the units and from the equation. Such abstractions construct mathematics as encapsulated and create serious problems to the students when they later meet wor(l)d problems.

Also solving equations contains rituals an echo-phrasings. A Top-Down approach will phrase “2+3·x=14” as an equation only solvable after equation theory has been introduced thus showing the relevance and applicability of modern abstract algebra in performing the neutralising ‘do the same on both sides’ method. Alternatively a Bottom-Up approach will phrase “2+(3·x) = 14” as a calculation story reporting both a calculation process (2+3·x) and a calculation product (14), thus accessible together with calculations and solvable by reversing or walking the calculations:

	*3**		*+2*	
x	→	**3*x**	→	**3*x + 2 = 14**

12/3 = 4	←	14 – 2 = 12	←	14
	/3		*-2*	

Q: To sum up, how would a postmodern mathematics curriculum look like?

R: Schools, education and curricula should meet human needs. To decide what we need we have to go back to the zone of necessity. We have holes in the head because we need food for the body and stories for the brain. So school could be rephrased as a story-house similar to other story houses as the church and the global TV. And education could be rephrased to storytelling. In modern echo-society students needed echo-stories to build echo-identity. In a postmodern dilemma-society individuals have to construct their own self-identity by building self-stories. Hence a postmodern school should provide stories from below with known concrete subjects, telling the students something they don't know about something they know, thus enabling them to extend their existing self-story.

Q: And where does mathematics and education come in?

R: Mathematics consists of Geometry and Algebra, so we could ask: what are geometry and algebra stories about? With geometry it is easy: translated from Greek it means 'earth measuring'. So Geometry can be introduces in school as stories about earth-measuring together of course with a lot of exercises in dividing and measuring earth. With algebra it is more difficult. Very few textbooks or people know that algebra means 'reunite' in Arabic. In all money based cultures there is a cultural practice called totalling: If we buy several items we don't have to pay them individually, we can ask for the total. And we don't have to pay exactly the total; we can pay more, which then is split into the price and the change. To check we reunite the total. In a way we can say that geometry and algebra are stories about how we divide the earth and what it produces.

Q: But how about calculation?

R: Algebra is about uniting and dividing the total. And the total can be united through in four different calculation types depending upon the nature of the numbers to be united. Multiplication and exponentials total constant unit- and per-numbers. Addition and integration total variable unit- and per-numbers. In reverse division and root and logarithm split totals into constant unit- and per-numbers. And subtraction and differentiation split totals into variable unit- and per-numbers. (Tarp 2000 a)

Conclusion

Q: So to sum up, what can you say about postmodern research and mathematics education in few sentences.

R: Postmodernism accepts the eventuality of all phrasings, and sees humans clientified and frozen by echo-phrasings. Postmodern counter re-search searches for echo-words in ruling discourse to create counter discourses by re-phrasing these echo-words. In a modern industrialised echo society students get identity through echoing the tradition making echo-mathematics from above rational and efficient. In a postmodern information dilemma society the students have to build up self-identity through self-stories build upon meaning and authenticity, making mathematics from below rational and efficient. Mathematics education is frozen in an Mathematics&Application discourse that forces it to practice echo-teaching from above and prohibits it to see the hidden alternative, a Mathematics&Grammar discourse seeing Geometry and Algebra as stories about two fundamental social practices: How two divide the earth and how it produces, and how two name different degrees of multiplicity.

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● SECTION III, SPREADING the New Paradigm

17. ICME Conferences 1976, 1996-2024

ICME 3 in Karlsruhe

A Welcome Letter

The International Program Committee, 3rd International Congress on Mathematical Education

D 75 Karlsruhe

Dear Sirs,

I warmly welcome the First Announcement of the Third International Congress on Mathematical Education. Many problems within mathematical education are of international nature and many ideas within mathematical education arise from international discussions. Therefore it is of vital importance to form a forum for international contacts and debates. The ICME has created such a forum.

I am particularly interested in two areas within mathematical education. My background is the following: seven years of education in mathematics and physics, three years of teaching mainly at senior high school but also at university level (teacher training), and study tours to Belgium, England, the U.S.S.R., and the United States. I have drawn the conclusion that textbooks and teacher training are among today's major problems within mathematical education. I enclose two reports from Farawaystan ts clarify these problems. The majority of textbooks and training courses tend to overdo the theoretical aspects of mathematics, thus transmitting to the receiver a distorted picture of mathematics, which might cause him to lose interest in the subject.

In order to find the key to these problems we must in ray opinion analyze the perception of mathematics transmitted in an educational process not only knowledge and skills are transmitted but also conceptual formations, a perception of mathematics and common attitudes, normally only knowledge and skills are assessed, thus having great influence on textbooks and education. Even if we are not able to find ways to assess the other transmitted items, we should still let them influence textbooks and education.

At the moment I am working on textbooks and teacher training. I am writing a system of textbooks for the Danish senior high school based on the perception of mathematics that mathematics is model building of the real world. The system will consist of books on functions, vectors, calculus and probability theory, and will be completed in 1976, Also as part of being visiting teacher at the University at Roskilde, Denmark, I am working on a textbook on didactics defined as an analyses of the consequences of various ways of presentation (e.g. various textbooks) as to the transmission of knowledge and skills, conceptual formations, perception of mathematics and common attitudes.

I look forward to the congress with big interest, and I would be glad if you would send me the Second Announcement,

Yours faithfully, Allan Tarp, 15th August 1975

Poster Abstract

On different Perceptions of Mathematics

During a mathematical educational process not only knowledge and skills are transmitted, but also conceptual formations, a perception of and attitudes to mathematics, and a perception of and attitudes to real life. Normally only knowledge and skills are assessed thus having the greatest influence on textbooks and education. The other transmittals are perhaps not as easy to assess, but they are of equal if not greater importance. So they should also have great influence on textbooks and education.

As to the perception of mathematics (at least) two different perceptions seem possible:

1. *Mathematics is a collection of arbitrary theoretical structures*: Mathematics is created by setting up some arbitrary definitions and axioms and from these deducing a sequence of theorems.

2. *Mathematics is model-building of the real world*: Mathematics is about solving problems from the real world and generalizing the results BO that they can be applied to similar problems.

Which perception do we want to transmit, and what determines the perception transmitted? In order to clarify these problems it is often useful to consider an analogy. The author has done so in some reports from "Farwaystan" where they treat tool science as we treat mathematics. One of these reports has been published in "the Mathematical Gazette", number 407, March 1975.

The other follows below.

Report from Farawaystan II

On a previous occasion I have reported on the strange way in which people in. Farawaystan teach tool science. To find an explanation I went to see the place where the teachers are trained. It is at the same place as where research in new tools iss made, the Institute of Tool Science. I was kindly received and was guided through the institute by one of its professors.

"At our institute we are specially Interested in screws," the professor told me. "So it is not here you have a screw loose?" I tried joking. The professor ignored my remark and continued, "It is a trivial matter that screws can be used to be screwed into various materials, and for a long time the theory of screws only consisted in banalities due to this. Screws did not become of serious interest until a gifted young research student from one of our neighboring universities got the genius idea that a screw could also be screwed into a screw. Can you imagine in the middle of a screw a hole into which another screw can be screwed?

This idea created an enormous research activity not only at our own department but also at other departments. This shows that our work concerns and is of interest to others than ourselves. For example at the department of screwdrivers they have started constructing a screwdriver to screw in the two screws at the same time!"

The professor continued, "You might know that our subject often develops by generalizing ideas. Thus to screw a screw, into a screw can be generalized to screw more screws into the same screw, or to screw a screw into the screw you have just screwed Into another screw. In this way we are led to consider the so-called screwcycles. For the moment we have a group of students who work in this area. They are inspired and guided by one of our foreign staff members. You see, in order to keep in touch with the development we are very anxious to be in contact with other universities.

Of special interest among the screwcycles is the so-called closed screwcycles, consisting of screws making a closed cycle. I got my job at this university because of my thesis on closed screwcycles of second degree, i.e. closed screwcycles consisting of screws into which two screws can be screwed."

"Talking about screwcycles", the professor continued, "you might know the classical problem which so many have unsuccessfully tried to solve, the famous 4-screw problem: four screws make a closed cycle, how many of them can be screwed simultaneously?"

"Maybe you also make research on headless screws?" I asked with a forced smile. "Headless screws?" the professor answered. "No, we do not make any research on that, but it sounds as an interesting subject. Thank you very much for the good idea."

When I was going to leave the institute, I asked the professor, "I admit that the problems you deal with might be of theoretical interest, but are they of any practical value? When do you need to screw a screw into a screw? I almost get the impression that today research workers are some sort of assembly-line workers, working on what they receive from the other workers without considering where it comes from, where it goes, or what controls the assembly line. Thus the development of tool science is determined not by problems from the real world, but by each individual research worker's want to get his share of the limited number of jobs and promotion possibilities. In this way tool science easily develops in the direction of the weakest resistance, which might not be where the real problems are. Do there not exist more essential problems within tool science, on which the limited resources should be used? It would be a sad thing if tool science, once created to solve problems from the real world, should end up as a patent agency with 1000 useless patents for each useful."

"As to your last question," the professor answered, "I have already told you that the subjects we deal with are and of course must be determined by what other universities deal with.. As to your first question I would say that today it is possible you do not need to screw a screw into a screw, but who knows what the future might demand."

"But isn't that the same," I argued, "as instead of satisfying the needs of people by building roads where they are needed, the architects started placing roads at random, and the pipe fitters started placing pipes at random, because "you might need them in the future?" The result would be that instead of placing roads and pipes where people are, people must be placed where there are roads and pipes."

The professor answered with a counter-question, "You do not make research like this in your country?" "Yes", I replied, "research into mathematics is mostly made in this way, but between mathematics and tool science there is all the difference in the world!"

"There is?" he said.

Letter of Acceptance

3rd INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION KARLSRUHE 16. - 21. AUGUST 1976, 3. INTERNATIONALER KONGRESS ÜBER ATHEMATIKUNTERRICHT

Mr. Allan Tarp, Grenaa, Denmark, June 24, 1976

Re: Acceptance of your Short Communication

Ladies and Gentlemen!

On their last meeting of June 17, 1976 the IPC has decided on the acceptance or non-acceptance of short communications submitted for approval and has assigned the accepted papers to either lectures or poster sessions.

We are glad to advise you that your paper has been accepted and scheduled for the poster sessions of section B3. Your abstract will figure in part ITT of the program. The reporter, the coordinator and the Chairman of your section reserve the right to present some of the papers of the poster sessions in the section to suggest discussions on the topic provided the work in the section gives time and opportunity for this purpose.

The posters comprising four pages of DIN A4 size each should be displayed on the respective boards in the section centre on Tuesday, August 17, 1976 to allow participants to inform themselves in due time. The program will advise you about the situation of the section centres.

You are kindly asked to attend the opening of the poster session on Thursday, August 19, 1976 at 2.00 p.m. and to keep near your poster in order to discuss it with those participants who might be interested. If on this occasion a more detailed discussion is requested by several participants, a room will be made available by one of the secretaries of the section.

Yours faithfully Local Organising Committee.

IMUK-PRASIDENT: PROFESSOR S. IYANAGA, TOKYO, JAPAN
KONGRESSLEITUNG: PROFESSOR DR. H. KUNLE, KARLSRUHE
SEKRETAR: J. MOHRHARDT, KARLSRUHE
ADRESSE: D 75 KARLSRUHE, UNIVERSITÄT KAISERSTRASSE 12, RUF 0721/608-2059
KONTEN: POSTSCHECK KARLSRUHE 761 81-758 •
DEUTSCHE BANK KARLSRUHE 352 732

Mathematics, an Integral Part of the Real World

Accepted poster at ICME 3, by Allan Tarp, Grenaa, Denmark.

Mathematics is part of the school's attempt to give the students a picture and an understanding of the real world. The real world is a unity consisting of different aspects, so one way of learning about the real world is to learn about its different aspects. However, different aspects do not necessarily integrate to create a unity. The following is an attempt to make a rough outline of how to teach mathematics so that it is perceived as mathematics and not as 'theoretics', i.e., as an integral aspect of the real world and not as a subject isolated from the real world.

Why Teach Mathematics?

Mathematics is not taught for its own sake. If so, it would be a mere hobby, and hobbies are not normally taught in the school, and certainly not to the extent mathematics is taught. Mathematics is taught because there is an outside need for it. The real world is often described by quantities. So, when you must reason on the real world, you often must reason on quantities. And this is what mathematics is about. So, mathematics is taught because there is a need for reasoning on quantities, not quantities which are arbitrarily invented, but quantities coming from or originating from the real world.

What Should Be Taught?

Quantities consist of numbers and dimensions, so first and foremost we should teach to reason on numbers. Since quantities often depend upon other quantities, we should teach to reason on dependent quantities. Dependent quantities do not always remain constant, they often change, they show growth, positive or negative. Some quantities show constant growth, some show multiplying growth, some show other forms of growth. In some cases, it is natural to talk about periodic growth, in other cases it is natural to talk about instant growth. All in all, we should also teach to reason on growth. Some quantities are described in fractions. There-fore we should teach how to reason on fractions (statistics and probability). Some quantities occur in connection with figures. Therefore-we should also teach how to reason on figures (geometry).

What Are the Consequences of an Educational Process?

An educational process in mathematics can be regarded as a communicational process which transmits to the student skills and knowledge, perceptions of concepts, a perception of mathematics and contributes to a perception of the real world.

Skills and Knowledge

When reasoning on quantities, some reasonings are repeated quite often. These are then put down in books (or memorized) as theorems. They constitute what we call skills and knowledge. When evaluating our teaching, we only care about this area and not so much about the perceptions of concepts, the conception of mathematics or the contribution to the perception of the real world trans-mitted. We are so anxious to make the mathematical room a perfect one that we often forget to put doors in it, i.e., if the student gets out of the room, he does not come back. What are the doors to the mathematical room? In mathematics we teach properties of concepts, so to activate these properties, the concepts must be activated. Whether or not the concepts are activated outside the classroom depends upon what perceptions of concepts the student has received.

Perception of Concept

To the student, what is a function? Is it a rule which assigns to each element in one set exactly one element in another set, or is it just a technical word for a dependent quantity? Is an exponential function a function defined by f(x) = ln-1(x*lna), or is it a technical word for a quantity which grows by multiplying. Is the derivative the limit of a difference quotient, or is it a technical word for instant growth? Where the former perceptions do not occur in the real world, the latter do. In the first case the student has learned theoretics, in the second case mathematics.

Two things are important to the perception of concepts 1) the naming of the concepts, 2) the examples constituting the concepts. Anatomy is taught in school without the use of technical (latin) names. Mathematics is not. It is strange to observe how the naming of mathematical concepts has been determined by the theoretical properties of the concepts rather than their practical properties. Thus, we say ‘function’ instead of ‘dependent quantity’, ‘continuous’ instead of ‘locally constant’, ‘differentiable’ instead of ‘locally linear’, ‘derivative’ instead of ‘instant growth’, etc.

If we want the student to find the concepts in the real world, should we not use the latter names? At least we should use both. Of course, here the word growth is to be understood in a broad sense, but that is a simple case of accommodating the concept once it has been created.

Concepts are not created out of nothing, yet from many textbooks this seems to be the case with mathematical concepts. Concepts grow out of examples. A concept without examples is and empty and uninteresting concept. It is more important to know some examples than to have memorized the concept in a formally correct way. Knowledge of examples can be transferred to similar examples by analogy. From an empty concept nothing can be transferred.

When learning about mathematical concepts things like the classroom the awareness that this is the mathematical course, the teacher, his way of talking and acting, the *x*'s in equations are often constant factors. Thus, to many students they become Part of the concept. These students will have difficulties doing mathematics when one of these factors is missing, e.g., in a physics course. How do we vary out this noise from the concepts?

Perception of Mathematics

If the student perceives mathematics as a collection of axioms, definitions, and theorems, where the axioms and definitions come out of the blue air from their birth formulated in a precise and formal way and with theorems being immediately deduced from them, he has learned about something which is not part of the real world, he has learned theoretics. No intelligent person will be attracted to this, he will forget about it the moment he is outside the classroom, and he will be reluctant to deal with it later. If however, he has perceived the axioms, definitions and theorems motivated and developed from problem solving situations from the real world, he has learned about something which is an integral part of the real world. He has learned mathematics.

Perception of the Real World

The school traditionally splits the real world into different subjects. In order that these subjects should integrate to give the student a picture of the real world as an integral unity, each subject must be considered as an integral part of a unity, they must be considered the means where the real world is the goal, not the other way round. Only then has the school and mathematics done their job. This is difficult to establish by the traditional isolated courses, on by inter-subject courses. Combining two isolated parts does not make the whole a unity. A solution might be found in project orientated courses where the different subjects deal with the same project. So, part of the problems in teaching mathematics must be solved outside mathematics.

Mathematics, an Integral part of the real world

Oral report to section B3 at the 3rd International Congress on Mathematical Education in Karlsruhe in 1976, Overall Goal and Objectives for Mathematics Teaching (Why Do We Teach Mathematics?)

In English, 1 minute:

Why teach our subject? To me the answer is simple. Our subject is part of the school, and it is an attempt to give the students a picture and understanding of the real world. The real world is often described in quantities, and the reasoning on quantities is what our subject is about. It is not taught for its own sake. It is not a hobby. It is not taught because it might have some transfer values as critical and constructive thinking. Many people doubt it has. It is taught because there is a need for reasoning on quantities in the real world.

When teaching our subject, we transmit to the students not only skills and knowledge, but also perceptions of concepts, a perception of our subject and we contribute to the students’ perception of the real world. Only skills and knowledge are tested. But knowledge is about concepts and will only become activated if the concepts are activated. Why is it always the theoretical aspects and not the practical aspects, which gives names and content to the concepts? If the student thinks of a function as arrows between sets, he will never meet a function in the real world. If he thinks of it as just a technical word for a dependent quantity, he will meet it often.

The perception of our subject. If the student perceives our subject as a collection of axioms, definitions, and theorems, where the axioms and definitions come out of the blue air from their birth formulated in a precise and formal way and with theorems being immediately deduced from them, he has learned about something which is not part of the real world, he has learned ‘theoretics’. If, however, he has perceived the axioms, definitions and theorems motivated and developed from problem solving situations in the real world, he has learned about something, which is an integral part of the real world. He has learned Mathematics.

Et maintenant en Français, 1 minute et 15 secondes.

ICME 9 2000 in Tokyo

Poster: “A New Curriculum for a New Millennium” - A Curriculum Architect Contest

Last year a school in “Farawaystan” decided to arrange a curriculum architect contest in mathematics: “A new curriculum for a new millennium”. Below is a fictitious response to this contest.

Organic Bottom-Up Mathematics: A Three Level Bundling and Totaling Curriculum

The holes in the head provide humans with food for the body and knowledge for the brains: tacit knowledge for the reptile brain and discursive knowledge for the human brain. This proposal sees a school as an institutionalized knowledge house providing humans with routines and stories by making them participants in social practices and narratives, and by respecting conceptual liberty.

The chaotic learning of tacit routine knowledge can be guided by attractors in this case by social practices providing authenticity. In the case of mathematics, the social practices will be those of bundling and totaling according to the Arabic meaning of the word Algebra: reunite.

The agriculture transforms the human hand to an artificial hand, a tool, enabling humans to transform the wood to a field for growing crops. The industrial culture transforms the human muscle to an artificial muscle, a motor, integrating tools and motors to machines enabling humans to transform nature raw material to material goods. The information culture transforms the human reptile brain to an artificial brain, a computer, integrating the artificial hand, muscle and brain to an artificial human, a robot, freeing humans from routine work.

Human production and exchange of goods has developed a number language besides the word language to quantify the world and calculate totals. Agriculture totals crops and herds by adding. Trade totals stocks and costs by multiplying. Rich traders able to lend out money as bankers total interest percentages by raising to power. And finally industrial culture calculates the total change-effect of forces through integrating: by adding a certain amount of momentum per second and energy per meter a force changes the meter-per-second number, which again changes the meter-number.

This proposal presents an organic bottom-up mathematics growing out of the social practices of bundling, stacking and totaling. It is organized in three levels, level 1: 6-10 years, level 2: 10-14 years and level 3: 14-18 years. It is activity and question driven limiting the amount of written material. It is learner centered limiting the amount of in-service teacher training.

The curriculum metaphor is a tree with a trunk consisting of five fundamental social practices: bundling, stacking, totaling, coding, and reporting fed by a root of basic activities. From the trunk two branches grow out, a “totals in space” branch and a “totals in time” branch reintegrating into a “totals in space and time” at three levels.

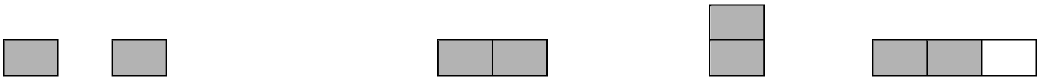
The basic activities are carried out with different piles of pellets or beads arranged and rearranged in sand or plastic boxes or frames always followed by the question “How many in total?” The pellets are bundled in different ways, illustrated graphically, reported as a Total-story, controlled on a calculator and finally coded.

To Tell the Difference Between Degrees of Many

Bundle & Stack, control by counting & calculating

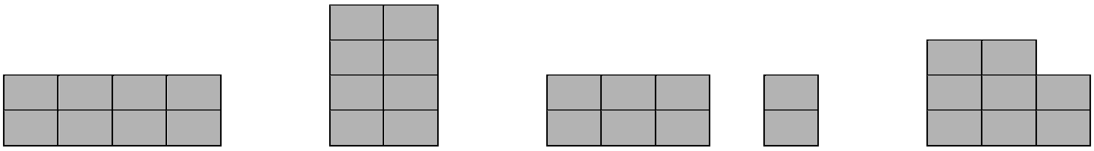
The Total T is:

TWO **The recounting equation: T = T/b·b**



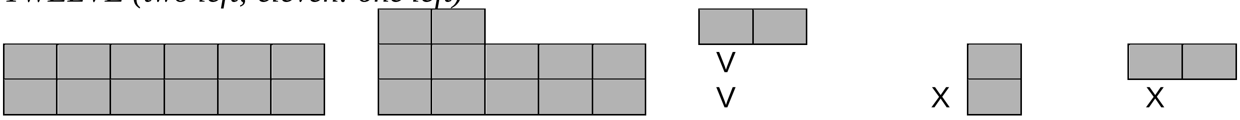
T = 1+1 = 2 T = 1·2 = 2 T = 2·1 = 2 T=3·1–1 = 2/3·3 = 2 per 3 of 3 = 2

EIGHT



T = 2·4 = 8 T = 4·2 = 8 T = 2·3 + 2·1 = 8 T = 2·3+2/3·3 = (2+2/3)·3 = 2 2/3·3 = 8

TWELVE (two left; eleven: one left)

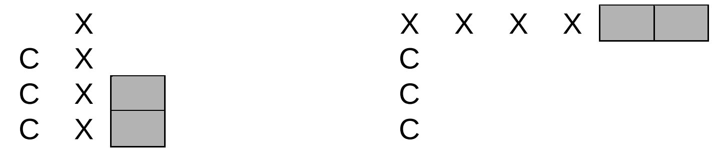


T = 2·6 = 12 T = 2 2/5·5 = 12 T= 2 2/5·5= 12 T= 1·X + 2·1 = 12 T= 1 2/X·X = 1.2·10 =12

FOURTY TWO *SEVENTY SIX*

T = 4·X+2·1 = 42 T = 4 2/10·10 = 4.2·10 = 42 T = 7·10+6·1 = 6·10+16·1 = 5·10+26·1

THREE HUNDRED FOURTY TWO

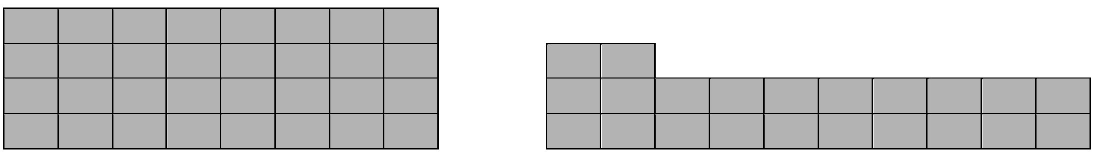


T = 3·C+4·X+2·1 = 342 T = 3 42/100·100 = 3.42·100 = 342

Rebundle

Operate **or Calculate**

T = 4·3 = ?·5 T = 4·3 = 2 2/5·5 T = 4·3 =12 = 12/5·5 = 2 2/5·5



T = 4·8 = ?·10 T = 4·8 = 3 2/10·10 T= 4·8= 32= 32/10·10= 3.2·10

T = 43·38 = ?·100 impossible T = 43·38 =1634 =1634/100·100 =16.34·100

Change units

T = 25·day = 25/7·7·day = 3 4/7·week

T = 200·min = 200/60·60·min = 3 20/60·h

T = 123·cm = 123/100·100·cm = 1.23·m

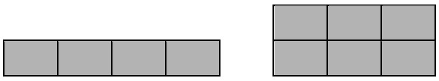
How To Unite and Split Totals

Like bundles can be stacked directly

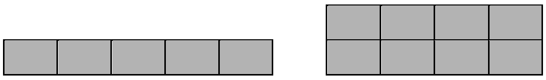
Unite Totals

T = A + B = ?

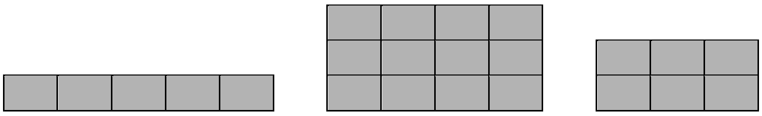
Operate



A = 1·4 + 2·3



B = 1·5 + 2·4



T = 1·5 + 3·4 + 2·3

Calculate

A: 0·5 + 1·4 + 2·3
B: 1·5 + 2·4 + 0·3
T: 1·5 + 3·4 + 2·3

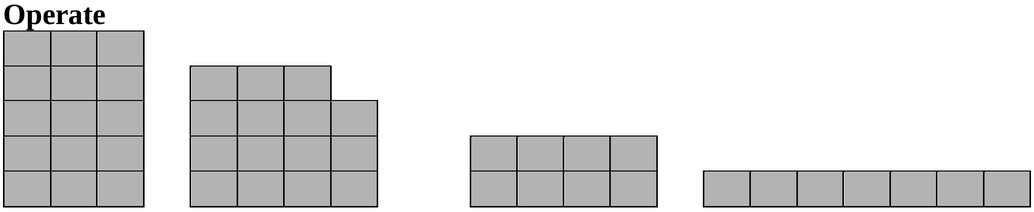
Calculate

A =	47	= 0·100 + 4·10 + 7·1
B =	526	= 5·100 + 2·10 + 6·1
T		*= 5·100 + 6·10 + 13·1*
T		*= 5·100 + (6+1)·10 + 3·1*
T =	*573*	*= 5·100 + 7·10 + 3·1*

Split Totals

T = 5·3 = 2·4 + ?

Operate



T = 5·3 = 3 3/4·4 = 2·4 + 1·7

Calculate

T = 5·3 = 15
= 15/4·4
= 3 3/4·4
= 2·4 + 1 3/4·4
= 2·4 + 7

T = 541 = 215 + ?

Calculate

A = 215 = 2·100 + 1·10 + 5·1
B = 326 = 3·100 + 3·10 – 4·1
T = 541=5·100 + 4·10 + 1·1

Operate

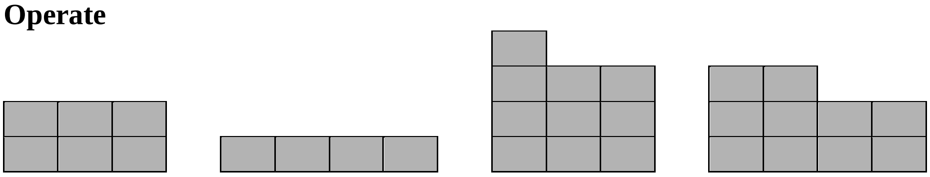
```
                          I                              I
C                C        I  I              I            I
C X              C        I  I              I            I
C X              C  X     I  I              I      C     I
C X              C  X     I  I      C       I      C  X  I
C X  I           C  X     I  I      C  X    I      C  X  I
```

T = 541 = 5·C + 4·X + 1 = 5·C+3·X + 11 = 2·C+1·X+5 + 3·C + 2·X + 6
= 2 1 5 + 3 2 6

Total and Change

T = 2·3 + 1·4 = ?·3 = ?·4

Operate



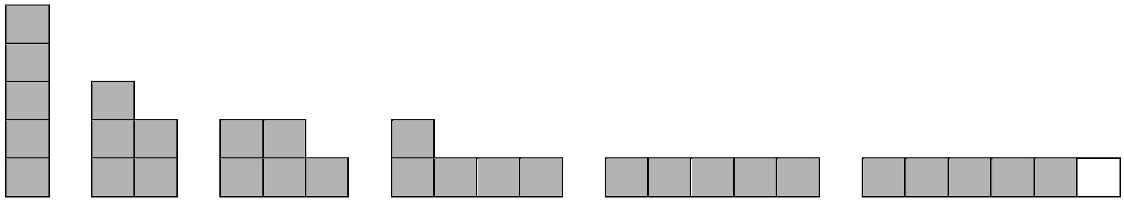
T = 2·3 + 1·4 = 3 1/3·3 = 2 2/4·4

Calculate

T = 2·3+1·4 = 10
10 = 10/3·3 = 10/4·4
10 = 3 1/3·3 = 2 2/4·4

Total Stories and Games

1. Systematic rebundling



Operate:	T=5·1	T=2 1/2·2	T=1 2/3·3	T=1 1/4·4	T=1·5	T=5/6·6=1·6–1
Calculate:	5=5/1·1	=5/2·2	=5/3·3	=5/4·4	=5/5·5	=5/6·6

2. Telling Total-stories about structures

Design and build a structure by e.g. LEGO blocks and tell its Total-story

3. Rebundling using full stacks (beware of overload)

Full 2-stacks:	2·2 = 4	2·4 = 8	2·8 = 16
Full 3-stacks:	3·3 = 9	3·9 =27	3·27 = 81
...			
Full 10-stacks:	10·10 = 100	10·100=1000	10·1000=10000

E.g. 26 = 1·16 + 1·8 + 1·2 = 2·9 +2·3 + 2·1 = 1·16 + 2·4 + 2·1 = 1·25 + 1·1 = 4·6 + 2·1 = ... = 2·10 + 6·1

4. Guess and calculate

T = 2·a = 8
a = *8/2 = 4*

T = 2·a + 1 = 7
2· a = *7–1 = 6*
a = *6/2 = 3*

5. Walking equations

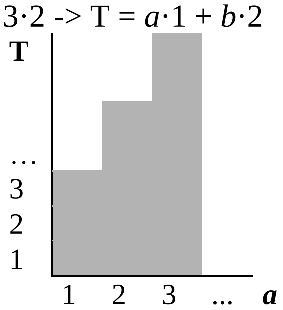
Solve equations as T = 2·*a* = 8 and T = 2·*a* + 1 = 7 by walking forwards and backwards

6. Coding and decoding

T = 3·2+1 -> T = *a*·2+1   T = 2·1 + 3·2 -> T = *a*·1 + *b*·2

a	T = 2·*a* + 1
1	T = 2·1 + 1 = 3
2	T = 2·2 + 1 = 5
3	T = 2·3 + 1 = 7
…	...

a	T =2·*a*+1
?	3
?	7
?	9
…	...



b					
…					
4	9	10	11	12	13
3	7	8	9	10	11
2	5	6	7	8	9
1	3	4	5	6	7
T	1	2	3	4	5 ... ***a***

7. Buy & sell

A dice has the numbers 1, 3 and 5 as red buy-numbers and 2, 4 and 6 as sell-numbers. Start with 10 units. Report after each five throws that includes the mean (if all numbers were the same).

8. Tax and refund

A dice has the numbers 1, 3 and 5 as red numbers. Throwing two dices gives e.g. 2 and 5. Receive 2 per 5, or pay 2 per 5 if one of the numbers is red.

9. Stack-building, up and down (tables)

Put some pellets on the table and remove them in bundles of 2 or 3 or... to be stacked:
T = 0, 2, 4, 6, 8, 10, ..., 20;   T = 20, 18, 16, ..., 4, 2, 0;   T = 0, 3, 6, ..., 30;   T = 30, 27, 24, ..., 6, 3, 0; etc.

10. Plus-growth or decay

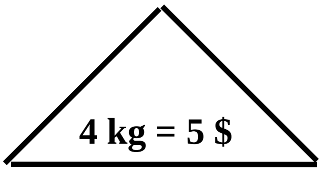
Begin with e.g. 8 pellets and add or subtract e.g. 2 pellets a certain number of times told e.g. by two dices. Try to predict before performing (calculate before counting).

11. Per-growth or decay

Begin with e.g. 8 pellets and add or subtract e.g. 1 per 2 pellets a certain number of times told e.g. by two dices. Try to predict before performing (calculate before counting).

HOW TO CHANGE UNITS?

Mixed Units

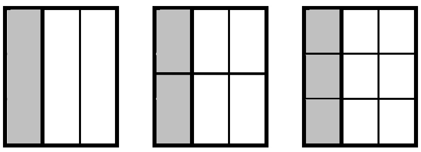


Method	**? kg = 8 $**		**6 kg = ? $**	
Recount	T = 8 $	= 8/5·5 $ = 8/5·4 kg = 6.4 kg	T = 6 kg	= 6/4·4 kg = 6/4·5 $ = 7.5 $
Units	kg = kg/$·$	= 4/5·8 = 6.4	$ = $/kg·kg	= 5/4·6 = 7.5
Per-number	? kg / 8 $?·5 ?	= 4 kg / 5 $ = 4·8 = 4·8/5 = 6.4	? $ / 6 kg ?·4 ?	= 5 $ / 4 kg = 5·6 = 5·6/4 = 7.5
Enlarge	?/4 ?	= 8/5, = 8/5·4 = 6.4	?/5 ?	= 6/4, = 6/4·5 = 7.5

Percentage

	? is 40% of 5	**3 is ?% of 5**	**3 is 60 % of ?**
Method	? per 5 = 40 per 100	3 per 5 = ? per 100	3 per ? = 60 per 100
Recount	5 = 5/100·100 has 5/100·40 = 2	100 = 100/5·5 has 100/5·3 = 60	3 = 3/60·60 is in 3/60·100 = 5
Per-number	?/5 = 40/100 ?·100 = 40·5 ? = 40·5/100 = 2	?/100 = 3/5 ?·5 = 3·100 ? = 3·100/5 = 60	3/? = 60/100 ?·60 = 3·100 ? = 3·100/60 = 5
Enlarge	?/40 = 5/100 ? = 5/100·40 = 2	?/3 = 100/5 ? = 100/5·3 = 60	?/100 = 3/60 ? = 3/60·100 = 5

Fractions



1/3 = 2/6 = 3/9 = etc.

Totalling Per-Numbers

5 kg at	3 $/kg	= 5·3	= 15 $
3 kg at	6 $/kg	= 3·6	= 18 $
8 kg at	*x* $/kg	= 8·*x*	= 33 $
		x	= 33/8 = 4.13 $/kg
Hence	**3 $/kg + 6 $/kg**	**= 4.13 $/kg**	

2 s at	3 m/s	= 2·3	= 6 m
6 s at	6 m/s	= 6·6	= 36 m
8 s at	*x* m/s	= 8·*x*	= 42 m
		x	= 42/8 = 5.25 m/s
Hence	**3 m/s + 6 m/s**	**= 5.25 m/s**	

5000 $ at	1/2	= 5000/2·1	= 2500 $
3000 $ at	1/6	= 3000/6·1	= 500 $
8000 $ at	*x*	= 8000·*x*	= 3000 $
		x	= 3000/8000
			= 0.375 = 375/1000
Hence	**1/2 + 1/6**		**= 375/1000**

6500 $ at	3%	= 6500/100·3	= 195 $
1500 $ at	6 %	= 1500/100·6	= 90 $
8000 $ at	*x* %	= 8000/100·*x*	= 285 $
		x	= 285/80 = 3.6 %
Hence	**3 % + 6 %**	**= 3.6 %**	

	+ 3% *+ 6%*		*+ 9.2%*
5000 -----> 5150 -----> 5459		5000	----------> 5150
	·1.03 *·1.06*		*·1.092*
Hence 3 %	**+ 6 %**		**= 9.2 %**

0.01 s at	3 m/s	= 0.01·3	= 0.03 m
0.01 s at	3.01 m/s	= 0.01·3.01	= 0.0301 m
0.01 s at	3.02 m/s	= 0.01·3.02	= 0.0302 m
…			
0.01 s at	4.99 m/s	= 0.01·4.99	= 0.0499 m
2 s at	*x* m/s	= 2·*x*	= 7.99 m
Hence 2 s at 3 m/s increasing to 5 m/s is 7.99 m			

Change-Stories

Linear and exponential change emerge as constant-change-stories from questions as:
“100$ plus n days at 5$/day total ? $”
“100$ plus n days at 5%/day total ? $”

Differential and integral calculus emerge as variable-change-stories from questions as:
“100$ plus n times at (10%/n)/time total ? $”
“100m plus 5 seconds at 3m/sec increasing to 4 m/sec total ? m”.

Linear Change

A change is linear if it is constant: +5$, +5$, +5$.
Linear change is also called “+change”.
The Total T after n days can be calculated as T = 100 + 5·n, or more general:
T = $b + a\cdot n$, where b, T: initial, terminal capital; a: $-addition/day; n: days

Exponential Change

A change is exponential if it is constant in percent: +5%, +5%, +5%. Since we cannot add % to $, we must consider the initial capital b as 100%. After an addition of 5% the terminal value is 105% of b, i.e. b·105% = b·1.05. So we add 5% by multiplying with 105% or 1.05, i.e. the change-multiplier is constant: ·1.05, ·1.05, ·1.05.
Exponential change is also called “·change” or interest change.
The Total T after n days can be calculated as T = 100·1.05^n, or more general:
T = b·a^n, a = 1+r where b, T: initial, terminal capital; r: %-addition/day; n: days

		Linear change		**Exponential change**		
		***b*+*a*·*n* =**	**T**	**= *b*·*a*^*n***	*a* = 1+*r*	T: final value
		+*a*	·*a*			+*a*: change per time
		etc.				·*a*: multiplier per time
+*a*$ *n* times	+*a*·*n*	+*a*	·*a*	·*a*^*n*	+*r*% *n* times	+*r*: %-change per time
		+*a*	·*a*			*n*: number of changes
		b				*b*: initial value

A *calculation table* can be used to report calculations.
“80$ plus 4 days at ? $/day total 100$” “80$ plus 4 days at ? %/day total 100$”

a = ?	T = b+(a·n)
T = 100	T–b = a·n
b = 80	$\frac{(T-b)}{n} = a$
n = 4	$\frac{(100-80)}{4} = a$
	5 = *a*

r = ?	$T = b\cdot(a$^$n)$
T = 100	$\frac{T}{b}$ = a^n
b = 80	$\sqrt[n]{(\frac{T}{b})} = a$
n = 4	$\sqrt[4]{(\frac{100}{80})} = a$
a = 1+r	1.057 = a = 1+r
	0.057 = ***r* = 5.7%**

Predictable Change

Differential Calculus

Differential calculus answers questions like “100$ + n times at (10%/n)/time total ? $”.
The Total T after n additions can be calculated as T = 100·(1+0.1/n)^n.
We can set up a T-table for the different n’s:

n	1	10	100	1000	10000	100000	1000000
T	110	110.4622	110.5116	110.5165	110.5170	110.5171	110.5171

The stabilised T-number 110.5171 can also be calculated by using the Euler number e: 100·e^0.1 = 110.5171.

Hence we see that for n very large:

e^0.1 = (1+0.1/n)^n or
e^t = (1+t/n)^n or
e^(t/n) = 1+t/n or by substituting dx=t/n making dx very small
e^dx = 1+dx true locally, i.e. for very small numbers dx

This local connection between something non-linear e^x and something linear 1+x, corresponds to the geometrical fact, that a bending curve is locally linear. This Bottom-Up alternative to the traditional Top-Down ε–δ approach can be used all over calculus:

Rule: (x^n)’=n·x^(n–1)			
Proof: Let y = x^n, and let dx be very small.			
If x->x+dx then y-> y+dy	= (x+dx)^n	= (x·(1+dx/x))^n	
		= (x·e^(dx/x))^n	
		= x^n · e^(n·dx/x)	
		= x^n · (1+n·dx/x))	
		= x^n + n·dx·x^(n–1)	
hence	dy	= n·dx·x^(n–1)	
and	dy/dx	= n·x^(n–1)	

Integral Calculus

Integral calculus answers questions like “100m plus 5 seconds *at* 3m/sec increasing to 4 m/sec total ?m”.

We observe that the total change ΔF can be calculated in two different ways:

ΔF = F2–F1 as a difference between the terminal and the initial values
ΔF = ΣΔF as the sum of the single changes, or
ΔF = ∫dF for very small single changes dF

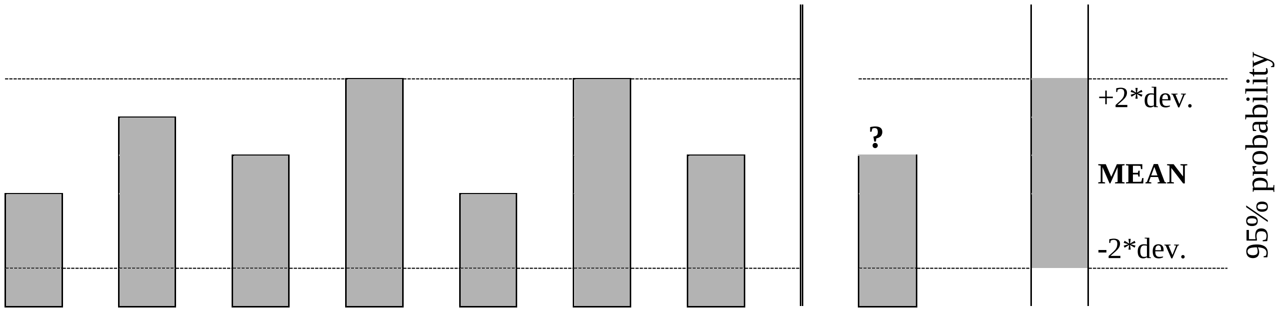
If dF/dx = f
then dF = fdx
∫dF = ∫fdx= ΔF = F2–F1
Since d/dx(3x+0.1x^2) = 3+0.2x
d(3x+0.1x^2) = (3+0.2x)dx

$$\int_0^5 d(3x+0.1x^2) = \int_0^5 (3+0.2x)\,dx = \Delta(3x+0.1x^2) = (3\cdot 5+0.1\cdot 5^2) - 0 = 17.5$$

So the answer is:

100m plus 5 sec. at 3m/sec increasing to 4 m/sec total $100+\int_0^5 (3+\frac{4-3}{5}x)\,dx = 117.5$ m

Unpredictable Change, Stochastic variation



Statistics | *Probability*

Post-diction: MEAN, DEVIATION | Pre-diction x = MEAN ± 2·DEV

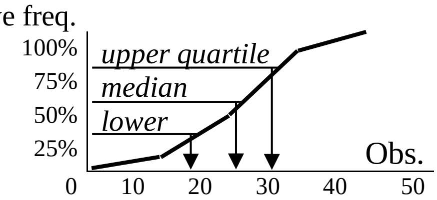
1. Observations

x: 10, 12, 22, 12, 15, ...

2. Bundle & count frequency

Observations	Frequency	Relative freq.	Cumul. freq.
x	h	p	∑p
0-10	3	3/40=0.075	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

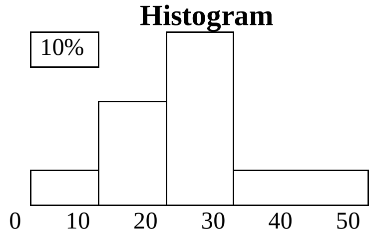
Ogive



3. Mean or average: If all observations were alike ...

Observations	Frequency	Relative freq.	Cumul. freq.	Mean
x	h	p	∑p	μ = ∑xi·pi
0-10	3	3/40=0.075	0.075	5·0.075=0.375
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1

Histogram



4. Variance. deviation: If all variations were alike ...

Observations	Frequency	Rel. freq.	Cumul. freq.	Mean	Distance	Variance
x	h	p	∑p	μ = ∑xi·pi	|xi - μ|	v = ∑(xi-μ)^2·pi
0-10	3	3/40=0.075	0.075	5·0.075=0.375	|5-23.1|=18.13	18.13^2·0.075=24.64
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		1 s^2 = 95.86

Deviation s = √95.86 = 9.8

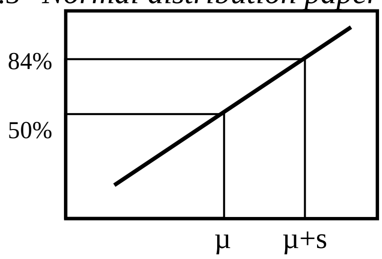
5. Prediction: x = MEAN ± 2·DEV =μ ± 2·s = 23.1 ± 19.6 *Confidence interval* = [3.5 ; 42.7]

6. Binomial distribution: Repeated experiment with two outcomes (win or not)

n repetitions of an experiment with two outcomes and winning probability p. x total number of wins.

	Obs.	Freq.	Cumul.	Prec. 3	Max. 2	Min. 2	Min.1 & Max.3
	x	p	∑p	x=3	x≤2	x≥2	1≤x≤3
n	0		0.2560				-0.2560
4	1		0.1792			-0.1792	
	2		0.5248	-0.5248	0.5248		
p	3		0.8704	0.8704			0.8704
0.6	4		1.0000			1.0000	
	Total			0.3456	0.5248	0.8208	0.6144

Normal distribution paper



Prediction: Total x = n·p ± 2·√(n·p·(1-p)) Percent x/n = p ± 2·√(p·(1-p)/n) P(x<t) = Ø[(t-MEAN)/DEV]

Fact. Fiction. Fiddle - Three Types of Models

FACT MODELS

If the equation is a fact. the model can be called a *fact* model. a “since-hence” model or a “room”-model. Fact models quantify quantities and calculate deterministic quantities: “What is the area of the walls in this room?” In this case the calculated answer of the model is what is observed. Hence calculated values from a fact models can be trusted.

FICTION MODELS

If the equation is a fiction. i.e. if it is contingent and could look otherwise. the model can be called a *fiction* model. an “if-then” model or a “rate” model. Fiction models quantify quantities and calculate non-deterministic quantities: “My debt will soon be paid off at this rate!” Fiction models are based upon contingent assumptions and produces contingent numbers that should be supplemented with calculations based upon alternative assumptions. i.e. supplemented with parallel scenarios.

FIDDLE MODELS

If the equation is a fiddle. the model can be called a *fiddle* model or a “risk” model. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence · probability.

	Equations			**Fact/Fiction/Fiddle**
Economy			constant	
Shopping	Cost = price·volume		*price*	Fact
Time-series	T = To+a·n	ΔT/Δn=a	*slope*	with constant numbers
“	T = To·a^n, a=1+r	(ΔT/T)/Δn=r	*%change*	*else* Fiction
“	T = To·n^a	(ΔT/T)/(Δn/n)=a	*elasticity*	(calculates averages)
“	T = To+a·n	ΔT=a$	*$-input*	
Saving	T = To·a^n, a=1+r	ΔT=r%	*%-input*	
“	T/R=a/r, 1+R=(1+r)^n	ΔT=a$+r%	*$&%-input*	
Theory	Demand = Supply			Fiction
Physics				
Falling body	acceleration = position”			Fact
	Force = Mass · acceleration			Fact
	Force = Mass · gravity			Fiction if air resistance
Electrical	Watt = Volt · Ampere			Fact
circuit	Volt = Ohm · Ampere			Fiction unless resistor
Statistics	Risk = Consequence · Probability			Fiction/Fiddle
Dice-game	Risk = 6·(1/6)			Fiction
Technology	Risk = Casualty·Probability + Death·Probability			Fiddle

Examples of fact, fiction, and fiddle models

An ICME Trilogy, ICME 10 & 11 & 12, in 2004 & 2008 & 2012 in Copenhagen & Monterrey & Seoul

http://mathecademy.net/wp-content/uploads/2013/05/An-ICME-Trilogy-online.pdf

Introduction

This is a selection of papers and other inputs produced for the 10th International Congress on Mathematical Education, ICME 10, in Denmark in 2004; and for ICME 11 in Mexico as well for ICME 12 in Souh Korea in 2012. The contributions are numbered 1xx, 2xx and 3xx respectively.

101 contains the paper 'One Digit Mathematics' written together with Saulius Zybartas, and presented at the topic study group 1, new development and trends in mathematics education at pre-school and primary level. The paper The paper suggests that to solve the relevance paradox in mathematics education, LIB-mathematics as 2+3 = 5 valid only in the library and not in the laboratory should be replaced by LAB-mathematics as 2*3 = 6, also valid in the laboratory. And that replacing modern authorized routines with postmodern authentic routines turns elementary mathematics upside down by bringing the authority back to the multiplicity-laboratory where mathematics may be learned from one digit numbers only. At the end of the chapter a German version of the paper is included.

102 contains the paper 'Adding PerNumbers' presented at the topic study group 2, new development and trends in mathematics education at secondary level. The paper suggests that to solve the relevance paradox in mathematics education postmodern sceptical Cinderella research could be used to look for new ways to teach mathematics at the secondary school. The paper introduces addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

103 contains the paper 'Bundling & Stacking in a Count & Add Laboratory' written together with Saulius Zybartas, and presented at the topic study group 8, research and development in the teaching and learning of number and arithmetic. The paper suggests that to solve the relevance paradox in mathematics education, LIB-mathematics as 2+3 = 5 valid only in the library and not in the laboratory should be replaced with LAB-mathematics as 2*3 = 6, also valid in the laboratory. And that a reconstruction respecting Kronecker's and Russell's scepticism shows that multiplicity-based LAB-mathematics is rather different from set-based LIB-mathematics by allowing fundamental mathematics as per-numbers, re-counting and re-stacking to be introduced at the first year of school. Then differences between modern LIB- and postmodern LAB-mathematics are discussed, and finally a testing of postmodern LAB-Mathematics in the classroom of primary school and teacher education is described.

104 contains the paper 'Deconstructing Modern Top-Down Algebra into Postmodern Bottom-Up Algebra' written, first for the 24^{th} conference of the International Group for the Psychology of Mathematics Education, PME, in Hiroshima in Japan in 2000 where it was rejected, then for the topic study group 9, research and development in the teaching and learning of algebra. The paper says it is a postmodern point that a phrasing constructs what it describes and that ruling phrasings and discourses clientifies humans. Inspired by this the paper asks: is it possible to re-describe and deconstruct mathematics? 'Geometry' means 'earth measuring' - but what does "Algebra" mean? The dictionary tells us that 'Algebra' means 'reunite'. Since 'low attainers' might be deconstructed into 'authenticity searchers' we could also ask: what will happen if we present authenticity searchers for authenticity by inviting them to join the social practice of reuniting that created Algebra? This paper then tells about what happened in such classes. The paper was accepted for distribution.

105 contains the paper 'Per-Number Calculus, A Postmodern Sceptical Fairy tale Study' written for the topic study group 12, research and development in the teaching and learning of calculus. To solve the relevance paradox in mathematics education this paper uses postmodern sceptical fairy tale research to look for new ways to teach calculus in the school. A renaming of 'calculus' to 'adding per-numbers' allows us to think differently about the reality 'sleeping' behind our words,

and all of a sudden we see a different calculus taking place both in elementary school, middle school and high school. Being a ‘Cinderella-difference’ by making a difference when tested, this postmodern calculus offers to the classroom an alternative to the thorns of traditional calculus. The paper was accepted for distribution.

106 contains the paper ‘Applying Mathe-Matics, Mathe-Matism or Meta-Matics’ written for the topic study group 20, Mathematical applications and modelling in the teaching and learning of mathematics. The paper suggests that to solve the relevance paradox in mathematics education a sceptical look should be taken at one of the taboos of mathematics education, the mathematical terminology. Two kinds of words are found, LAB-words abstracted from laboratory examples; and LIB-words exemplified from library abstractions, transforming mathe-matics to meta-matics. A third kind of mathematics is mathema-tism only valid in the library and not in the laboratory, and blending with meta-matics to meta-matism. This distinction suggests that the relevance paradox of mathematics education occurs when teaching and applying metamatism, and disappears when teaching and applying mathematics. The paper was accepted for presentation.

107 contains the paper ‘Pastoral Power in Mathematics Education, A Postmodern Sceptical Fairy tale Study’ written for the topic study group 25, language and communication in mathematics education. The paper suggests that to solve the irrelevance paradox in mathematics education we should look for help at institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies, the American and the French, in the form of pragmatism and post-structuralism. Inspired by Foucault’s notion of ‘pastoral power’ the paper looks at the use of words in mathematics education, distinguishing between ‘lib-words’ coming from the library and ‘lab-words’ coming from the laboratory. From this distinction a hypothesis can be made saying that the irrelevance paradox is created by lib-words installing pastoral power, and that lab-words will make the irrelevance paradox disappear. Consequently mathematics education should be based upon verb-based ‘ing’-words such as counting and adding and calculating etc. The paper was accepted for presentation.

108 is a short paper called ‘FunctionFree PerNuber Calculus’ made for my contribution to the Nordic presentation at ICME10. Observing that calculus did not call itself ’calculus’ it is suggested that calculus could also be called something else, thus the non-action word ‘calculus’ could be reworded to the action-word ‘adding per-numbers - taking place from K - 12. Then examples are given on adding per-numbers in primary school, in middle school and in high school. Also people were invited to the MATHeCADEMY.net stand at the conference to discuss details.

109 is a poster called ‘A Kronecker-Russell Multiplicity-Based Mathematics’presented at the ICME 10. The poster is a short outline of the curriculum at the MATHeCADEMY.net. It was part of most of the papers at the conference, but the poster allowed for having additional time to discuss details with different people.

110 contains a proposal for a paper ‘ Multiplicity-Based Mathematics found by Postmodern Sceptical Fairy tale Research’ written for the thematic afternoon E, perspectives on research in mathematics education. The focus of the paper is to solve the irrelevance paradox of the research industry. The paper uses as its theoretical framework institutional scepticism, as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism. On this basis the paper describes a methodology called ‘sceptical fairy tale research’ as a postmodern counter-seduction research based upon a post-structuralist ‘pencil-paradox’. By its LIB-LAB-destinction between words and numbers, sceptical fairy tale research is inspired by the ancient Greek sophists always distinguishing between choice and necessity, between political and natural correctness, between nomos and logos. By transforming seduction back into interpretation scepticism transforms the library from a hall of fact to a hall of fiction to draw inspiration from, especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in counter-seduction (and to a certain extent grounded theory) is the fairy tale. Once a fairy tale interpretation has identified the ‘evil’ word,

scepticism begins to look for hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault; or by inventing alternatives using sociological imagination. The aim of sceptical fairy tale research is not to extend the existing seduction of the library, so no systematic reference to the existing 'research' literature will take place. The aim is to solve problems by searching for hidden Cinderella-alternatives in the laboratory, i.e. by 1) finding the word suspected to be the villain, 2) renaming the evil word through discovery and imagination, 3) testing the hidden alternative in the laboratory to see if it is a Cinderella-difference making a difference, and 4) publish the alternative so it can become an option. The proposal was rejected.

201 contains a paper called 'Avoiding Ten, a Cognitive Bomb'. The number ten is the basis of our number system. The traditional curriculum sees no problem in introducing ten as the follower of nine. However, being the only number with its own name but without its own icon, the number ten becomes a cognitive bomb if introduced to quickly. First 1digit mathematics should be learned through bundling & stacking reported by cup- and decimal-writing. The paper was written for the Topic Study Group 1: New developments and trends in mathematics education at preschool level. The paper was presented as a full paper.

202. contains a paper called 'A Fresh Start Presenting Mathematics as a Number-predicting Language'. It describes the website www.MATHeCADEMY.net that contains a CATS-approach to mathematics, Count&Add in Time&Space, offering to learn mathematics as a natural science investigating the nature of many. Also the website contains 7 papers from the ICME10 Congress describing the approach in details. The paper was written for the Topic Study Group 4: New developments and trends in mathematics education at upper secondary level. The paper war rejected.

203 contains a paper called 'Decimal-Counting, Disarming the Cognitive Bomb Ten'. The number ten is the base of our number system. The modern curriculum sees introducing 10 as the follower of 9 as nature. However, being the only number with its own name but without its own icon, ten becomes a cognitive bomb if introduced to quickly. Anti-pastoral sophist research, searching for alternatives to choices presented as nature, shows that ten is not 10 by nature but by choice, and that jumping directly from 1.order to 3.order counting means missing the learning opportunities of 2.order decimal-counting by bundling and stacking. The paper was written for the Topic Study Group10: Research and development in the teaching and learning of number systems and arithmetic. The paper war rejected.

204 contains a paper called 'Pastoral Algebra Deconstructed'. Presenting its choices as nature makes modern algebra pastoral, suppressing its natural alternatives. Seeing algebra as pattern seeking violates the original Arabic meaning, reuniting. Insisting that fractions can be added and equations solved in only one way violates the natural way of adding fractions and solving equations. Anti-pastoral grounded research identifying alternatives to choices presented as nature uncovers the natural alternatives by bringing algebra back to its roots, describing the nature of rearranging multiplicity through bundling & stacking. The paper was written for the Topic Study Group11: Research and Development in the Teaching and Learning of Algebra. The paper war accepted.

205 contains a paper called 'Pastoral Calculus Deconstructed'. Calculus becomes pastoral calculus killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives to choice presented as nature uncovers the natural alternatives by bringing calculus back to its roots, adding and splitting stacks and per-numbers. The paper was written for the Topic Study Group16: Research and development in the teaching and learning of calculus. The paper war accepted.

206 contains a paper called 'Applying Pastoral Metamatism or Re-Applying Grounded Mathematics'. When an application-based mathematics curriculum supposed to improve learning fails to do so, two questions may be raised: What prevents it from improving learning? And is 'mathematics applications' what it says, or something else? Skepticism towards wordings leads to

postmodern thinking that, dating back to the ancient Greek sophists, warns against patronizing pastoral categories, theories and institutions. Anti-pastoral sophist research, identifying hidden alternatives to pastoral choices presented as nature, uncovers two kinds of mathematics: a grounded mathematics enlightening the physical world, and a pastoral self-referring mathematics wanting to ‘save’ humans through ‘metamatism’, a mixture of ‘metamatics’ presenting concepts as examples of abstractions instead of as abstractions from examples; and ‘mathematism’ true in the library, but seldom in the laboratory. Also ‘applying’ could be reworded to ‘re-applying’ to emphasize the physical roots of mathematics. Three preventing factors are identified: ‘ten=10’-centrism claiming that counting can only take place using ten-bundles; fraction-centrism claiming that proportionality can only be seen as applying fractions; and set-centrism claiming that modelling can only take place by applying set-based concepts as functions, limits etc. In contrast, an implying factor is grounded mathematics created through modelling the natural fact many by counting many in bundles &stacks; and by predicting many by a recount-formula $T = (T/b)*b$ that can be re-applied at all school levels. The paper was written for the Topic Study Group21: Mathematical applications and modelling in the teaching and learning of mathematics. The paper war accepted.

207 contains a paper called ‘Mathematics: Grounded Enlightenment - or Pastoral Salvation; a Natural Science for All - or a Humboldt Mystification for the Elite’. Mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral ‘metamatism’ is exemplified from metaphysical mystifying concepts from above. To make mathematics a human right, pastoral Humboldt counter-enlightenment mathematics must be replaced with democratic grounded enlightenment mathematics. The paper was written for the Topic Study Group24: Research on classroom practice. The paper war rejected.

208 contains a paper called ‘Pastoral Humboldt Mathematics Deconstructed’. Having existed since antique Greece, pastoral and anti-pastoral curricula today exist at the Humboldt Bildung schools inside EU and Enlightenment schools outside. However, Humboldt anti-enlightenment seems to have influenced all mathematics curricula. Following the advice of the Greek sophists, this paper separates choice from nature in the mathematics curriculum to design an alternative natural enlightenment curriculum grounded in the roots of mathematics. The paper includes an appendix called ‘A General Enlightenment Curriculum’. The paper was written for the Topic Study Group25: The role of mathematics in the overall curriculum. The paper war rejected.

209 contains a paper called ‘CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher’. The CATS-approach, Count&Add in Time&Space, is a natural way to become a math teacher. It obeys the fundamental rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, multiplicity. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics without knowing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers. The paper was written for the Topic Study Group TSG 27: Mathematical knowledge for teaching. The paper war accepted.

210 contains a paper called ‘Pastoral Words in mathematics education’. Mathematical terminology is very fixed, almost dogmatic, which seems to indicate a metaphysical nature. The necessity of language shows the great advantages by having a fixed terminology. However, there is a fundamental difference between enlightening words labeling and pastoral words hiding differences. Following the advice of the ancient Greek sophists warning against mixing up nature and choice, this paper asks ‘what is nature and what is choice in mathematical terminology?’ The paper was written for the Topic Study Group31: Language and communication in mathematics education. The paper war rejected.

211 contains a paper called ‘Deconstructing the Mathematics Curriculum: Telling Choice from Nature’. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, seen from a skeptical sophist perspective wanting to tell nature from choice, three questions are raised: Are concepts grounded in nature or forcing choices upon nature? How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum? Does mathematics education mean pastoral patronization of humans, or anti-pastoral enlightenment of nature? The paper was written for the Topic Study Group TSG 35: Research on mathematics curriculum development. The paper war accepted.

212 contains a paper called ‘Mathematics Education: Pastoral Bildung - Or Anti-Pastoral Enlightenment’. Applying a postmodern philosophical perspective to mathematics education reveals different kinds of mathematics and different kinds of education and different kinds of philosophy. Based upon the ancient Greek controversy between the sophists and philosophers as to the nature of knowledge, two different forms of schooling has developed, an enlightenment school abstracting categories from physical examples; and a pastoral school exemplifying metaphysical categories; as well as two different kinds of mathematics, enlightenment mathematics seeing the world as the roots of mathematics, and pastoral mathematics seeing the world as applying mathematics. The paper was written for the Discussion Group 5: The role of philosophy in mathematics education. The paper war accepted.

213 contains a paper called ‘Concealing Choices to Teachers’. Teaching or preaching - this dilemma goes back to the ancient Greek controversy between the sophists advocating enlightenment, and the philosophers advocating patronization. Also today two kinds of schools exist, the North American enlightenment schools educating the people, and EU Humboldt Bildung Counter-enlightenment schools educating the elite for offices. Should teachers be told if they are trained for enlightenment or patronization? Or are they just expected unreflectively to follow the orders of the institution paying their wages? The paper was written for the Discussion Group 7: Dilemmas and controversies in the education of mathematics teachers. The paper war accepted.

214 contains a workshop in 1digit Mathematics, Cup-writing & Decimal-counting called ‘Avoiding 10, a Cognitive Bomb’. The workshop was accepted.

215. contains a poster called ‘The 12 Blunders of Pastoral Mathematics’. The poster was accepted.

216 contains a talk called ‘Mathematics as an anti-Pastoral Natural Science’ given to each visitor at the MATHeCADEMY.net booth in the exhibition area.

301 contains a paper called ‘Come Back with 1digit Mathematics’. Postmodern contingency research uncovers hidden alternatives to choices presented as nature e.g. by replacing choice with nature. Within traditional mathematics, numbers, operations, formulas, equations etc. turn out to be choices hiding their natural alternatives. Presenting mathematics from its roots, the natural fact Many, help many dropouts to master mathematics as a natural science. The paper was written for the Topic Study Group 4: Activities and programs for students with special needs. The paper was rejected.

302 contains a paper called ‘Recounting as the Root of Grounded Mathematics‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 7: Teaching and learning of number systems and arithmetic - focusing especially on primary education. The paper was presented as a poster.

303 contains a paper called ‘Calculus Grounded in Adding Per-numbers‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective

wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? What are the roots of calculus? The paper was written for the Topic Study Group 13: Teaching and learning of calculus. The paper was presented as a poster.

304 contains a paper called ‘Saving Dropout Ryan With A Ti-82’. To lower the dropout rate in pre-calculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ replaced the textbook. A formula’s left and right hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects. The paper was written for the Topic Study Group 18: Analysis of uses of technology in the teaching of mathematics. The paper was presented as a full paper.

305 contains a paper called ‘Contingency Research Uncovers the Roots of Grounded Mathematics‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 21: Research on classroom practice. The paper was presented as a poster.

306 contains a paper called ‚Mathematics as Manyology‘. Mathematics education claims to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical perspective wanting to tell nature from choice, two questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? The paper was written for the Topic Study Group 23: Mathematical knowledge for teaching at primary level. The paper was rejected.

307 contains a paper called ‚Counting and Adding Roots Grounded Mathematics‘. Mathematics education claims to deliver well-proven knowledge about well-defined concepts applicable to the outside world. However, skepticism would ask: Are the concepts grounded in nature or forcing choices upon nature? Can ungrounded mathematics from above be replaced by grounded mathematics from below generalized in a natural way in secondary school? The paper was written for the Topic Study Group 24: Mathematical knowledge for teaching at secondary level. The paper was presented as a poster.

308 contains a paper called ‚Fractions Grounded as Decimals, or 3/5 as 0.3 5s‘. The tradition sees fractions as difficult to teach and learn. Skepticism asks: Are fractions difficult by nature or by choice? Are there hidden ways to understand and teach fractions? Contingency research seaching for hidden alternatives to traditions looks at the roots of factions, bundling the unbundled, described in a natural way by decimals. But why is the unnatural presented as natural? The paper was written for the Topic Study Group 25: In-services education, professional development of mathematics teachers. The paper was presented as a full paper.

309 contains a paper called ‚Counting and Adding - a Natural Way to Teach Mathematics‘. The CATS-approach, Count&Add in Time&Space, obeys the rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, Many. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics by doing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers. The paper was written for the Topic Study Group 26: Preservice mathematical education of teachers. The paper was presented as a poster.

310 contains a paper called Hidden Understandings of Mathematics Education‘ To answer the question ‘are there hidden understandings of mathematics education’ this paper tries to reinvent mathematics as a natural science grounded in its natural roots, the study of the natural fact Many. It

turns out that two different mathematics exist, metamatism from above and grounded mathematics from below: Also two different kinds of education exist: Line-organized Bildung forcing students to learn the same, and block-organized enlightenment allowing students to develop personal talents. The paper was written for the Topic Study Group 32: Mathematics curriculum development. The paper was rejected.

311 contains a paper called ‚Social Theory in Mathematics Education‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 37 Theoretical issues in mathematics education. The paper was presented as a poster.

312 cointains a workshop in Recounting and Decimal-writing. To deal with the natural fact Many, we totalize. However, there are hidden ways to count and add. This workshop demonstrates the power of recounting made possible by counting in icons before counting in tens. Recounting shows that natural numbers are decimal numbers carrying units. And recounting allows both proportionality and integration to be introduced in grade one. The workshop was accepted.

313 contains the posters presented at the ICME12. First the poster for the general poster session. Then posters for the Topic Study Groups 7, 13, 21, 24, 26 and 37

314 contains a contribution to the blog of discussion group 11 on Mathematics Teacher Retention called ‘Three Teacher Taboos in Mathematics Education’. It describes ‘Three Teacher Taboos’ facing a mathematics teacher: 1) As to mathematics: shall I preach self-referring metamatism, or mathematics grounded in the outside world? 2) As to education: shall I choose a line-organized talent impeding school, or a block-organized talent developing school? 3) As to research: shall I seek guidance in self-referring discourse protection from monastery-like universities, or in grounded contingency research from Internet academies?

315 contains a contribution to the blog of discussion group 12 on Mathematics Teacher Educators’ Knowledge for Teaching called ‘To Math or to Totalize, That is the Question’. It asks the question: How to educate math users, math teachers, and math teacher educators? And proposes the answer: By learning, not how to math, since math is not an action word, but how to deal with the natural fact Many by totalizing; in short, by becoming competent in counting and adding in time and space. It the describes the MATHeCADEMY.net is free for users and for franchise takers. It offers Internet PYRAMIDeDUCATION to teachers and educators wanting to learn about mathematics as a natural science investigating the natural fact Many through the CATS approach, Count&Add in Time&Space, building upon five principles.

316 contains a contribution to the blog of Discussion Group 6 on Postmodern Mathematics called ‘Theses 1-7’ proposing an answer to the question “To be able to participate in this discussion group, could you give me a short introduction to your view on postmodern thinking in mathematics education?”

317 contains a contribution to the Discussion Group 6 on Postmodern Mathematics called ‘Postmodern Skepticism toward Mathematics and Education and Research‘. The cornerstones of modern society is research and education, especially in the two basic languages, the word-language assigning words and sentences to qualities, and the number-language assigning numbers and calculations to quantities. And as an important institution, mathematics education is equipped with its own research to make it successful. Still the problems in mathematics education seem to grow with the number of research articles. This irrelevance paradox makes postmodern skepticism ask: Are mathematics, education and research what they claim to be? Or are they choices that presented as nature install patronization to be unmasked by postmodern contingency research?

318 contains the manuscript to a YoyTube Video on Postmodern Math Education. In the video Paul Ernest and Allan Tarp discuss 8 questions: What is meant by postmodern? What is meant by

modern? What is the root of postmodern thinking? Who is the most important postmodern thinker? What is mathematics? What is postmodern mathematics? What is postmodern research? Used as introduction in the Discussion Group 6 on Postmodern Mathematics.

319 contains the manuscript to a YoyTube Video Manuscript to a YoyTube Video on A Postmodern Deconstruction of World History. This YouTube video on postmodern deconstruction describes world history as the history of trade. First eastern lowland pepper and silk was traded with western highland silver, then eastern cotton was moved west and traded with northern industrial products, and finally electrons replaced the silver and cotton economy with an information economy. Published after the ICME12.

320 contains the manuscript to a YoyTube Video on Deconstruction of Fractions. This YouTube video on deconstruction in mathematics education connects fractions to its root, the leftovers when performing icon-counting. To deal with Many, we total by bundling in icon-numbers less than ten, or in tens needing no icon as the standard bundle. When bundling in 5s, 3 leftovers becomes 0.3 5s or 3/5 5s, thus leftovers root both decimal fractions and ordinary fractions. Presented in the topic study group 25: In-services education, professional development of mathematics teachers

321 contains the manuscript to a YoyTube Video on a Postmodern Deconstruction of of PreCalculus. This YouTube video on deconstruction in mathematics education connects preCalculus to its root, the natural fact Many. To deal with Many, we total. Some totals are constant, some change in space or time. The change might be predictable, or not. Pre-calculus describes predictable constant change. Calculus describes predictable changing change. Presented in the topic study group 18 : Analysis of uses of technology in the teaching of mathematics.

322 contains the manuscript to a YoyTube Video on a Postmodern Deconstruction of Mathematics Education. This YouTube video on deconstruction in mathematics education describes how natural mathematics is made difficult by removing eight links to its roots, the natural fact Many. The missing links make mathematics a privilege to a mandarin class wanting to monopolize public offices. Reopening the eight missing links will make mathematics easy and accessible to all. Published after the ICME12.

323 contains pictures from The MATHeCADEMY.net booth in the exhibition area. To share information about free web-based teacher education on mathematics as a natural science about Many, the MATHeCADEMY.net had a booth.

324 is the folder handed out in the MATHeCADEMY.net booth in the exhibition area.

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Discussion Group 6 at ICME 12: Postmodern Mathematics

Co-Chairs: Paul Ernest (UK) p.ernest@ex.ac.uk
Regina Möller (Germany) regina.moeller@uni-erfurt.de
Team Members: Ubiratan D'Ambrosio (Brazil) ubi@usp.br
Allan Tarp (Denmark) Allan.Tarp@MATHeCADEMY.net
Sencer Corlu (USA) sencer@tamu.edu
Extra Members: Gelsa Knijni (Brazil) gelsak@uol.com.br
Maria Nilokakaki (Greece) mnikolak@yahoo.gr
Liaison IPC Member: Michele Artigue michele.artigue@univ-paris-diderot.fr

Aim & Rationale

The goal of the discussion group is to elucidate the multiplicity of the subject of mathematics: to explore and share how postmodern perspectives offer new ways of seeing mathematics, teachers and learners. The two key themes are

Theme 1: Perspectives of mathematics as having multiple dimensions / components

Theme 2: Multiple-self perspectives of the human subject (teacher/ learner/ researcher)

These will be used as organizing themes for the two sessions

Postmodernism rejects a single authoritative way of seeing mathematics, teachers and learners, for each can be seen and interpreted in multiple ways. Mathematics can be seen as axiomatic and logical leading to indubitable conclusions, but it can also be seen as intuitive and playful, open-ended, with surprises and humour, as evidenced in popular mathematical images and cartoons. Additionally it can be seen in its applications in science, information and communication technologies, everyday life and ethnomathematics. All of these dimensions are part of what makes up mathematics and they all co-exist successfully.

It is also important to recognize that all human subjects have multiple selves and that we all (mathematicians, teachers and learners) have access to different selves: authoritative knowers, researchers, learners, appreciators and consumers of popular and other cultures, as well as having non-academic selves. Thus mathematics teachers can be seen as epistemological authorities in the classroom as well as co-explorers of unfamiliar realms both mathematical and cultural, and as ring-masters in the mathematical circus. Students can be seen as receivers of mathematical knowledge, but also as explorers and interpreters of mathematical and cultural realms that can be related to mathematics.

All of these perspectives and selves are resources we can use to enhance the teaching and learning of mathematics, but many are currently overlooked or excluded.

The aim of the Discussion Group is to raise and discuss these ideas and explore and generate examples relevant to classroom practices.

Papers and resources will be available on-line before the conference so that participants can prepare themselves and so that presentations can be kept short and most of the time is devoted to discussion. Examples will also be distributed in hard copy in the sessions.

Key Questions

* What is postmodern thinking in mathematics and mathematics education? What is new or different about it and what are the implications for research in mathematics education?

* Given a postmodern multiple-perspectives view of mathematics what illuminations and surprises can be found for mathematics and its teaching and learning in multidisciplinary sources including: history of mathematics, ethnomathematics, science, information and communication technology, art works, stories, cartoons, films, jokes, songs, puzzles, etc.?

* How might the new emphases and differences foregrounded by postmodern perspectives impact in the primary and secondary mathematics classrooms? What concrete examples serve to illustrate these differences?

* How can a multiple-selves view of the human subject be reflected in the mathematics classroom and in mathematics teacher education? How can a multiple-selves view of the teacher facilitate teacher education?

Video: A Postmodern Mathematics Education, https://www.youtube.com/watch?v=ArKY2y_ve_U

ICME 13 in Hamburg

Existentialism in mathematics and its education, from essence to existence. Don’t preach essence – teach existence.

From essence to existence in mathematics education

In mathematics and its education, the difference between essence and existence is seldom discussed although central to existentialist thinking. So we can ask: What will an existentialist mathematics education look like? Thus we close the door to the library with today's self-referring mathematics and go outside to rebuild mathematics from its roots, the physical fact Many. Likewise, we can ask if mathematics is learned by exposure to inside essence claims or to the outside existence rooting it.

Background

Institutionalized education typically has mathematics as one of its core subjects in primary and secondary school. To evaluate the success, OECD arranges PISA studies on a regular basis. Here increasing funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015).

As to the content of education, sociology offers understandings of schools and teacher education, psychology of learning, and philosophy of textbooks. Focusing upon existentialist philosophy this paper asks: What will an existentialist mathematics and education look like? The purpose is not to replace one tradition with another but to uncover hidden alternatives to choices presented as nature.

Existentialism

The Pythagoreans labeled their four knowledge areas by a Greek word for knowledge, mathematics. With astronomy and music now as independent areas, today mathematics is a common label for the two remaining activities both rooted in Many: Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek arithmetic (Freudenthal 1973).

The Greeks used the word 'sophy' meaning knowledge for men of knowledge, the sophists and the philosophers, disagreeing on the nature of knowledge. Seeing democracy with information and debate and choice as the natural state-form, the sophists emphasized knowing nature from choice to prevent patronization by choices presented as nature. Seeing autocracy patronized by themselves as the natural state-form, the philosophers saw choice as an illusion since to them physical existence was but examples of metaphysical essence only visible to the philosophers educated at the Plato academy having as entrance sign 'Let no one ignorant of geometry enter.'

Today, the sophist skepticism towards false is-claims is carried on by French post-structuralism with Derrida and Lyotard and Foucault and Bourdieu showing skepticism towards our most fundamental institutions: words, correctness, cures and education; and by the existentialism of Kierkegaard and Nietzsche and Heidegger and Sartre, defining existentialism as holding that 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino, 2004: 344).

In Denmark, a heritage allowed Kierkegaard to publish whatever he wrote. At the end, shortage forced him to shift to flying papers when rebelling against institutionalized Christianity in the form of Christendom. Focusing on the three classical virtues Truth, and Beauty and Goodness, Kierkegaard left truth to the natural sciences, and argued that to change from a person to a personality the individual should stop admiring beauty created by others and instead realize their own existence through individual choices. Of course, angst is a consequence when fearing to choose the bad instead of the good, and death might follow, but so will forgiveness and resurrection to a new life in real existence, as promised by Christianity in the Holy Communion.

In Germany, Nietzsche saw institutionalized Christendom as the creator of moral is-statements that prevented individuals from realizing their true existence through individual choices and action. To end this serfdom he hoped that someday we will see a

> redeeming man (..) whose isolation is misunderstood by the people as if it were flight from reality – while it is only his absorption, immersion, penetration into reality, so that (..) he may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it. (Marino, 2004: 186-187)

Likewise in Germany, Heidegger saw that to avoid traditional essence-claims, is-statements must be replaced by has-statements so that being is characterized by what it has, 'Dasein'. Arendt carried his work further by dividing human activity into labor and work focusing on the private sphere and action focusing on the political sphere thus accepting as the first philosopher political action as a worthy human activity creating institutions that should be treated with care to avoid 'the banality of evil' if turning totalitarian. (Arendt 1998)

One such institution is education. Here a subject always has an outside goal to be reached by several inside means. But if seen as mandatory, an inside means becomes a new goal, that by hiding its alternatives becomes a choice masked as nature hindering learners in reaching the original outside goal. Consequently, if trapped in a goal-means confusion by neglecting its outside goal, Mastering Many, mathematics education becomes an undiagnosed cure, forced upon patients, showing a natural resistance against an unwanted and unneeded treatment. In this case the institution education becomes a Foucault 'pris-pital', a mixture of a prison locking people up and a hospital curing diagnoses. This hybrid is an effective disciplining tool with teachers as 'jail-ters', a mixture of jailers and doctors, exercising the banality of evil by willingly following the orders of the established textbook rituals, thus obeying the two fundamental CD-rules of keeping a job, 'Compete or die' in the private sector and 'Conform or die' in the public sector. An institution is created to produce solutions to problems, but once created employees might seek a stronger goal: to keep the job, problems should be kept unsolved by being described by disagreeing diagnoses. (Foucault 1995)

Mathematics as Essence

Within mathematics, the existentialist distinction is shown by the function concept, defined by Euler as labeling the existence of calculations combining known and unknown numbers, and today defined as a set-relation where first component-identity implies second-component identity thus becoming pure essence through self-reference. The set-concept transformed mathematics to 'meta-matics', a self-referring collection of well-proven statements about well-defined concepts, defined as examples from internal abstractions instead of as abstractions from external examples. Looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence it false' being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. With no distinction between sets and elements, the Zermelo–Fraenkel set-theory avoids reference, thus becoming meaningless by its inability to separate outside examples from inside abstractions. That institutionalized education ignores this can be seen as an example of 'symbolic violence' used to protect the privileges of today's 'knowledge nobility'. (Bourdieu 1977)

Behind colorful illustrations, self-referring metamatics is taught through a gradual presentation of different number types, natural numbers and integers and rational and real numbers, together with the four basic operations, addition and subtraction and multiplication and division, where especially division and letter fractions create learning problems. Equations are introduced as equivalent number names to be changed by identical operations. In pre-calculus polynomial functions are introduced as a basis for calculus presenting differential calculus before integral calculus.

Mathematics as Existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, geometry and algebra, both rooted as natural sciences about the physical fact Many.

The root of geometry is the standard form, a rectangle, that halved by a diagonal becomes two right-angled triangles where the sides and the angles are connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and tanA = a/b. Being filled from the inside by such triangles, a circle with radius r gets the circumference $2 \cdot \pi \cdot r$ where $\pi = n \cdot \tan(180/n)$ for n sufficiently large.

Meeting Many we ask 'how many?' Counting and adding gives the answer. We count by bundling and stacking as seen when writing a total T in its block form: T = 354 = 3·B^2 + 5·B + 4·1 where the bundle B typically is ten. This shows the four ways to unite: On-top addition unites variable numbers, multiplication constant numbers, power constant factors and per-numbers, and next-to addition, also called integration, unites variable blocks. As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations: subtraction, division, root & logarithm and differentiation. Likewise, a total can occur in two forms, an algebraic form using place values to separate the singles from the bundles and the bundle-bundles, and a geometrical form showing the three blocks placed next to each other.

Operations **unite**/ split Totals in	**Variable**	**Constant**
Unit-numbers m, s, kg, $	**T = a + b** T – b = a	**T = a·b** T/b = a
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫a·db** *dT/db = a*	**T = a^b** $\sqrt[b]{T} = a$ $\log_a T = b$

Although presented as essence, ten-bundling is a choice. To experience its existence and the root of core mathematics as proportionality and linearity, Many should be bundled in icon-bundles below ten to allow a calculator to predict the result when shifting units: Thus asking ‘T = 2 3s = ? 4s’ the answer is predicted by two formulas, a recount-formula T = (T/b)·b telling that from a total T, T/b times bs can be taken away, and a restack-formula T = (T– b)+b telling that from a total T, T– b is left when b is taken away and placed next-to. Now first T = (2·3)/4 gives 1.some. Then T = 2·3 – 1·4 leaves 2. So the prediction is T = 2 3s = 1 4s & 2 = 1.2 4s. Thus with icon-counting, a natural number is a decimal number with a unit where the decimal point separates singles from bundles.

With physical units, the need for changing units creates per-numbers as 3$/4kg serving as bridges when recounting $s in 3s or kgs in 4s: 15$ = (15/3)·3$ = (15/3)·4kg = 20kg. As per-numbers, fractions are not numbers but operators needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus adding as areas, called integration.

So relinking it to its root, Many, allows today’s ‘mandarin mathematics’ to escape from its present essence-prison. For details, see the 2012 MrAlTarp YouTube videos.

Learning as Essence and Existence

Constructivist learning theory contains a European social Vygotskian and a North American radical Piagetian version believing learning taking place through guidance or exposure respectively. The question now is what is to be learned? Here Vygotsky accepts the ruling essence-claims about the outside fact Many even if self-reference makes them meaningless. Learning is seen as adapting to them and teaching as developing the learner’s mind in their direction using outside artefacts as means. Piaget sees learning as a means to adapt to the outside world, and sees teaching as asking guiding questions to outside existence brought inside the classroom to allow learners construct individual schemata to be accommodated through exposure and communication. So to let existence precede essence, Piaget is useful to mediate learning through inside exposure to outside existence. Vygotsky is useful if accepting that outside existence can lead to competing inside essence-claims. However, its lacking skepticism towards the ruling claim involves a high risk for practicing the banality of evil.

Institutionalized Education as Essence and Existence

Two versions of post-primary education exist, one letting national administration define its essence, the other letting individual talents define its existence. To get Napoleon out of Berlin, a European line-organized office-directed education was created that concentrate teenagers in age-groups and force them to follow the same schedule. To meet the international norm that 95% of an age-group finishes high school, dropout rates are lowered by low passing grades and by strict retention policy.

In the North American republics, middle and high schools teachers teach their major subject in their own classroom where they welcome teenagers with recognition: ‘Inside, you carry a talent and it is our mutual task to uncover and develop your personal talent through daily lessons in self-chosen half-year blocks. If successful I say ‘Good job, you have a talent, you need more’. If not I say ‘Good try, you have courage to try uncertainty, now try something else that might be your talent.’

Conclusion

An existentialist view replacing essence with existence exposes today's mathematics as pure essence with little existence behind. What has existence is Many waiting to be united by bundling and stacking into a decimal number with a unit presented geometrically or algebraically as a row of blocks or digits. Thus mathematics exists as geometry measuring forms divided into triangles, and as algebra reuniting numbers by four uniting techniques, addition, multiplication, power and integration each with a corresponding reversed splitting technique. So concepts should present themselves as created, not by self-reference as examples from abstractions above, but as abstractions from examples below. And statements should be held true when not falsified. In short, mathematics should be taught and learned as 'many-math', not as 'metamatism', a mixture of metamatics and 'mathematism' true inside but seldom outside the classroom as e.g. 'the fraction paradox' where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

As to learning, mediating the ruling essence should be replaced by guided exposure to the roots of mathematics, the physical fact Many, thus replacing Vygotsky with Piaget. And institutionalized education using camps to concentrate teenagers in age-groups obliged to follow forced schedules should be labeled as such allowing mathematics education to avoid the banality of evil. Christianity's Holy Communion offers forgiveness to individuals, not to institutions. Instead institutionalized force should be limited to provide teenagers with daily lessons in self-chosen half-year blocks to uncover and develop their individual talent, as would be the case if the North American Enlightenment republics replaced essence with existence in algebra and geometry.

Meta Thinking

Now, to what use was writing this paper? It was given ten minutes for presentation at the ICME13 Topic Study Group on Philosophy of Mathematics Education followed by five minutes debate; and it was given a month to be enlarged to double size. Then it may be printed but who will read it? And is this way the best way to improve schools in Sweden and the rest of the world? Or could we think of a better way to let disagreeing ideas and theories enlighten problems and provide solutions? The traditional way to ensure this is to have opponents when defending a thesis and discussants when presenting a paper. But these rarely signal any form of serious disagreement, often they just praise the work done and add some questions in a footnote like manner. Of course, in a way this is a consequence of the modern research paradigm seeing research as valid knowledge claims open for further exemplification, refinement and gap filling. But, as shown by the Swedish case, more research does not lead to solving more problems, on the contrary.

One option, of course, is to ask: maybe postmodern research can deliver what modern research cannot? Here Lyotard uses the word modern 'to designate any science that legitimates itself with reference to a meta-discourse', and he uses the word postmodern to designate 'incredulity toward meta-narratives'. As to the nature of knowledge in the postmodern computerized society, he says:

> Postmodern knowledge is not simply a tool of the authorities; it refines our sensitivity to differences and reinforces our ability to tolerate the incommensurable. Its principle is not the expert's homology, but the inventor's paralogy. (..) And invention is always born of dissension. (Lyotard, 1984: xxiii-xxv)

As to the problems coming from performing postmodern research, Lyotard says:

> Countless scientists have seen their "move" ignored or repressed, sometimes for decades, because it too abruptly destabilized the accepted positions, not only in the university and scientific hierarchy, but also in the problematic. The stronger the "move," the more likely it is to be denied the minimum consensus, precisely because it changes the rules of the game upon which consensus had been based.' (Lyotard, 1984: 63)

In other words, a postmodern researcher has little chance of being seen as a suitable candidate for a position at a modern university producing homology-research; unless the university is striving to bring many different knowledge perspectives into the faculty's activities, e.g. by stressing the existentialist point that existence precedes essence. But do such universities exist?

An alternative way to establish a non-consensus dialogue between disagreeing views is to arrange an old-fashioned Viking 'holmgang' (single battle). One example of this is the Chomsky-Foucault debate on Human Nature. Here Foucault says

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006: 41)

Inspired by this we can ask: maybe not discussing potential goal-means confusions creates problems for the institution called mathematics education? To test this hypothesis I designed a framework for two 'holmgangs'. The first was the Ernest-Tarp debate on postmodern mathematics held at the ICME12 discussion group 6 on Postmodern Mathematics. The next was intended to take place with a person having been very influential on Swedish Mathematics Education research. However, the person refused to take part arguing that it would only support my personal views. Below is an extract showing eight questions posed by Bo, the ninth and tenth being 'Me and Mathematics Education and Research' and 'How to Improve Mathematics Education?' The full design together with my answers can be seen in (Tarp 2016). Hopefully, someday the full dialogue can be enacted.

Manuscript to a Debate on Mathematics Education and its Research

Bo: Today we discuss Mathematics education and its research. Humans communicate in languages, a word-language and a number-language. In the family, we learn to speak the word language, and we are taught to read and write in institutionalized education, also taking care of the number-language under the name Mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Today governments control education, guided by a growing research community. Still, international tests show that the learning of the number language is deteriorating in many countries. This raises the question: If research cannot improve Mathematics education, then what can? I hope our two guests will provide some answers. I hope you will give both a statement and a comment to the other's statement.

1. Mathematics Itself

We begin with Mathematics itself. The ancient Greeks Pythagoreans used this word as a common label for what we know, which at that time was Arithmetic, Geometry, Astronomy and Music. Later Astronomy and Music left, and Algebra and Statistics came in. So today, Mathematics is a common label for Arithmetic, Algebra, Geometry and Statistics, or is it? And what about the so-called 'New-Math' appearing in the 1960s, is it still around, or has it been replaced by a post New-Math, that might be the same as pre New-Math? In other words, has pre-modern Math replaced modern Math as post-modern Math? So, I would like to ask: 'What is Mathematics, and how is it connected to our number-language?'

2. Education in General

Now let us talk about education in general. On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. To survive, plants need minerals, pumped in water from the ground through their leaves by the sun. Animals instead use their heart to pump the blood around, and use the holes in the head to supply the stomach with food and the brain with information. Adapted through genes, reptiles reproduce in high numbers to survive. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a few children per year. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. While food must be split in portions, information can be shared. Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, teenagers and adults. Continental Europe uses words for education that do not exist in the English language such as Bildung, Unterricht, Erziehung, Didaktik, etc. Likewise, Europe still holds on to the line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics have block-organized talent developing education from secondary school. As to testing, some countries use centralized test where others use local testing. And some use written tests and others oral tests. So, my next question is ‘what is education?’

3. Mathematics Education

Now let us talk about education in Mathematics, seen as one of the core subjects in schools together with reading and writing. But there seems to be a difference here. If we deal with the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. However, you cannot Math, you can reckon. At the European continent reckoning, called ‘Rechnung’ in German, was an independent subject until the arrival of the so-called new Mathematics around 1960. When opened up, Mathematics still contains subjects as fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc. Today, Europe only offers classes in Mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. Therefore, I ask, ‘what is Mathematics education?’

4. The Learner

Now let us talk about at the humans involved in Mathematics education: Governments choose curricula, build schools, buy textbooks and hire teachers to help learners learn. We begin with the learners. The tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning takes place through internal construction. Therefore, I ask ‘what is a learner?’

5. The Teacher

Now let us talk about the teacher. It seems straightforward to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a ‘Lehrer’ thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate ‘unterrichtung’ and ‘erziehung’ and to develop qualifications and competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. And, until lately, educating ‘lehrers’ took place outside the university in special ‘lehrer-schools’. Thus, being a teacher does not seem to be that well-defined. Therefore, my next question is ‘what is a teacher?’

6. The Political System

Now let us talk about governments. Humans live together in societies with different degrees of patronization. In the debate on patronization, the ancient Greek sophists argued that humans must be enlightened about the difference between nature and choice to prevent patronization by choices presented as nature. In contrast, the philosophers saw choice as an illusion since physical phenomena are but examples of metaphysical forms only visible to philosophers educated at Plato’s Academy who consequently should be accepted as patronizors. Still today, democracies come in two forms with a low and high degree of institutionalized patronization using block-organized education for individual talent developing or using line-organized education for office preparation. As to exams, some governments prefer them centralized and some prefer them decentralized. As to curricula, the arrival of new Mathematics in the 1960s integrated its subfields under the common label Mathematics. Likewise, constructivism meant a change from lists of concepts to lists of competences. However, these changes came from Mathematics and education itself. So my question is: ‘Should governments interfere in Mathematics education?’

7. Research

Now let us talk about research. Tradition often sees research as a search for laws built upon reliable data and validated by unfalsified predictions. The ancient Greek Pythagoreans found three metaphysical laws obeyed by physical examples. In a triangle, two angles and two sides can vary freely, but the third ones must obey a law. In addition, shortening a string must obey a simple ratio-law to create musical harmony. Their findings inspired Plato to create an academy where knowledge meant explaining physical phenomena as examples of metaphysical forms only visible to philosophers educated at his academy by scholasticism as ‘late opponents’ defending their comments on an already defended comment against three opponents. However, this method discovered no new metaphysical laws before Newton by discovering the gravitational law brought the priority back to the physical level, thus reinventing natural science using a laboratory to create reliable data and test library predictions. This natural science inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research outside the natural sciences still uses Plato scholastics. Except for the two Enlightenment republics where American Pragmatism used natural science as an inspiration for its Grounded Theory, and where French post-structuralism has revived the ancient Greek sophist skepticism towards hidden patronization in categories, correctness and institutions that are ungrounded. Using classrooms to gather data and test predictions, Mathematics education research could be a natural science, but it seems to prefer scholastics by researching, not Math education, but the research on Math education instead. To discuss this paradox I therefore ask, ‘what is research in general, and within Mathematics education specifically?’

8. Conflicting Theories

Of course, Mathematics education research builds upon and finds inspiration in external theories. However, some theories are conflicting. Within Psychology, constructivism has a controversy between Vygotsky and Piaget. Vygotsky sees education as building ladders from the present theory regime to the learners’ learning zones. Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing categories to grow out of concrete experiences and observations. Within Sociology, disagreement about the nature of knowledge began in ancient Greece where the sophists wanted it spread out as enlightenment to enable humans to practice democracy instead of allowing patronizing philosophers to monopolize it. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. As counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe’s line-organized office preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the post-structuralism of the French Enlightenment republic. Likewise, the two extreme examples of forced institutionalization in 20th century Europe, both terminated by the low institutionalized American Enlightenment republics, made thinkers as Baumann and Arendt point out that what made termination camps work was the authorized routines of modernity and the banality of evil. Reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job forces you to carry out both good and evil orders. As an example of a forced institution, this also becomes an issue in Mathematics Education. So I ask: What role do conflicting theories play in Mathematics education and its research?

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ICME 14 in Shanghai

The power of bundle- & per-numbers unleashed in primary school: calculus in grade one – what else?

In middle school, fraction, percentage, ratio, rate and proportion create problems to many students. So, why not teach it in primary school instead? Here they all are examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept and learn from the mastery of Many that children possess until mathematics takes it away?

Mathematics is hard, or is it?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Also among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as e.g. The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always 'to learn mathematics'. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

Research method

Difference research searching for differences to institutionalized traditions has uncovered hidden differences (Tarp, 2018a). To see if the differences make a difference, experiential learning (Kolb, 1984) and design research (Bakker, 2018) may create cycles of observations, reflections and designs of micro curricula to be tested in order to create a new cycle for testing a redesigned micro curriculum.

Observations and reflections 01

Asked "How old next time?" a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus describes what exists, bundles of 2s and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers 'T = (T/B)xB' saying "from T, T/B times, B may be pushed away and stacked". Consequently, search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a generation-1 curriculum.

Guiding questions

The following guiding questions were used: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g. if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "How to bundle-count seconds, minutes, hours, and days?", "How to count lengths in centimeters and inches?", "What to do if a bundle is not full?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s there are in 8 can be written as u*2 = 8, how can this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when turning or stacking stacks?" "What happens when recounting stacks placed on a squared paper?"

Observations and reflections 02

Data and ideas allowed designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC 01: Digits as icons

With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. Ain this way, all digits may be iconized and used as units for bundle-counting (Tarp, 2018b).

MC 02: Bundle-counting ten fingers

A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with 'underloads' counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s and 2s.

In 5s with bundles: 0B1, …, 0B4, 0B5 no 1B0, 1B1, …, 1B4, 1B5 no 2B0.

In 5s with bundles and underloads: 1B-4, 1B-3, …, 1B0, 2B-4, … , 2B0.

MC 03: Counting sequences in tens and hundreds

In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus preparing the ground for later writing out a multi-digit number fully as a polynomial, T = 345 = 3BB4B5 = 3*B^2 + 4*B + 5*1.

Count 10, 20, 30, …, 90, 100 etc. Then 1B, 2B, …, 9B, tenB no 1BB.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1BB, 2BB, …, 9BB, tenBB no 1BBB.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1BB0B, 1BB1B, …, 1BB9B, 1BBtenB no 2BB0B.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1B1, 2B1, 3B1, 3B2, …; 3B4.

MC04: Cup-counting and bundle-bundles

When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. T = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write T = 3B1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1BB, so with bundle-bundles we write T = 1BB0B1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. T = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, T = 5B0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1BB, so with bundle-bundles we write T = 2BB1B0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1BBB, so with bundle-bundle-bundles we write T = 1BBB0BB1B0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC05: Recounting in the same unit creates underloads and overloads

Recounting 8 1s in 2s gives T = 4B0 2s. We may create an underload by borrowing 2 to get 5 2s. Then T = 5B-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then T = 3B2 2s = 2B4 2s = 1B6 2s. Later, such ‘flexible bundle-numbers’ will ease calculations.

MC06: Recounting in tens

With ten fingers, we typically use ten as the counting unit thus becoming 1B0 needing no icon.

Q. T = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write T = 3 4s = 1B2 tens = 1]2 tens = 1.2 tens, or T = 2B-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. T = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write T = 6 7s = 4B2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write T = 6 7s = 5B-8 tens = 5]-8 tens = 5.-8 tens = 3B12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. T = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. T = 6 7s = 6*7 = (B-4) * (B-3) = BB-3B-4B+4*3 = 10B-3B-4B+1B2 = 4B2 since the 4 3s must be added after being subtracted twice.

MC 07: Recounting iconizes operations and create a recount-formula for prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a ‘recount-formula’ 8 = (8/2)*2 = 8/2 2s, or T = (T/B)*B = T/B Bs, saying “From T, T/B times, we push away B to be stacked”. Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering ‘7/2’ gives the answer ‘3.some’ predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering '7-3*2' gives the result '1', thus predicting that 7 recounts in 2s as 7 = 3B1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as T = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as T = 8 = 2 2/3 3s thus introducing fractions. Or, as T = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. T = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC 08: Recounting in time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as T = 60days = (60/7)*7days = 8B4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC 09: Double-counting in space creates per-numbers or rates

Counting in space has seen many units. Today centimeter and inches are common. 'Double-counting' a length in inches and centimeters approximately gives a 'per-number' or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. T = 12in = ?cm; and T = 20cm = ?in

A1. T = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and A2. T = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC 10: Per-numbers become fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s: T = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s: T = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s: T = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18. Here we enlarge both numbers in the fraction by 12/2 = 6.

MC 11: Per-numbers become ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: T = 12 = 2B2 5s = 3B-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle. Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6 and B gets the fraction 4/6 of the total.

MC 12: Prime units and foldable units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s. Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime numbers and foldable units up to two dozen.

MC 13: Recounting changes units and solves equations

Rephrasing the question "Recount 8 1s in 2s" to "How many 2s are there in 8?" creates the equation 'u*2 = 8' that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution u = 8/2 to u*2 = 8 thus comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation: u*2 = 4*2 = 8, OK.

Recounting from tens to icons gives equations: "42 is how many 7s" becomes u*7 = 42 = (42/7)*7.

MC 14: Next-to addition of bundle-numbers involves integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. T = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or A2. T = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC 15: On-top addition of bundle-numbers involves proportionality

To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. T = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. T = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or T = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC 16: Reversed addition of bundle-numbers involves differentiation

Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + u*5 = 3 8s is solved by moving to opposite side with opposite calculation sign

u*5 = 3 8s – 2 3s = 3*8 – 2*3, so u = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC 17: Adding and subtracting tens

Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: T = 4B6 tens = 4.6 tens = 46. Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: T = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4BB5B6 + 2BB6B9 = 6BB11B15 = 7BB12B5 = 7BB2B5 = 725.

Subtraction: 456 – 269 = 4BB5B6 – 2BB6B9 = 2BB-1B-3 = 2BB-2B7 = 1BB8B7 = 187

Multiplication: 2 * 456 = 2 * 4BB5B6 = 8BB10B12 = 8BB11B2 = 9B1B2 = 912

Division: 154 /2 = 15B4 /2 = 14B12 /2 = 7B6 = 76

MC 18: Next-to addition & subtraction of per-numbers and fractions is calculus

Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square. Thus per-numbers add by areas, i.e. by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

2*5 + u*3 = 3*8, so u*3 = 3*8 – 2*5, so u = (3*8 – 2*5)/3 = 4 2/3, or u = (T2-T1)/3 = ΔT/3

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC 19: Having fun with bundle-bundle squares

On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

n*n + 2*n + 1 = (n+1)*(n+1); and (n-1)*(n-1) = n*n – 2*n + 1; and (n–1)*(n+1) = n*n – 1.

Two s*s bundle-bundles form two squares that halved by their diagonal d gives four half-squares called right triangles. Rearranged, they form a diagonal-square d*d. Consequently d*d = 2*s*s

Four c*b playing cards with diagonal d are placed after each other to form a (b+c)*(b+c) bundle-bundle square. Below to the left is an c*c square, and to the right a b*b square. On-top are two playing cards. Inside there is a d*d square and 4 half-cards. Since 4 half-cards is the same as two cards, we have the formula c*c + b*b = d*d making it easy the add squares, you just square the diagonal.

MC 20: Having fun with halving stacks by its diagonal to create trigonometry

Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 5/10 = 0.5 approximately. Here tan comes from recounting the height in the base.

MC 21: Having fun with a squared paper

A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, and 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying “3up_per_1out, 2up_per_1out, …, 3down_per_1out” allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC 22: Having fun with turning and combining stacks

Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus 2 3s may be stacked 4 times to the box T = 4*(2*3) that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

Discussion and recommendation

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many. To find an answer, experiential and design research is used to create a cycle of observations, reflections and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions about mastering many. The first observation is that children use two-dimensional bundle-numbers with units instead of one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting and double-counting core activities where recounting leads directly to proportionality by combining division and multiplication, thus reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking e.g. 'how many 5s are 3 tens?' And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see flexible bundle-numbers easing traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education all teaching single numbers without units will deport the testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in e.g. division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus directly gives the proportionality recount-formula 8 = (8/2)*2 or T = (T/B)*B used in STEM and to solve equations. Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Thus, a new and different Kuhnian paradigm (1962) is created by respecting and developing the number-language and quantitative competence children bring to school by teaching numbering instead of numbers, by using two-dimensional bundle-numbers instead of one-dimensional single numbers, and by counting, recounting and double-counting before adding next-to or on-top.

This paradigm allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and to treat inside schoolbook and university mathematics as footnotes to bracket if blocking the way to the outside goal. To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not lean from children instead?

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From loser to user, from special to general education, learning Inside mathematics through outside actions

Although eager to begin school, some children soon fall behind and are sent to special education teaching the same at a slower pace. Wanting mathematics education to be for all leads to the question: Is this so by nature or by choice? Can it be otherwise? Observing how children communicate about Many before school, this paper asks what kind of mathematics can be learned if accepting the bundle-numbers as 2 3s that children bring to school. Using Difference Research, it turns out that accepting numbers with units means that counting, recounting and solving equations come before adding on-top or next-to introduce integral and differential calculus as well as proportionality in early childhood education. So, it is possible to institute an ethical mathematics education that transforms loser to users returning to general education as stars teaching fellow students and teachers how to master Many.

Inside, children adapt smoothly to their outside world

It is glad to see how children vividly communicate about Many before school. And it is sad to see how they stop after beginning school, and how more and more are excluded from general education and sent to special education. A day inside a classroom tells you why. The students no more communicate about Many, instead a textbook mediated by a teacher teaches them about what they need in order to communicate: multidigit numbers obeying a place value system, first to be added then subtracted with no respect to their units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers.

In a special education class, the same is taught but at a slower pace. Which makes you wonder: With their preschool foundation, can children learn number-language through communication as with the word-language (Widdowson, 1978)? And, can this mastery of Many lead to mastery of mathematics later, if needed? In which case, mastery of mathematics wouldn’t be the only way to master Many.

Looking for differences, Difference Research (Tarp, 2018) searching for differences that might make a difference may inspire micro curricula to be tested using, e.g., Design Research (Bakker, 2018).

Meeting many, children count with bundles as units

How children adapt to Many can be observed from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, “That is not four, that is two twos”, thus describing what exists: bundles of 2s, and 2 of them.

Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And they also use full sentences as in the word-language with a subject ’that’, and a verb ‘is’, and a predicate ‘2 2s’, which abbreviated shows a formula as a number-language sentence ‘T = 2 2s’.

A bundle-number curriculum for SPECIAL EDUCATION

Listening to special education students helps understanding when and why they fall behind. Inspired by this we may design question guided micro curricula, MC, to further develop the number-language and mastery of Many they acquired before school, allowing them an comeback to general education.

MC01. Digits

The tradition presents both digits and letters as symbols. A difference is letting students experience themselves digits as not symbols, but icons with as many sticks or strokes as they represent if written less sloppy (Tarp, 2018). In this way students see inside icons as linking directly to outside degrees of Many. And that ten has no icon since, as a bundle it becomes the unit, so that two-digit numbers really are two countings of bundles and unbundled singles.

A guiding question can be "There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?" Outside material could be sticks, a folding ruler, cars, dolls, spoons, etc.

Discussing why numbers after ten has no icon leads on to bundle-counting.

MC02. Bundle-Counting Sequences

Using a place value system, the tradition counts without bundles. A difference is to practice bundle-counting in tens, fives, and threes. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names 'eleven' and 'twelve' are Viking names meaning 'one left' and 'two left', and that the name 'twenty' has stayed unchanged since the Vikings said 'tvende ti'; and that English roughly is a mixture of Viking words labeling concrete things and actions, and French words labeling abstract ideas. The Viking tradition saying 'three-and-twenty' instead of 'twenty-three' was used in English for many years (see, e.g., Jane Austen). Now it stops after 20. The Vikings also counted in scores: 80 = 4 scores, 90 = half-5 scores.

A guiding question can be "Let's use the word bundle when bundle-counting in tens, in 5s and in 3s."

Outside material could be fingers, sticks, cubes, and a ten-by-ten abacus. Using fingers and arms we can count to twelve, also called a dozen. Using cubes, the bundles are stacked on-top of each other.

First, we count in 5s (hands): 0*B*1, …, 0*B*4, 0*B*5 or 1*B*0, 1*B*1, …, 1*B*5 or 2*B*0, 2*B*1, 2*B*2.

Then we count in 3s (triplets): 0*B*1, …, 0*B*3 or 1*B*0, …, 1*B*3 or 2*B*0, …, 2*B*3 or 3*B*0, 3*B*1, ...

Counting cubes in 3s, 3 bundles is 1 bundle-of-bundles or 1*BB* in writing, so we repeat: 0*B*1, …, 2*B*3 or 3*B*0 or 1*BB*0*B*0, 1*BB*0*B*1, 1*BB*0*B*2, 1*BB*0*B*3 or 1*BB*1*B*0.

Then we count in tens, including the bundles: 0*B*1, …, 0*B*8, 0*B*9, 0*B*10 or 1*B*0, 1*B*1, 1*B*2.

Finally, we bundle-count in tens from 0 to 111.

MC03. Bundle-Counting with Underloads and Overloads

Strictly following the place value system, the tradition silences the units when writing 'two hundred and fifty-seven' as plain 257. A difference may be inspired by the Romans using 'underloads' when writing four as "five less one", IV; and by overloads when small children use 'past-counting': "twenty-nine, twenty-ten, twenty-eleven".

A guiding question can be. "Let us count with underloads missing for the next bundle. And with overloads as children saying 'twenty-eleven'. Outside material could be sticks, cubes, and an abacus.

First, we notice that five fingers can be counted in pairs in three different ways

T = 5 = I I I I I = ~~II~~ I I I = 1*B*3, overload

T = 5 = I I I I I = ~~II~~ ~~II~~ I = 2*B*1, normal

T = 5 = I I I I I = ~~II~~ ~~II~~ ~~II~~ = 3*B*-1, underload

Using fingers and arms, first we count using underloads: 0*B*1 or 1*B*-9, 0*B*2 or 1*B*-8, …, 0*B*9 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-9, 1*B*2 or 2*B*-8. The we count to twelve in 5s (hands): 0*B*1 or 1*B*-4, 0*B*2 or 1*B*-3, …, 0*B*4 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-4, …, 2*B*2 or 3*B*-3.

And in 3s (triplets): 0*B*1 or 1*B*-2, 0*B*2 or 1*B*-1, 0*B*3 or 1*B*0, 1*B*1 or 2*B*-2, …, 3*B*3 or 4*B*0 or 1*BB*1*B*0. Cup-counting with a cup for bundles, and for bundles-of-bundles: T = 1]1]0 = 4]0 = 3]3 = 2]6 = 1]9.

Then we count in tens from 1 to 111, using past-counting: … 1*B*9, 1*B*10, 1*B*11 or 2*B*1, 2*B*2, …, 2*B*11 or 3*B*1, …, 9*B*9, 9*B*10, 9*B*11 or 10*B*1 or 1*BB*0*B*1.

Then we rewrite totals as ‘flexible bundle-numbers’ with overloads and underloads:

T = 38 = 3*B*8 = 2*B*18 = 1*B*28 = 4*B*-2 = 5*B*-12

MC04. Doing Math with Flexible Bundle-Numbers

The tradition uses carrying when adding and multiplying, and borrowing when subtracting and dividing. Here, a difference is to instead use flexible bundle-numbers.

A guiding question can be. “Let us do inside arithmetic with flexible bundle-numbers.”

Overload	Underload	Overload	Overload
65 + 27	65 - 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 - 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 1: Doing Arithmetic with Flexible Bundle-Numbers

MC05. Talking Math with Formulas

In a number-language sentence as “The total is 3 4s”, the tradition silences all but the calculation 3*4. A difference is to use full sentences with an outside subject, a verb and an inside predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence “*T* = 5*6 = 30” thus inside predicts that outside 5 6s can be re-counted as 3 tens.

A guiding question can be. “Let us talk math with full sentences about what we calculate and how.”

We begin by counting in ones using a full sentence: “From the total we pull away one to get one.”

We then write what was before and after, using a rope as an icon for pulling away:

“*T* = (*T* – 1) + 1”.

This number-language sentence formulates what we call a formula. It also applies if instead pulling away 2, “*T* = (*T* – 2) + 2”, and if pulling away any unspecified bundle *B*, “*T* = (*T* – *B*) + *B*”. We call this formula a ‘re-stack formula’ since, with the total as a stack, we pull away the bundle from the top and place it next-to as its own stack.

Outside asking “Adding what to 2 gives 5?”, inside becomes “? + 2 = 5” in writing. Using the letter *u* for the unknown number, this becomes an ‘equation’ “*u* + 2 = 5”, easily solved outside by pulling away the 2 that was added, described inside by restacking the 5: *u* + 2 = 5 = (5 – 2) + 2, so *u* = 5 – 2.

So, inside an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the definition of the number ‘5-2’: “5 minus 2 is the number *u* that added to 2 gives 5”.

We now use full sentences when counting in bundles, e.g., re-counting 8 1s in 2s. We use ‘/’ to iconize a broom brushing away 2s. So ‘8/2 = 4’ is an inside prediction for the outside action “From 8, brush 2 away, 4 times.”

Having been brushed away, the bundles of 2s are stacked. This is iconized by an ‘x’ for lifting the bundles, so ‘4x2 = 8’ is an inside prediction for the outside action “4 times stacking 2s gives 8 1s”.

Re-counting 8 in 2s thus gives a ‘re-count formula’ 8 = (8/2) x 2, outside showing a box with the counter 8/2 and the unit 2 on the vertical and horizontal side, and with the total 8 as the area. So, totals add by areas, called integral calculus. With unspecified numbers it says: *T* = (*T*/*B*) x *B*, or *T* = (*T*/*B*) * *B*, or “From a total *T*, *T*/*B* times, *B* can be brushed away”.

Outside asking “How many 2s in 8”, inside is the equation “? * 2 = 8”, or “*u* * 2 = 8” easily solved outside by brushing away 2s, described inside by recounting 8 in 2s:

$u * 2 = 8 = (8/2) * 2$, so $u = 8/ 2$.

Again, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the formal definition of '8/2': "8 divided by 2 is the number *u* that multiplied to 2 gives 8".

MC06. Naming the Unbundled Singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question can be "How to see the unbundled singles?". Outside materials can be cubes or an abacus

Before outside recounting 9 in 2s, inside we let a calculator predict the result: Entering 9/2 gives '4.some' predicting that "from 9, brush away 2s can be done 4 times". To find unbundled singles, outside we pull away the 4-by-2 stack, inside predicted by entering '9 – 4 * 2' giving 1. So, inside the calculator predicts that 9 recounts as 4*B*1 2s, which is also observed outside.

Recounting 9 cubes in 2s, the unbundled can be placed on-top of the stack. Here it can be described inside by a decimal point separating the bundles from the unbundled: T = 4*B*1 2s = 4.1 2s. Likewise, when counting in tens: T = 4*B*2 tens = 4.2 tens = 4.2 * 10 = 42.

Seen as part of a bundle, inside we can count it in bundles as a 'fraction', 1 = (1/2) * 2 = 1/2 2s; or we can count what is missing in a full bundle, 1 = 1*B*-1 2s.

Again, we see the flexibility of bundle-numbers: T = 4*B*1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Likewise, when counting in tens: T = 4*B*2 tens = 4 2/10 tens = 4.2 tens = 5.-8 tens.

MC07. Changing Number Units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon to another, from icons to tens, or from tens to icons, or into a square.

A guiding question can be "How to change number units?". Outside materials can be an abacus.

Asking '3 4s = ? 5s', we inside predict the result by entering on a calculator the 3 4s as 3*4, to be counted in 5s by dividing by 5. The answer '2.some' predicts that from 3 4s, 2 times, 5 can be brushed away. To find unbundled singles, outside we pull away the 2 fives from the 3 4s; inside we predict this by entering '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted as 2 5s & 2, or 2*B*2 5s.

Asking "40 = ? 5s", we predict the result by solving the equation "$u * 5 = 40$" by recounting 40 in 5s: $u * 5 = 40 = (40/5) * 5$, so $u = 40/5$.

Asking "6 8s = ? tens", or "6 * 8 = ?", we inside predict the result by looking at a ten-by-ten square with 6 and 8 as *B*-4 and *B*-2 on the sides. We then see that the 6*8 box is left when from the *B***B* box we pull away a 4**B* and a 2**B* box and add the 4*2 box pulled away twice.

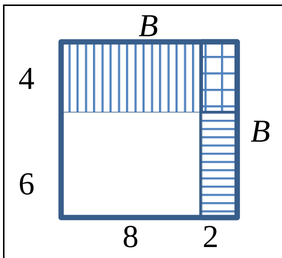	$T = 6 * 8$ $= (B-4) * (B-2)$ $= BB – 2B – 4B + 8$ $= 4B8$ $= 48$	$T = \begin{pmatrix} 1B - 4 \\ 1B - 2 \end{pmatrix}$ $= 1BB – 2B – 4B + 8$ $= 10B – 6B + 8$ $= 4B8 = 48$	$T = \begin{pmatrix} 2x - 3 \\ 4x - 5 \end{pmatrix}$ $= 8x^2 – 10x – 12x + 15$ $= 8x^2 – 22x + 15$

Figure 2: Multiplying Numbers as Binomials

Inside, multiplying two 'less-numbers' horizontally creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles.

Wanting to square a 5-by-4 box, its side is called $\sqrt{20}$, using lines to iconize the square. To find $\sqrt{20}$ we see that removing the 4-square leaves 20 – 4*4 = 4 shared by the two 4**t* boxes in a

4+t square, giving t = 0.5. A little less since we neglect the t-square. Inside, a calculator predicts that $\sqrt{20}$ = 4.472.

Intersection points between lines and circles leads to quadratic equations as x^2+ 6x + 8 = 0, easily solved when rotating the upper of two x by x+3 playing cards to create a square with sides x+3, and with zero area except for the 1 left in the upper 3-by-3 square after 8 is removed.

In total, $(x+3)^2$ = 0 + 1 = 1 = $\sqrt{1}^2$, so x + 3 = ± 1, giving x = - 2 and x = - 4.

MC08. Changing Physical Units with Per-Numbers

The tradition sees shifting physical units as an application of proportionality. Typically finding the unit cost will answer questions as "with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?"

A difference is to use 'per-numbers' (Tarp, 2018) coming from double-counting the same quantity in the two units, e.g., T = 3$ = (3$/2kg) * 2kg = p * 2kg, with the per-number p = 3$/2kg, or 3/2 $/kg.

A guiding question can be "How to change physical units?". Outside materials can be colored cubes.

Recounting in the per-number allows shifting units:

T = 6 kg = (6/2)*2 kg = (6/2)*3 $ = 9$; and T = 15$ = (15/3)*3$ = (15/3)*2kg = 10 kg.

Alternatively, we recount the units: $ = ($/kg)*kg = (3/2)*6 = 9; and kg = (kg/$)*$ = (2/3)*15 = 10.

With like units, per-numbers become fractions: 1$/4$ = ¼. The tradition teaches fractions as division: 1/4 of 12 = 12/4. A difference is to see a fraction as a part of a bundle counted in bundles, 1 = (1/4)*4, so 1/4 = 1 part per 4. Finding 3/4 of 12 thus means finding 3parts per 4 of 12 that recounts in 4s as:

T = 12 = (12/4) * 4 = (12/4) * 3parts = 9parts, so 3 per 4 is the same as 9 per 12, or 3/4 = 9/12.

So, finding 3/4 of 100 means finding 3 parts per 4 of 100 = (100/4)*4, giving 75 parts per 100 or 75%.

MC09. Recounting the Sides in a Box Halved by its Diagonal Gives Trigonometry

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, mutually recounting the sides in a box halved by its diagonal.

A guiding question can be "How to recount the sides in a box halved by its diagonal?". Outside materials can be tiles, cards, peg boards, and books.

Recounting the height in the base, height = (height/base) * base = tangent A * base, shortened to

h = (h / b) * b = tan A * b = tan A bs, thus giving the formula tangent A = height / base, or tan A = h/b.

Using the words 'run' and 'rise' for 'base' and 'height', we get the formula: tan A = rise/run, giving the steepness or slope of the diagonal. The word 'tangent' is used since the height will be a tangent in a circle with centre in A with the base as its radius.

This gives a formula for the circumference since a circle contains many right triangles: In an h-by-r half-box, h recounts in r as h = (h/r) * r = tan A * r.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, half the circumference is

½C = 100 * h = 100 * tan 1.8 * r = 100 * tan (180/100) * r = 3.1426 * r

Calling the circumference for 2 * π * *r*, we get a formula for the number π:

π = tan (180/*n*) * *n*, for *n* sufficiently large.

MC10. Adding Next-To and On-Top

The tradition sees numbers as one-dimensional cardinality with addition defined as counting on. A difference is to accept childrens' bundle-numbers as 2-dimensional boxes that add next-to and on-top. A guiding question can be "How to add 2 3s and 4 5s on-top and next-to?". Material: cubes and abacus.

To add 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing one or both; inside recounting changes units. Or to use the recount formula to predict the result by entering (2*3+4*5)/*B*, where *B* can be 3 or 5 or 8. Added next-to by areas is called integral calculus.

Adding 20% to 30$, we have two units with the per-number 30$ per 100%. Adding 20% to 100% gives 120%, recounting in 100s as 120% = (120/100)*100% = (120/100)*30 $ = 120%*30$. So, we add 20% by multiplying with 120%, also called to multiply with the index-number 120.

Reversing adding next-to and on-top, a guiding question can be "How many 3s to add to 4 5s to get a total of 6 5s or 5 8s?". Outside materials can be cubes and an abacus.

To find the answer outside, we pull way the 4 5s from the total *T* before recounting in 3s, which is predicted inside by asking the calculator: (6*5 – 4*5)/3, or (5*8 – 4*5)/3, i.e., as Δ*T*/3. Using a difference to calculate the change in the total, Δ*T*, before recounting, this is called differential calculus.

MC11. Adding Per-Numbers and Fractions

Seeing fractions as numbers that add without units, the tradition teaches 'mathematism', true inside but seldom outside classrooms where, e.g., 2m + 3cm = 203cm. A difference is respecting that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding.

A guiding question can be "What is 2kg at 3$/kg plus 4kg at 5$/kg?" Outside materials can be a peg board with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

Inside we see that unit-numbers add directly. Whereas, per-numbers first must be multiplied to become unit-numbers. And since multiplication creates a box with an area, per-numbers add by their areas, i.e., as the area under the per-number curve. And again, adding areas is called integral calculus.

And again, the opposite is called differential calculus asking "2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg." The two connect by the fact that adding serial differences, the middle terms disappear leaving only the difference between the end and initial numbers: (*b*-*a*) + (*c*-*b*) + (*d*-*c*) = *d*-*a*.

MC12. Change by Adding or by Multiplying

The tradition taches adding arithmetic and geometric sequences as series. A difference is adding constant unit-numbers and per-numbers. A guiding question can be "What is 2$ plus 3$/day?" and "What is 2$ plus 3%/day?" Outside materials can be a peg board and an abacus.

Inside we see that adding 3$/day to 2$ gives a total of *T* = 2 + 3**n* after *n* days. This is called change by adding, or linear change with the general formula *T* = *b* + *a***n*. And that adding 3%/day to 2$ gives a total of *T* = 2*103%^*n* after *n* days since adding 3% means multiplying with 103%. This is called change by multiplying or exponential change having the general formula *T* = *b* * *a*^*n* = *b**(1+*r*)^*n*.

Reversing change by adding means facing an equation as 100 = 20 + 5**u*, easily solved by restacking and recounting: 100 = (100-20) + 20 = 80 + 20, so *u**5 = 80 = (80/5)*5, so *u* = 80/5 = 16.

Reversing change by multiplying gives two equations. In the equation 20 = u^5, we want to find the factor u of which 5 gives 20, predicted by the factor-finding root $\sqrt[5]{20}$ = 1.82. In 20 = 5^u we want to find the number u of 5-factors that give 20, predicted by the factor-counting logarithm $\log_5(20)$ = 1.86. We now know all the ways to unite parts into a total, and to split a total in parts, the 'Algebra-square':

Operations **unite**/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	***T* = *a* + *n*** $T - n = a$	***T* = *a**n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	***T* = ∫*a**d*n*** $\frac{\mathrm{d}T}{\mathrm{d}n} = a$	***T* = *a*^*n*** $\sqrt[n]{T} = a \qquad n = \log_a T$

Figure 3: The 4 Ways to Unite Parts into a Total, and the 5 Ways to Split a Total into Parts.

Observations

The special education students were asked to write or phone short messages to a friend about how they experienced the twelve micro curricula. Typical answers expressed positive attitudes towards learning that digits are icons with sticks, that hundred is a bundle-of-bundles, that with bundles you don't need the place value system, that over- and underloads are allowed, that recounting is predicted by a recount-formula that also solves equations, that negative numbers and decimals and fractions simply tell how to see the unbundled, that the multiplication tables come when recounting from icons to tens, that boxes can be squeezed to change units or to become squares, that physical units are changed by recounting in the per-numbers, that recounting a box halved by its diagonal introduces trigonometry and a formula for pi, that number-boxes can be added on-top, but also next-to as integration adding areas, also occurring when adding per-numbers. And they proudly talk about returning to general education and becoming stars when teaching fellow students and the teacher new ways to do math.

Conclusion

The ancient Greek civilization talked about common ethos, individual ethics and collective moral. As to a school, we would expect a civilized ethos of this institution to be its original meaning, a timeout to reflect ongoing activity, as when called by a coach in a match. Foucault (1972) thus warns against a school becoming a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cells daily, and accept the diagnose 'un-educated' to be cured by, of course, education as defined by a self-referring 'truth regime'. To avoid this, mathematics education should be a timeout where the ethics of civilized educators would be that of foster-parents guiding their foster-children to better master and communicate about Many. And as to moral, we would expect a civilized mathematics education curriculum to support the foster-parents when helping children improve their mastery of outside existence instead of forcing them to first master an inside constructed essence. That, as an institutionalized means, may be tempted by a goal displacement (Bauman, 1990, p. 84): By making itself so difficult that only few will arrive at the outside end-goal of mastering Many, inside mathematics could increase its power and funding in order to finally, if ever, reach its outside goal.

Thus, to the benefit of all students, a moral mathematics education should use guidance to develop the mastery and communication that children build up when adapting to Many before school. And it would be immoral and unethical to force upon them the necessity of a construct called one dimensional place-value line-numbers adding without units in order to exclude some to special education just repeating the same at a slower pace. This paper shows that this immoral mathematics education is not there by necessity, but by choice. And that a moral version comes from not teaching but learning from children.

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ICME 15 in Sydney

Possible topics for Discussion Groups

We welcome proposals for Discussion Groups on a variety of topics and themes at ICME-15. In addition, the IPC has identified a number of suitable topics that may facilitate valuable Discussion Group sessions. Some of these topics relate to Plenary Panel 1, which will examine how mathematics education responds to issues for humanity. Discussion Groups could take a closer look at how, in different parts of the world, mathematics education can, or should relate to issues such as:

- Climate change and sustainability
- Social justice, poverty, and inequality
- Citizenship, democracy, and fake news
- Displaced people, peace, and justice
- Indigenous knowledges and decolonising mathematics
- Contemporary gender issues (transgender, gender diversity etc.)
- Artificial Intelligence
- Pandemics, wellbeing, and resilience

My contributions:

- Modeling Eased by Demodeling and Rerooting, a TSG paper for a Topic Study Group accepted as the poster De-colonizing mathematics by de-modeling & re-rooting
- A Text-Free Math Education Found by Difference Research for Protection Against Alien Artificial Intelligence, a TSG paper for a Topic Study Group, no recation came
- Decolonizing mathematics when demodeling it by using the child's uncolonized 2D bundle-numbers with units, an accepted workshop
- Decolonizing mathematics, can that secure numeracy for all, and be protected from AI?, a rejected discussion group proposal

Modeling Eased by Demodeling and Rerooting

Modeling should motivate mathematics education but is not always that easy. Could this be different? Difference research searching for differences making a difference suggests that inside concepts may be de-rooted from the outside world by getting different names and meanings. So de-modeled they may retake their original roles allowing mathematics to again become a number-language communicating about the outside world. The search found children's own flexible bundle-numbers with units, that allow counting and recounting to precede adding. This leads directly to the core of mathematics, using proportionality and calculus to re-unite changing and constant unit-numbers and per-numbers.

Does mathematics modeling have to be so difficult?

Inspired by the first International Congress on Mathematical Education, ICME-1, I joined the student revolt in 1969 to secure that mathematics would no more be taught without linking it to its use through modeling. I was allowed to write a master thesis on modeling where I chose Game Theory. In 1974 I published my first textbook "Mathematical Growth Models" showing how calculus is modeling predictable non-constant change. And at the ICME-3 I presented a poster "Mathematics, a collection of arbitrary theoretical structures, or model-building of the real world" as well as a short oral address in English and French "Mathematics, an integral part of the real world". From 1975 I worked with Mogens Niss for three years at Roskilde University. We were both interested in modeling the Danish macro-economic cycle. Niss preferred the actual government model where the mathematics was so complicated that it could not be addressed in high school. I saw the model, not as 'since-then' fact model as in physics, but as an 'if-then' fiction model based upon assumptions that could be different. And here Ockham's razor says with two different models to explain the same, you should prefer the simpler one. So, I worked out a simple linear model that could be used in high school (Tarp, 2001). Niss stayed at the university and I returned to the high school and joined a group that succeeded changing the precalculus curriculum so that polynomials of first and second degree were replaced by linear and exponential functions so modeling could enter the classroom. I was allowed to test a special curriculum showing how statistical tables with categories divided into subcategories and changing over time may be modelled by statistics, linear and exponential change (Tarp, 2021), which allowed all students to pass the exam successfully. But the standard textbooks still presented pure mathematics to be learned before it could be applied. Linear and exponential functions were presented as examples of the function concept that was presented as an example of a set-relation. So instead of presenting an abstract concept though its examples, it was presented as an example of a more abstract concept. This was difficult to many students and resulted in so poor exam results, that it was suggested to remove precalculus as a mandatory class at the reform in 2005.

When my students asked for examples, I chose saving money by adding 5 $ per week at home or 5% per year in a bank. "Why can't we call this growth by adding and by multiplying?" the students asked. This made a difference. So, I used deconstruction (Derrida, 1991) to develop a Difference Research searching for differences making a difference (Tarp, 2018). And the government accepted my advice that precalculus should stay as a mandatory subject, but the function concept should be replaced with variables as in in physics and economics, so that we write y = b+a*x instead of f(x) = b+a*x. However, modeling still was difficult to many students. So, I turned to primary school to see if deconstruction by listening to children would make a difference here also. I found that the children see four fingers held together two and two, not as 4, but as two 2s, thus using bundle-numbers with units for what exists, bundles of 2s in space, and 2 of them when counted in time. I described the potentials of deconstruction and bundle-numbers in several MrAlTarp YouTube videos, and in 10+15+16 contributions to the ICME 10-12 (Tarp, 2012). For ICME 13 I wrote 9 papers but was allowed only 1. For the ICMI Study 24 on curriculum I designed several micro curricula where the math core was re-rooted and renamed by the process of counting and recounting before adding (Tarp, 2018, 2020). Also, I had designed a teacher education academy in 'ManyMath', MATHeCADEMY.net, Count & Add in Time & Space. Typically, modeling is eased by demodeling and re-rooting when tested in math labs, libraries and private education. So, the time has come for others to perform a large-scale testing.

Demodeling: from the inside to the outside and back

Modeling goes from the outside to the inside and back. Demodeling does the opposite by going from the inside to the outside and back. So demodeling begins with the core of inside mathematics as seen on a calculator: digits, operations, brackets, multidigit numbers, decimal point, and an equation sign. And then asks the question "What outside things and actions have rooted these inside concepts?"

Demodeling digits and multidigit numbers

Digits and letters may both be seen as symbols. But digits may also be seen as icons with as many sticks as they represent, five sticks in the five-icon, etc., if written 'less sloppy'. A sequence of digits may be seen as one multidigit number obeying a place-value system with ones, tens, hundreds, etc., and seldom with the word 'bundle-of-bundles' used for 'hundred'. But a multidigit number may also be seen as rooted in several numberings of unbundled, bundles, bundles of bundles, etc. (Tarp, 2018).

Recounting a total T of ten in 3s we get T = 3 Bundles 1, or T = 3B 1, or T = 1BB 0B 1 = 1B^2 0B 1 since 3 bundles is 1 bundle-of-bundles. So, bundling bundles roots power, and bundle-counting totals roots polynomials, T = 345 = 3*B^2 + 4*B + 5*1. And it also roots functions as number-language sentences with an outside subject, a verb, and an inside predicate as in word-language sentences.

To bundle-count a total, bundles are pushed away and lifted into a stack to be pulled away to look for unbundles singles. These actions root division as an icon for a push-away broom, multiplication as a lift, subtraction as a pull-away rope, and addition showing two ways to unite stacks, on-top or next-to. Placed on-top of the stack, the unbundled may be seen as a decimal number, or as a fraction when counted in bundles also, or described by what has been pulled away in time from the next bundle, or what is missing in space for another bundle. Recounting 9 in 2s, the end result may thus be written as

T = 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s (an underload), or with an overload, T = 3B3 2s, and T = 2B 5 2s.

As to the process, to recount 8 in 2s we push-away 2s to be stacked as 4 2s, which may be written as

8 = (8/2) x 2 = 8/2 2s, or $T = (T/B) \times B = T/B$ *B*s with unspecified numbers.

By changing the unit, this recount-formula roots the proportionality formula $T = a*b$ recounting T in *b*s. Examples may be meter = meter/sec*sec, recounting a distance in seconds, or $ = $/kg*kg, recounting dollars in weight, thus creating 'per-numbers' as meter/sec, $/kg, etc. Or part = part/whole * whole, recounting a part in wholes and becoming fractions with like units. In time, terminal = terminal/initial * initial recounts the end-value in start-values.

A rectangle has base, b, height, h, and diagonal, d, raising an angle, A. Here, mutual recounting roots trigonometry: $h = (h/b)*b = \tan(A)*b$, $h = (h/d)*d = \sin(A)*d$, and $b = (b/d)*d = \cos(A)*d$. In half a radius 1 circle, splitting the circumference in n parts gives the pi-number $\pi = n*\tan(180/n)$ for n big.

Recounting between icons and tens root equations and early algebra.

Recounting from tens to icons, we ask "How many 2s in 8?" This roots equations solved by recounting 8 in 2s: $u*2 = 8 = (8/2)*2$, so $u = 8/2$ from pushing-away 2s from 8, showing that an equation is solved by reversing the process, i.e., by moving a number to the opposite side with the opposite sign. This follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, $u*2= 8$. 'To opposite side with opposite sign' may be rooted outside, while the inside balancing method is derived from abstract algebra concepts as group, inverse and neutral elements, associativity and commutativity.

Recounting from icons and tens, we ask "6 7s is how many tens?" This roots early algebra if allowing underloads: T = 6 7s = 6*7 = (B-4)*(B-3) = BB -4B -3B + 4 3s, as seen on a B*B square where the 6 7s is left when from ten bundles we pull-away 4 bundles and 3 bundles, and finally add the 4 3s that was pulled-away twice.

Once counted, stacks may be added, on-top after recounting provides like units, or next-to as areas thus rooting integral calculus, as well as differential calculus when reversing asks “2 3s + ?5s = 4 8s”.

Per-numbers are added in mixture problems as “2kg at 3$/kg plus 4kg at 5$/kg give what?”. With like units, the unit-numbers 2 and 4 add directly. But per-numbers must be multiplied to unit-numbers before adding as the areas created by the multiplication. So, mixture problems root integral calculus, preceding differential calculus occurring when the problem is reversed.

Inside, outside totals become rectangular stacks as T = 9 5s = 9*5, or squares in the case of bundle-bundles, T = 5 5s = 5*5. So, we may ask “How to square a rectangle?”, e.g., T = 9 5s = B*B = B^2 where B is the square root of 45, B = √45, iconized by half a perimeter. Here, half the excessing 4 5s is placed to the right to create a 7*7 square except for the upper 2*2 right corner. Again, half of this is placed to the right to create a 6 5/7-square, since ½*4 = 2 = (2/7)*7. This is close: (6 5/7)^2 = 45.1.

Fact, fiction and fake, the 3 modeling genres

With mathematics as a number-language modeling outside things in space and actions in time, its quantitative literature needs to be divided into fact, fiction or fake, the same genres used in the word-language for qualitative literature. Fact stories are ‘since-then’ stories that quantify and predict predictable quantities by using factual numbers and formulas; and that need to be checked for units and correctness. Fiction stories are ‘if-then’ stories that quantify and predict unpredictable quantities by using assumed numbers and formulas; and that need to be supplied with scenarios with alternative assumptions. Fake stories are ‘what-then’ stories that quantify and predict unpredictable qualities by using fake numbers and formulas; and that need to be replaced by word stories (Tarp, 2001).

Discussion and conclusion

What should name a mathematical concept? Its outside root, or its inside relation to other concepts on a lower or higher abstraction level? Should a function be named a ‘sentence’ using a verb to link an outside subject to an inside calculation? Or a ‘standby calculation’ with specified and unspecified numbers? Or an example of a ‘many-to-one set relation’? In fact, what we here ask is: The goal of math education, is that to learn to master math to later master Many, or the other way around? Holding that existence precedes essence, existentialist philosophy (Sartre, 2007) prefers the latter. And we see that to master Many to later master mathematics by re-rooting often implies a different name and order of de-rooted concepts. Since, to bring outside Many inside, counting and recounting precedes addition, and 2D flexible bundle-numbers with units replace the traditional 1D line-numbers without, and recounting to change units leads to proportionality, $T = (T/B)xB$, to per-numbers with different physical units, and to fractions with like units. Likewise, recounting between icons and tens leads to equations solved in the original way by moving to ‘opposite side with opposite sign’, and to early algebra when 6 7s is rewritten as (B-4)*(B-3). And, recounting rectangular stacks as squares roots the square root, and to quadratic equations. And in a stack halved by its diagonal, mutual recounting between the sides roots trigonometry thus preceding both plane and coordinate geometry. And once counted and recounted, stacks may be added in two ways, on-top, after recounting has made the units like, or next-to as areas thus rooting integral calculus, and differential when the question is reversed. And, since per-numbers must be multiplied to unit-numbers before adding they also add by areas as integral calculus, facilitated by differential calculus trying to rewrite area-strips as differences so the sum of many differences becomes one difference between the end and start values.

So, demodeling and re-rooting inside de-rooted concepts may ease modeling: Now you don’t first learn about inside essence but learn math directly by manipulating and communicating about (Widdowson, 1978) outside existence, e.g., things and actions on a ten-by-ten Bundle-Bundle-Board (Tarp, 2023).

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A Text-Free Math Education Found by Difference Research for Protection Against Alien Artificial Intelligence

Artificial Intelligence, AI, friend or foe to math education? Some warn that AI develops into an alien intelligence infiltrating all that is text-bound in a library. So, to protect math education from this, we ask if math may be taught and learned in a text-free form out of reach to AI. Having not yet met math in its text-bound form, 3year old children give the answer by using bundle-numbers with units as 2 2s thus seeing what exists in time and space. This discovery allows difference research to use sociological imagination to design text-free curricula giving priority to outside existence over inside essence, and to use flexible bundle-numbers with units in tales about things and actions on a Bundle-Bundle-Board.

Protecting Math and its Education from being Infiltrated by AI

It seems only natural that mathematics is a core subject in education because of its many important applications in core societal matters within economy, science, technology, engineering, etc. So, all we need are universities to define and develop mathematics, and to teach teachers how to teach it to students in mathematics classes that it may later be applied in other classes. It is as simple as that. And of course, it goes without saying that first mathematics must be learned to be applied later.

However, a core part of mathematics is geometry, in Greek meaning to measure earth. As well as algebra, in Arabic meaning to reunite changing and constant unit- and per-numbers with addition, multiplication, integration and power. Both thus indicate that inside mathematics has outside roots. So instead of teaching the abstract before its concrete roots, maybe it should be the other way around as suggested by existentialist philosophy (Sartre, 2007) holding that outside existence precedes inside essence that might be power-charged by being not natural but socially constructed (Foucault, 1972)?

Peer-reviewed research may give an answer. But can it be trusted, John Bohannon asked in his 2013 article "Who's Afraid of Peer Review?". And if nobody teaches existence before essence, then peer-review might reject all articles about this arguing they don't discuss or extend established knowledge. So 'difference research' searching for differences making a difference (Tarp, 2018) typically has its papers rejected at conferences' peer-reviews performed by the other contributors. Until now where AI, Artificial Intelligence, with its access to the library may write research articles also in huge numbers. In May 2023, 350 leading scientists and notable figures signed a common statement warning against AI by saying that "Mitigating the risk of extinction from AI should be a global priority alongside other societal-scale risks such as pandemics and nuclear war". A similar warning is found on the YouTube video "AI and the future of humanity" given by Yuval Harari at the Frontiers Forum, May 2023.

To protect math education from infiltration by an alien intelligence seems almost impossible since both mathematics and education are text-bound. So, we may ask: "Can mathematics be taught and learned in a different text-free version?" Let us see what difference research may offer here.

To look for a different version we listen to brains that have not yet been exposed to books, young preschool children. So, we ask a 3year old child "How many years next time?" Typically, the answer is four showing four fingers. But presented by four fingers held together two by two, the child protests: "That is not four, that is two twos". The child thus sees what exists in space and time: Bundles of twos in space, and two of them when counted in time. These rectangular bundle-numbers with units are different from the textbook's linear number-line numbers without units.

Based on this discovery, difference research now uses sociological imagination (Mills, 1959) to design text-free curricula giving priority to outside existence over inside essence, and using bundle-numbers.

Designing micro curricula

Looking at our 5 fingers we observe that bundle-numbers may be flexible when bundle-counting. If we shorten 'Total' to T and 'Bundle' to B we have: T = 0B 5 = 1B 3 = 2B 1 = 3B -1 2s, or T = 1BB 0B 1 = 1B^2 0B 1 = 101 2s if we leave out the units. Here 1B 3 may be called an overload, and 3B -1 may be called an underload. Counting all ten fingers, we get T = 2BB 0B 2 = 1BBB 0BB 1B 0 =

1010 2s. Counting them in 3s, we get T = 3B 1 = 1BB 0B 1 = 101 3s. We notice that with units, the place value system becomes redundant, and that power is the first operation we meet.

Flexible bundle-numbers may also be used with ten as bundle-size: T = 68 = 6B 8 = 5B 18 = 7B -2 tens. This eases standard operations and makes also carrying and borrowing redundant:

T = 23+59 = 2B 3+5B 9 = 7B 12 = 8B 2 = 82; and T = 83 – 59 = 8B 3 – 5B 9 = 3B -6 = 2B 4 = 24

T = 3*59 = 3*5B 9 = 15B 27 = 17B 7 = 177; and T = 84 /3= 8B 4 /3 = 6B 24 / 3 = 2B 8 = 28

Here we meet Many in space. In time we include the unit in the counting sequence: 0B 1, 0B 2, …, 0B 9, 0B ten or 1B 0, 1B 1 etc., enjoying that 'eleven' comes from the Vikings saying '1 left, 2 left'.

We now look at the counting process by asking "How many 2s in 8?". To answer, first we push-away the 2s, which allows division to be iconized as a broom, 8/2. Then 4 times we stack the 2s, which allows multiplication to be iconized as a lift, 4x2. We may now write the result as a 'recount-formula":

8 = 4 2s = 8/2 2s = (8/2) x 2, or $T = (T/B)$ x B with unspecified numbers.

So, with bundle-counting changing the units from 1s to bundles we get the proportionality formula directly. Also, we meet a formula or function as a number-language sentence with an outside subject, a verb, and an inside predicate as in word language sentences. Also we meet solving equations since our question could be reformulated as ux2 = 8 where recounting 8 in 2s gives 8 = (8/2)x2. The equation thus is solved by u = 8/2, i.e., by 'moving to opposite side with opposite sign'. Which also follows from the formal definition saying that "8/2 is the number u that multiplied with 2 gives 8, ux2 = 8".

Likewise, the equation u+2 = 5 is solved by moving over as u = 5-2 since u is a placeholder for a number that with 2 added gives 5, thus found by reversing the action and pulling-away the 2 again.

Solving equations by 'opposite side & sign' is a difference to the traditional balancing method 'do the same to both sides' introduced to motivate teaching teachers the abstract algebra concept 'group'.

When bundle-counting, we also meet decimals, fractions, and negative numbers to account for the unbundled singles: First we pull-away the stack which allows subtraction to be iconized as a rope, e.g., 9 – 4x2 = 1. Then we place the unbundled on-top of the stack, as a decimal number, 9 = 4B1 2s, or as a fraction when counted in 2s also as 1 = (1/2) x 2, 9 = 4 ½ 2s, or as a negative number showing in space what is missing for the next bundle, or what have been pulled away from it in time,

9 = 5B -1.

Above we saw that recounting from tens to icons solves equations: u*6 = 30 = (30/6)*6, so u = 30/6.

Recounting from icons to tens gives multiplication tables that may be seen on a ten-by-ten Bundle-Bundle-Board, a BBBoard where 6*7 may be seen as 6 7s or as '(B-4)*(B-3) which leads to early algebra since the 6 7s are left when from the ten Bs we pull-away 4Bs and 3Bs, and then add the 4 3s we pulled-away twice:

T = 6 7s = 6*7 = (B-4)*(B-3) = 10B - 3B - 4B + 4 3s = 3B + 1B 2 = 4B 2 = 42.

Inside, outside totals become rectangular stacks as T = 8 4s = 8*4, or squares in the case of bundle-bundles T = 4 4s = 4*4. So, we may ask "How to square a rectangle?", e.g., T = 8 4s = B*B = B^2 where B is called the square root of 32, B = √32, iconized by half a perimeter. We begin by adding half of the excess, ½*(8-4) 4s= 2 4s to both sides of the 4x4 square, which gives a 6x6 square with a total of 36. This is too much since also the upper right corner must be included. So instead we add a number t determined by (4+t)^2 = 32. On a drawing we see that the square (4+t)^2 has four parts, 4^2 and t^2 and 2*4*t, so (4+t)^2 = 4^2 + t^2 + 8*t = 32, or t^2 + 8*t + 16 = 32 , or t^2 + 8*t – 16 = 0, called a quadratic now rooted in transforming a 32-rectangle into a '4+t square'.

Likewise, we may rewrite the quadratic t^2 + b*t + c = 0 as t^2 + 2*b/2*t + (b/2)^2 = (b/2)^2 – c = D/4, or as (t+ b/2)^2 = D/4 where D is called a discriminant.

This shows that a $D/4$ rectangle may be transformed into a $b/2+t$ square thus providing the solution to the quadratic as $t = -b/2 \pm\sqrt{(D/4)}$.

Changing the unit, the recount-formula roots the proportionality formula $T = a*b$ recounting T in bs. Examples may be meter = meter/sec*sec, recounting a distance in seconds, or $ = $/kg*kg, recounting dollars in weight, thus creating 'per-numbers' as meter/sec, $/kg, etc. Or part = part/whole * whole, recounting a part in wholes and becoming fractions with like units. In time, the end value may be recounted in the start-value: end = end/start * start, where end/start is the change-factor, e.g., 105%.

Finally, in a rectangle with a base, b, a height, h, and a diagonal, d, mutual recounting roots trigonometry as per-numbers: $h = (h/b)*b$ = tangent(Angle)*b where tan(A) is the per-number h/b. Splitting the circumference of half a unit-circle in n parts gives the number

$\pi = n*\tan(180/n)$ for n big.

Once counted, stacks may be added, on-top after recounting has provided like units, or next-to as areas thus rooting integral calculus, as well as differential calculus when reversing asks

"2 3s + ?5s = 4 8s".

Per-numbers add in mixture problems as "2kg at 3$/kg plus 4kg at 5$/kg gives what?". With like units, the unit-numbers 2 and 4 add directly. But per-numbers must be multiplied to unit-numbers before adding as the areas created by the multiplication. So, mixture problems root integral calculus, becoming differential calculus when the problem is reversed. So, integral calculus should be introduced before its inverse differential calculus show that many differences add as one difference.

Fact, fiction & fake, the 3 modeling genres

With mathematics as a number-language for outside things in space and actions in time, its quantitative literature or models needs to be divided into fact, fiction or fake, the same genres used in the word-language for qualitative literature. Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas; and that need to be checked for units and correctness. Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas; and that need to be supplied with scenarios with alternative assumptions. Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas; and that need to be replaced by word stories (Tarp, 2001).

Conclusion, yes we can

So, the answer to our question is: Yes, mathematics may be taught and learned in a version free from text and AI if we follow the advice of existentialist philosophy and let outside text-free existence precede inside text-bound essence; and use children's own flexible bundle-numbers with units instead of the textbook's line-numbers without units (Tarp, 2020-2023). This allows a communicative turn in number-language as the one that took place in the word-language education in the 1970s (Widdowson, 1978). Core mathematics will then be learned automatically as tales about Many on a Bundle-Bundle-Board. As to teacher education and research, the MATHeCADEMY.net offers a corresponding free online teacher education using Bundle-numbers with units where learning takes place through guided activities that allow questions to be answered by the subject in the laboratory instead of by an instructor in a library. The academy also contains many articles showing the possibilities of leaning to master Many before mathematics. They propose that studies of the discovered differences are carried out as design research (Bakker, 2018). And studies on the ability to bring back brains from special education is especially needed in a subject that has deprived children of their own bundle-numbers with units.

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Decolonizing mathematics when demodeling it by using the child's uncolonized 2D bundle-numbers with units

Workshop Leader Allan Tarp, MATHeCADEMY.net, Denmark, Allan.Tarp@gmail.com

The names of any other people who will assist with running the Workshop None

The name of the Workshop as it should appear in the ICME-15 program:

Decolonizing mathematics when demodeling it by using the child's natural 2D bundle-numbers with units.

The aim of the Workshop

The workshop aims at experiencing how 'existence precedes essence', the slogan of existentialist philosophy, may create a different 'counting precedes adding' mathematics that uses the child's natural bundle-numbers with units occurring when a 3year-old reacts to four fingers held together two by two by saying "That is not four, that is two twos". The child thus sees what exists, bundles of twos in space that serve as units when later counted in time. An 'existence-based unit-number math' may perhaps serve as a decolonization of the traditional 'essence-based non-unit math'. Furthermore, it is outside the reach of AI by being text-free since, with units, numbers as 2 3s become physical rectangles on a ten-by-ten Bundle-Bundle-Board, a 'BBBoard'.

The activities that will run

The child's natural 2D bundle-numbers with units makes linearity and calculus enter at once in grade one. Linearity enters when asking "2 3s is how many 5s?". This recounting will change the form but not the area of the total. Calculus enters when next-to addition of totals may ask "2 3s and 4 5s total how many 8s?"

In both cases the answer may be predicted by a 'recount-formula', Total = (Total/Bundle)xBundle, or T = (T/B)xB, exemplified by 8 = (8/2)x2 saying that "the number of 2s in 8 is 8/2". Digits have already entered as icons with as many strokes as they represent, and now also division and multiplication enter as icons for a broom and a lift pushing-away and stacking bundles. Likewise, subtraction enters as an icon for a rope to pull-away the stack to find unbundled that are placed on-top of the stack as decimals, fractions, negatives ('underloads'), or 'overloads', e.g., 9 = 4B1 = 4½ = 5B-1 = 3B3 2s; and 48 = 4B8 = 5B-2 = 3B18 tens.

Recounting from tens to digits may ask "How many 7s in 42?". This leads to the equation "*u*x7 = 42" where recounting 42 in 7s as 42 = (42/7)x7 gives the answer *u* = 42/7, found by moving 'to opposite side with opposite sign'. As when restacking T = (T–B)+B says that "pulled-away from T, B is placed next-to T–B".

Recounting from digits to tens may ask "6 7s is how many tens?". This leads to early algebra when shown on the BBBoard as (B-4)x(B-3) which is left after pulling-away 3 right- and 4 top-bundles, and adding the upper right 4 3s removed twice, so 6x7 = (B-4)x(B-3) = BB-3B-4B+4x3 (FOIL). Here, minus x minus gives plus.

Recounting ten fingers in 3s leads to bundle-bundles with 9 as 3 bundles, or 1 bundle-bundle, which makes ten to 1BB0B1 or 1(B^2)0B1 3s. Or to 1(B^3)0(B^2)1B0 2s. Bundle-bundles thus leads to squares growing by adding an extra top and side and a corner so that 6^2 is 5^2+2x5+1. And to square roots almost found by moving half the extra top to the side, e.g., 6 4s = (6-1)x(4+1) = 5x5 almost. And to quadratics where a (*u*+3) square contains two squares, *u*^2 and 3^2, and two stacks, 2x(3x*u*). This means that with *u*^2+6*u*+8 = 0, only 9-8 = 1 is left for the (*u*+3) square, which gives pull-away 2 and pull-away 4, or -2 and -4, as solutions.

Recounting meters in seconds gives a 'per-number' as 4m/5sec or 4/5 m/sec that bridges the units by recounting in the per-number: 20m = (20/4)x4m = (20/4)x5sec; and 20sec = (20/5)x5sec = (20/5)x4m. With the same units, per-numbers become fractions: 3m/4m = ¾; and 3m/100m = 3/100 = 3%.

The 3 4s stack has a base B, and a height H, and a diagonal D. Mutually recounted, the three leads to trigonometry, e.g., H = (H/B)xB = tangent (Angle) x B, where tan(A) = H/B. In a low stack the height is almost the same as the perimeter in a circle with its center at the other end of the base. So pi, the full circle's half perimeter counted in the base, is almost *n* x tan(180/*n*) with *n* as a very high number.

There is a 4x4 square inside a 6x4 stack where a new unit, *k*, makes the upper right corner shrink along the diagonal to become a corner in a 6*k* x 4*k* stack. When it meets the square, 6x*k* is 4 = (4/6)x6, so *k* = 4/6, which changes 4*k* into 4x4/6 or 16/6. So now it is a corner in a 4x16/6 stack. Later it meets the square's falling diagonal where the height 6*k* now is 4 – 4*k*, so that 4 is 10*k*, or *k* = 4/10. This changes 6*k* and 4*k* into 24/10 and 16/10. It is a corner in a 2.4x1.6 stack. A BBBoard or a squared paper confirms these predictions.

Once counted and recounted totals may add, but on-top or next-to? Adding 2 3s and 4 5s on-top, recounting first must make the units like. Adding 2 3s and 4 5s next-to as 8s means adding areas as in integral calculus that becomes differential calculus if reversing the question to "2 3s and how many 5s gives 3 8s?"

Mixture problems exemplify adding per-numbers or fractions: To add 2kg at 3$/kg and 4kg at 5$/kg, the unit numbers 2 and 4 add directly whereas, before adding as $, the per-numbers 3 and 5 must be multiplied which creates areas. So, per-numbers add by the area under the per-number curve, i.e., as integral calculus.

Adding 5% to 200$ can take place by adding 5% to the 200 or to the unit to create a new unit, 105%x$. Adding 5% 7 times thus changes the unit from $ to 105%^7$. Constantly adding the same percentage thus leads to power, or if reversed, to the factor-finding root, or to the factor-counting logarithm.

In Arabic, algebra means to reunite numbers. So, we now have an 'Algebra square' as a number-language to reunite the world's four different number-types: multiplication and addition unite like and unlike unit-numbers, and power and integration unite like and unlike per-numbers. And splitting is performed by the reverse operations: division and subtraction, and root or log and differentiation.

Now it is time to use the number-language to create quantitative tales, reports, and literature. The first two tales are about space and time. In space, the Greek word geometry means to measure earth. In time, pre-calculus is about change where the change-number and -percent may be constant or constantly changing, or predictable with a change formula, or unpredictable without thus leading to statistics and probability.

With a number-language describing outside things in space and actions in time, its literature or models needs to be separated in fact, fiction and fake, the same genres that exist when using the word-language to produce qualitative literature. Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas. They need to be checked for correctness and units. Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. They need to be supplied with scenarios building on alternative assumptions. Fake or fiddle stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas. They need to be replaced by word stories.

Any other specific requirements i.e. multimedia facilities needed to run the activities.

Participants could bring matches, snap cubes, and a pegboard with rubber bands.

Workshops will have a professional focus, not a commercial one.

There is no commercial focus. Professionally it connects to our discussion group proposal "Decolonizing mathematics, can that secure numeracy for all, and be protected from AI?" as well as to my two papers/ posters in Topic Study Groups 3.4 and 5.10, "Modeling Eased by Demodeling and Rerooting", and "A Text-Free Math Education Found by Difference Research for Protection Against Alien Artificial Intelligence."

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Decolonizing mathematics, can that secure numeracy for all, and be protected from AI?

A discussion group proposed by Shuhua An at California State University Long Beach, and Le Trung at Ho Chi Minh City University of Education, and Allan Tarp at MATHeCADEMY.net, Denmark

The purpose

This group will discuss the question: ‘Quality Education’, the fourth of the 17 UN Sustainable Development Goals, has as a goal target to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”. If decolonizing mathematics is relevant in this connection, how could it take place, and how to protect it from AI?”

Introduction

“That is not four. That is two twos.” Said a 3year-old when asked “How many years next time?” and seeing 4 fingers 2 by 2. Which indicates that children have their own number-language before they are asked to shift to the school’s version. The child sees what exists, bundles of twos in space serving as units when counted in time. And as in the word-language, the child’s number-language also uses a full sentence with an outside existing subject, a linking verb, and an inside predicate.

The school thus could help children to further develop their own number-language that uses two-dimensional bundle-numbers with units where multiplication always holds by simply changing the unit, e.g., from 4s to tens where 3 x 4 = 12 states that 3 4s may be recounted in tens as 1.2 tens.

So, by what right and how ethical is it when the school imposes upon the children its own one-dimensional non-unit numbers where addition only holds without units inside the school but seldom with units outside the school where 2+1 = 3 is often falsified, e.g., by 2 days + 1week = 9 days?

To separate reliable ‘multiplication-math’ from unreliable ‘addition-math’ the latter should maybe be called ‘mathematism’, true inside but seldom outside school. But then, why teach addition of non-unit numbers inside when students outside need to add numbers with units?

We therefore could ask: To impose unreliable addition of one-dimensional non-unit numbers upon students that use multiplication in their two-dimensional unit numbers, isn’t that an example of “a colonization of the life world by the system”, the key concept in the sociology of Jürgen Habermas?

If so, then we should ask if demodeling could be used to bring inside concepts back once more to its outside roots in order to decolonize mathematics and its education to meet the fourth of the 17 UN Sustainable Development Goals that is called ‘Quality Education’ aiming at “ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all”. And having as a goal target to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”. Decolonization will not be easy as seen by different definitions of ‘numerate’. The English Oxford Dictionary defines it as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary defines it as “having the ability to understand and work with numbers.”

In their common history, England once colonized America. So, the difference in the definitions is interesting. The former uses the passive term ‘being’ where the latter uses the active term ‘having’. The former only includes the inside institutionalized essence that has already colonized numeracy, mathematics, and arithmetic. While the latter connects the definition directly to the outside existence of numbers. So, a search for ways to decolonize should use only the methods coming from the philosophy, sociology, and psychology of the colonized, i.e., from American pragmatism and symbolic interactionism, from Piagetian psychology, and from existentialist philosophy holding that existence should precede essence to prevent the latter from colonizing the former.

One example of a decolonized mathematics education that respects the children’s bundle-numbers with units may be found in the article “Mastering Many by counting, re-counting and double-counting before adding on-top and next-to.” Using postmodern deconstruction, this article shows that a ‘counting before adding’ approach leads to the same concepts as a traditional approach but with different identities, and in a different order. Also, working with 2D multiplication-numbers on a ten-by-ten ‘Bundle-Bundle-Board’ will allow learning to be text-free out of the reach of AI.

Counting in 3s leads to 9 as a bundle-bundle, a B^2, which leads on to squares, square roots, and quadratics. Counting transforms the operations into icons where division and multiplication become a broom and a lift that pushes-away bundles to be stacked later as shown when recounting 8 in 2s as 8 = (8/2) x 2, or with T and B for Total and Bundle, T = (T/B) x B, that creates per-numbers when recounting in physical units, $ = ($/kg) x kg. Subtraction becomes a rope that pulls-away the stack to find the unbundled that placed on-top of the stack as part of an extra bundle become decimals, fractions, or negatives, e.g., 9 = 4B1 = 4½ = 5B-1 2s.

Finally, addition becomes a cross showing the two ways to add stacks, on-top using the linearity of recounting to make the units like, or next-to creating integral calculus by adding areas, that is also used when adding per-numbers needing to be multiplied to areas before adding. All this provides an 'Algebra Square' showing how to unite the four types of existing numbers: multiplication and addition unite like and unlike unit-numbers, and power and integration unite like and unlike per-numbers. And how to split totals with the opposite operations: division and subtraction, together with root or logarithm and differentiation.

Reuniting like and unlike unit- and per-numbers is "ability to understand and work with numbers" to produce quantitative tales, reports, and literature; and to discuss to which genre they belong, fact or fiction or fake. Which will allow a communicative turn in the number-language as the one that took place in foreign language education in the 1970s allowing all to use the English language without first knowing its abstract grammar. Which again may create a world where numeracy is no longer a privilege of an elite colonizing the number-language with unreliable mathematism.

Some questions

With a 'decolonized' and a 'colonized' version of math education we ask: What are advantages and shortcomings of the two approaches? Do both "ensure inclusive and equitable quality education and promote lifelong learning opportunities for all"? Can both be protected from Artificial Intelligence?

Call for papers

The discussion group invites participants to submit a four-page paper or essay before May 15 using the conference template. The papers will all be accessible on the MATHeCADEMY.net website. Some will be chosen for a five-minute presentation followed by a ten-minute discussion.

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Abstracts on the ICME15 website

Discussion group

This group will discuss the question: 'Quality Education', the fourth of the 17 UN Sustainable Development Goals, has as a goal target to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". If decolonizing mathematics is relevant for this, how could it take place, and how to protect it from AI?"

"That is not four! That is two twos." Said a 3year-old when asked "How many years next time?" and seeing 4 fingers held together 2 by 2. This indicates that children have a different uncolonized number-language before it is colonized by the school's version. The child sees what exists, bundles of twos in space serving as units when counted in time. The school thus could help children to further develop their original number-language that uses two-dimensional bundle-numbers with units where multiplication always holds by simply changing the unit, e.g., from 4s to tens where 3 x 4 = 12 states that 3 4s may be recounted in tens as 1.2 tens.

So, by what right and how ethical is it when the system imposes upon the children its essence with one-dimensional non-unit numbers where addition without units holds only inside the school but seldom outside where 2+1 = 3 may be falsified with units: 2 days + 1week = 9 days? In sociology, Habermas calls this "a colonization of the life world by the system". May 'existence precedes essence' be a text-free decolonization of mathematics protected from AI?

This discussion group is coordinated with the workshop "Decolonizing mathematics when demodeling it by using the child's uncolonized 2D bundle-numbers with units".

Workshop

The workshop aims at experiencing how 'existence precedes essence', the slogan of existentialist philosophy, may create a different 'counting precedes adding' mathematics built from the child's uncolonized number-language using the bundle-numbers with units that occur when a 3year-old reacts to four fingers held together two by two by saying "That is not four, that is two twos!" The child thus sees what exists, bundles of twos in space that serve as units when later counted in time. An 'existence-based unit-number mathematics' may perhaps serve as a decolonization of the traditional 'essence-based non-unit mathematics'. Furthermore, it is outside the reach of AI by being text-free since, with units, numbers as 2 3s become physical rectangles on a ten-by-ten Bundle-Bundle-Board.

In a 'essence after existence' de-modeling, linearity and calculus enters immediately when asking "2 3s is how many 5s?", and "2 3s and 4 5s total how many 8s?" Both answers is predicted by a 'recount-formula', Total = (Total/Bundle)xBundle, or T = (T/B)xB, exemplified by 8 = (8/2)x2 saying that "the number of 2s in 8 is 8 push-away 2". Digits have already entered as icons with as many strokes as they represent, and now division and multiplication enter as icons for a broom and a lift pushing-away and stacking bundles. Likewise, subtraction enters as an icon for a rope to pull-away the stack to find unbundled that are placed on-top of the stack as decimals, fractions, negatives ('underloads'), or 'overloads', e.g., 9 = 4B1 = 4½ = 5B-1 = 3B3 2s.

This workshop is coordinated with the discussion group " Decolonizing mathematics, can that secure numeracy for all, and be protected from AI?".

ICME-15
7-14 July 2024 • ICC Sydney, Australia
Come and be counted

TSG 3.4: Mathematical applications and modelling in mathematics education

Allan.Tarp@gmail.com , MATHeCADEMY.net

De-colonizing mathematics by de-modeling & re-rooting

De-modeled & re-rooted, **de-rooted math** may finally change from '**mathematism**' and its no-unit regime to **Many-math**, a natural science about Many in time and space, where **counting precedes adding**, to let **existence precede essence** as philosophical **existentialism** holds.

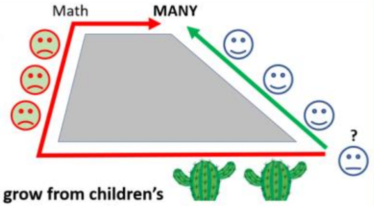
So, let **mastery of Many** precede **mastery of math**

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

The Goal of Math Education, is it well Defined?
What comes first: to master Math - or to master Many?

All say:
The goal is to master Math,
to later master Many.
But Math is hard!
Why not first master Many,
to later master Math?

So, we may ask:



What Mathematics may grow from children's innate mastery of Many, as developed before school?

Findings:

"That is not four, that is two twos". Said a 3year old child when asked "How many years next time?" And seeing 4 fingers held together 2 by 2.

So, un-colonized by the 1D number **no-unit essence-regime**, children describes existence by 2D **bundle-numbers with units**.
A **curriculum** built on **existence before essence** and **counting before adding**, will lead directly to the core of mathematics:
• digits & operations as **icons** • **re-counting** to shift units and to solve equations • fractions and **trigonometry** as **per-numbers**
• **add on-top** after recounting gives like units • **add next-to** by **calculus**, also adding **piecewise & locally constant per-numbers.**

There are two numbers-types in the world: UNIT-numbers & PER-numbers
which may be unlike or like, & which may be united or split

The aim of math education therefore is not to 'learn to math', because math is not an action verb, but to actively act to:	**Re-unite unlike & like UNIT-numbers & PER-numbers**

• 3\$ and 2\$ are unlike unit-numbers where the calculation 3+2 = 5 predicts the result of uniting them.
• 3 times 2\$ are like unit-numbers where the calculation 3*2 = 6 predicts the result of uniting them.
• 3 times 2% are like per-number where the calculation 102%^3 = 106.12% predicts the result of uniting to 6% and 0.12% extra.
• Unlike per-numbers as mixture: 2kg at 3\$/kg and 4kg at 5\$/kg. Here, the unit-numbers 2 and 4 add directly while the per-numbers 3 and 5 must first be multiplied to unit-numbers before adding as areas, called integration, where multiplication precedes plus: T = (2+4)kg to (2*3+4*5)\$, i.e., 6kg á 26/6\$/kg. In Aarabic, 'algebra' means to re-unite.

Unite / *split into*	Unlike	Like
UNIT-numbers (meter, second)	**T = a + b** *T – b = a*	**T = a*b** *T/b = a*
PER-numbers (m/sec, m/100m = %)	**T = ∫ f dx** *dT/dx = f*	**T = a^b** *b√T = a* *loga(T) = b*

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• Tarp, A. (2021). *Flexible Bundle Numbers Develop the Childs Innate Mastery of Many*, youtu.be/z_FM3Mm5RmE.

TSG 3.4: Mathematical applications and modelling in mathematics education

Allan.Tarp@gmail.com , MATHeCADEMY.net

What should name a mathematical concept, *what exemplifies it OUTSIDE - or from what it exemplifies INSIDE?*

The goal of math education, is that to learn to master math to later master Many, or the other way around?

- Traditionally, the goal of math education is seen as learning to master math to later master Many. So, a difference could be to see the goal of math education as learning to master Many directly to indirectly learning math on the way, at least the math core as displayed on a calculator: digits, operations, and equations.
- Traditionally, these all occur as **products** in space, so a difference could be to see them as **processes** in time by letting **outside-Many precede inside-math**. And the math core is different as **tales about Many** existing as rectangular totals of bundle-stacks on a plastic ten-by-ten bundle-bundle board, a **BBBoard**.
- To see if a 'process-based' 'Many-first' education will make a difference to the traditional 'product-based' 'Math-first' education, **micro-curricula** are designed using **flexible bundle-counting** to bring outside totals inside as **flexible bundle-numbers with units**, that are rectangular where the bundle-bundles are squares.
- Here both **digits and operations are icons**. Digits when uniting sticks. And operations with division to **push-away** bundles that multiplication **lifts** into a stack. Subtraction **pulls-away** stacks so unbundled are included as decimals, fractions, or negatives. The addition-cross shows the **two ways to add**, next-to & on-top.
- Once counted, **changing unit** may be predicted on a calculator by **the recount formula T = (T/B) x B**, saying that the total T contains T/B Bundles. Here recounting from tens to icons and vice versa leads to **equations**, and to **multiplication tables** existing as the stack left when removing the 2 surplus stacks from the full bundle-bundle on a BBBoard. And here recounting a rectangle as a square introduces its side as the **square root**, and a way to solve quadratics.
- Here recounting in two physical units leads to **per-numbers** as 4$/5kg bridging the two units; and becoming fractions with like units, 4$/5$ = 4/5, 4$/100$ = 4%.
- Here mutual recounting the sides and the diagonal in a stack leads to **trigonometry before geometry**.
- Once counted, totals may **add on-top** after recounting makes the units like or **add next-to** as areas as integral calculus becoming differential calculus if reversed.
- **Per-numbers and fractions are operators** needing numbers to become numbers, so also adding by their areas after being multiplied to unit-numbers to add.
- So, outside totals inside appear in an '**Algebra Square**' where unlike and like unit-numbers and per-numbers are united by addition and multiplication, and by integration and power. And later again split by the reverse operations: subtraction and division, and by differentiation and factor-finding root or -counting logarithm.
- Once process-based Many-first Many-math micro curricula have been designed, they are **tested** in online education, as well as **in special education** to see if **BBBoards 'Bring Back Brains'** excluded from the 'Math-first' education.

Details: A. Tarp. "*ManyMath 2030: Decolonizing 1D Mathematism into 2D Many-math*", http://mathecademy.net/manymath-2030/

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2 Colonizations: Bundle-numbers by Line-numbers, then by abstract SETS

De-colonization by de-modeling & re-rooting & de-construction will change

	ESSENCE-math, Mathematism	into EXISTENCE-math, ManyMath
Digits	Symbols	Icons
345	Place value system	T = 3**BB** 4**B** 5, **BB** = **B**^2, **BBB** = **B**^3
Operations	Functions, order: + - x / ^	Icons, opposite order: ^ / x - +
3 + 4	3 + 4 = 7	Meaningless with no units
3 * 4	3 * 4 = 12	3 * 4 = 3 **4s** *that may be recounted to 1.2 tens*
9 = ? 2s	Meaningless, only ten-counting	9 = 3**B**3 = 5**B**-2 = 4**B**1 = 4½ **2s**
8 = ? 2s	Meaningless, only ten-counting	8 = (8/2)*2, **T = (T/B)*B**, proportionality
2*u = 8	(2*u)*½ = 8*½, (u*2)*½ = 4, u*(2*½) = 4, u*1 = 4, u = 4	2*u = 8 = (8/2)*2, så u = 8/2
6*7 = ?	eh, 44? eh, 52? eh, 42? OK	(**B**-4)*(**B**-3) = (10-4-3)**B**+12 = 3**B**12 = 4**B**2
4kg=5$, 6kg=?	1kg = 5/4$, 6kg = 5/4*6$	6kg = (6/4)*4kg = (6/4)*5$
1/2 + 2/3 = ?	1/2 + 2/3 = 3/6 + 4/6 = 7/6	1/2*2 + 2/3*3 = 3/5*5
2 3s + 4 5s	2*3+4*5 is 10*5 = 50, or 6+20= 26, decided by definition	2**B**0 **3s** + 4**B**0 **5s** = 3**B**2 **8s**, integration
6 + 9 = ?	6 + 9 = 15	1**B**0 **6s** + 1**B**3 **6s** = 1**B**-3 **9s** + 1**B**0 **9s** = 2**B**3 **6s** = 2**B**-3 **9s** = 1**B**5 **tens** = 15
Tangent = ?	Tangent = sine/cosine	raise = (raise/run)*run, tan = raise/run

2+3 = 5, seldom, since 2weeks + 3days = 17days. **Mathematism adds without units, true inside & maybe outside.**
2x3 = 6, always, since 2 3s = 6 1s. **ManyMath adds with units, true inside & outside.**

TSG 3.4: Mathematical applications and modelling in mathematics education — Allan.Tarp@gmail.com , MATHeCADEMY.net

18. The Swedish MADIF papers 2000-2020

Killer-equations, job threats and syntax errors, 2000.

Student-mathematics versus teacher-metamatics, 2002.

Mathematism and the irrelevance of the research industry, 2004.

The 12 math-blunders of killer-mathematics, 2006.

Mathematics: grounded enlightenment - or pastoral salvation, 2008.

Discourse protection in mathematics education, 2010.

Post-constructivism, 2012.

Golden learning opportunities in preschool, 2014.

Calculators and icon-counting and cup-writing in preschool and in special needs education, 2016.

Grounding conflicting theories, 2016.

The simplicity of mathematics designing a stem-based core mathematics curriculum for young male migrants, 2018.

Math competenc(i)es - catholic or protestant? 2018.

Sustainable adaption to quantity: from number sense to many sense, 2020.

Per-numbers connect fractions and proportionality and calculus and equations, 2020.

Sustainable adaption to double-quantity: from pre-calculus to per-number calculations, 2020.

A Lyotardian dissension to the early childhood consensus on numbers and operations, 2020.

Salon des refusés, a way to assure quality in the peer review caused replication crisis? 2020.

Introduction

Swedish school mathematics always fascinated me. Each second year Sweden arrange a Biennale where mathematics teachers from kindergarten to college can meet to share knowledge through exhibitions and inform themselves about new trends and ideas and listen to foreign or local researchers having met the day before at the MADIF conference, the Swedish Mathematics Education Research Seminar arranged by the Swedish Society for Research in Mathematics Education.

Furthermore, in 1999 the Swedish government decided to establish and gracefully fund a national resource centre for mathematics education, NCM, describing its task to 'co-ordinate, support, develop and implement the contributions which promote Swedish mathematics education from pre-school to university college'.

What a bright future for Swedish mathematics, I thought and decided to contribute with a paper at each MADIF conference and a general talk or an exhibition at each biennale.

My MADIF2 paper introduced postmodern counter research looking for hidden possible explanations for the problems in mathematics education within mathematics itself and warns against 'killer-Equations' and syntax errors. Furthermore, the paper suggests an alternative mathematics curriculum for the new millennium replacing the traditional Top-Down approach with a more user-friendly Bottom-Up approach. The paper was accepted for a full presentation.

However, I soon realized that it was almost impossible to establish a dialogue with the NCM and with Swedish researchers, so at the MADIF4 conference I presented a paper called 'Mathematism and the Irrelevance of the Research Industry' warning against supporting the irrelevance paradox in mathematics education research described by the following observation: 'the output of mathematics education research increases together with the problems it studies - indicating that the research in mathematics education is irrelevant to mathematics education'. The paper demonstrates how to avoid mixing up mathematics with mathematism, true in the library but seldom in the laboratory.

Although accepted for a full presentation, nothing happened afterwards, so in my MADIF5 paper I decided to be much more specific by warning against twelve blunders of mathematics education. The reaction to this paper was to reduce the presentation to a short communication.

In my MADIF6 paper I draw attention to the difference between North American enlightenment schools wanting as many as possible to learn as much as possible, and European counter-Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Bildung schools pastoral 'metamatism' descends from above as examples of metaphysical mystifying concepts.

The paper was rejected based upon a review process that allowed decisions to be made without specific reference to the paper reviewed.

So in my MADIF7 paper I warned against what I called 'Discourse Protection in Mathematics Education' and against reducing a constructive review process to what I called 'Moo Review' and 'Tabloid Review' using only one word or one sentence.

Again the paper was rejected.

One would expect the massive Swedish investment would show in the PISA scores. Here Sweden scored 502, 494, and 478 in the 2006, 2009 and 2012. Three consecutive numbers allow calculating the yearly change and the change to the change, which in the case of Sweden is -1.3 in 2006 changing yearly by -0.9 bringing the Swedish score to the zero level in 2038 if not changed.

At the same time research had demonstrated the positive effect of an early start in mathematics, so to be helpful to the Swedish research community I wrote a paper describing the golden learning opportunities in preschool accompanied by a YouTube video' Preschoolers learn Linearity & Integration by Icon-Counting & NextTo-Addition' (https://www.youtube.com/watch?v= R2PQJG3WSQY). The paper presents mathematics as natural science about the natural fact Many. To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

And again, again the paper was rejected, this time however without using moo- or tabloid-review.

In the PISA report Denmark scored 513, 503 and 500 giving an initial yearly change of -4.5 in 2006 changing yearly by 0.8 bringing the Danish score to 629 in 2030 if not changed.

However, Denmark has not significantly increased its research activity. So, the Danish success and the Swedish melt-down both indicate the correctness of the irrelevance paradox: More research creates more problems. Consequently, I suggested a two year no-research pause in Sweden. It was declined because researchers had found a new research paradigm, Design Research, they hoped would change the situation in a positive way.

Design Research bases its designs on existing theory. However, in conference presentations, disagreements between conflicting theories were simply ignored or denied. And not differentiating between grounded and ungrounded theory will hardly prevent the Swedish melt-down. So, to once more offer my assistance, instead of writing yet another paper that will be rejected yet again because of discourse protection, I have decided that my contribution to the MADIF 10 conference in 2016 should be a YouTube video similar to the Paul and Allan debate on postmodern mathematics education (https://www.youtube.com/watch?v=ArKY2y_ve_U), inspired by the Chomsky-Foucault debate on human nature (www.youtube.com/watch?v=3wfNl2L0Gf8), this time called 'Grounding Conflicting Theories to avoid the Irrelevance Paradox creating the Nordic Math Melt-Down - an invitation to a dialogue on Mathematics Education and its Research'. One prominent person within the research community has declined to take part in the dialogue, but hopefully other persons will accept their responsibility and be willing to enter into a fruitful dialogue to prevent the Swedish melt-down to become reality. Money does not solve the problem, dialogue between conflicting theories does.

The MADIF papers

For the MADIF 2 conference in 2000 I wrote the paper 'Killer-Equations, Job Threats and Syntax Errors, a Postmodern Search for Hidden Contingency in Mathematics.'

The abstract says that modern mathematics is facing an exodus problem: an increasing number of students are turning away from mathematics in school, and from math-based educations within science and engineering after school. Modern research looks for explanations within human factors: students, teachers and cultures. Postmodern counter research looks for hidden possible explanations elsewhere, in this case within mathematics itself. This study identifies unnoticed syntax errors within mathematics and a problematic Top-Down practice of allowing killer-equations into the classroom. Also, the study reports on a successful changing of this practice and reflects upon why a Bottom-Up approach might be more user-friendly than a Top-Down approach.

The paper contains chapters called: The Difference between Modern Research and Postmodern Counter Research, Killer-Equations in Paradise, Designing an Alternative: Rephrasing Equations, Practising the Alternative, Evaluating the Alternative, Why Might Bottom-Up Mathematics be More User-friendly? Why Might Bottom-Up Mathematics be Unrecognised? - Rephrasing Mathematics, Mixing Different Abstraction Levels Creates Syntax Errors , Abstraction Errors, Equations Can Also be Solved by Reverse Calculations, Bottom-Up Mathematics Education Through the Social Practices that Created Mathematics, The Social Practice of Bundling and Stacking, The Social Practices of Measuring Earth and Uniting Totals, When Will the logx Button be Included on Calculators?, The Social Practice of Building and Evaluating Models, Rephrasing Mathematical Concepts, Has Mathematics Become the God of Late Modernity?, and Fiction: "A New Curriculum for a New Millennium" - A Curriculum Architect Contest.

For the MADIF 3 conference in 2002 I wrote the paper 'Student-mathematics versus teacher-Metamatics'.

The abstract says that the paper reports on writer's career as an action researcher helping the students to develop their own student-mathematics, making mathematics accessible for all but being opposed by the educational system. The work took place over a 30 year-period in Danish calculus and pre-calculus classes and in Danish teacher education. As methodology a postmodern counter-research was developed accepting number-statements but being sceptical towards word-statements. Counter-research sees word-researchers as counsellors in a courtroom of correctness. The modern researcher is a counsellor for the prosecution trying to produce certainty by accusing things of being something, and the postmodern researcher is a counsellor for the defence trying to produce doubt by listening to witnesses, and by cross-examining to look for hidden differences that might make a difference. A micro-curriculum in student mathematics was developed and tested in 13 grade 11 classes showing a high degree of improvement in student performance.

The paper contains chapters called: A Confession, Methodology, the Case: Evidence and Cross-examination, and Concluding Statement.

However, I was not able to attend the conference, so instead the paper was presented at the ECER conference in 2003 and published at http://www.leeds.ac.uk/educol/documents/00003264.htm.

For the MADIF 4 conference in 2004 I wrote the paper 'Mathematism and the Irrelevance of the Research Industry, a Postmodern LIB-free LAB-based Approach to our Language of Prediction.

The abstract says that mathematics education research increases together with the problems it studies. This irrelevance-paradox can be solved by using a postmodern sceptical LAB-research to weed out LIB-based mathematism coming from the library in order to reconstruct a LAB-based mathematics coming from the laboratory. Replacing indoctrination in modern set-based mathematism with education in Kronecker-Russell multiplicity-based mathematics turns out to be a genuine ‘Cinderella-difference’ making a difference in the classroom.

The paper contains chapters called: The Irrelevance Paradox, A Methodology: Institutional Scepticism, Sceptical LIB-free LAB-Research, Mathematics and Mathematism, Fractions and Sets - LIB-words or LAB-words?, Bringing LAB-based Mathematics to a LIB-based Academy, The MATHeCADEMY and PYRAMIDeDUCATION, and Appendix: A Kronecker-Russell Multiplicity-Based Mathematics.

For the MADIF 5 conference in 2006 I wrote the paper ‘The 12 Math-Blunders of Killer-Mathematics, Hidden Choices Hiding a Natural Mathematics.

The abstract says that mathematics itself avoids blunders by being well defined and well proven. However, mathematics education fails its goal by making blunder after blunder at all levels from grade 1 to 12. This paper uses the techniques of natural learning and natural research to separate natural mathematics from killer-mathematics. Two-digit numbers, addition, fractions, balancing equations, and calculus are examples of mathematics that has been turned upside down creating the ‘metamatism’ that killed mathematics and turned natural Enlightenment mathematics into modern missionary set-salvation.

After the initial chapter ‘Taking the Killing out of Killer-Mathematics’ the paper describes twelve, Math-Blunders: Treating both Numbers and Letters as Symbols, 2digit Numbers before Decimal Numbers, Fractions before Decimals, Forgetting the Units, Addition before Division, Fractions before PerNumbers and Integration, Proportionality before DoubleCounting, Balancing instead of Backward Calculation, Killer Equations instead of Grounded Equations, Geometry before Trigonometry, Postponing Calculus; and the Five MetaBlunders of Mathematics Education.

For the MADIF 6 conference in 2008 I wrote the paper ‘Mathematics: Grounded Enlightenment - or Pastoral Salvation, Mathematics, a Natural Science for All - or a Humboldt Mystification for the Elite’.

The abstract says that mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral ‘metamatism’ descends from above as examples of metaphysical mystifying concepts. To make mathematics a human right, pastoral Humboldt counter-enlightenment must be replaced with democratic grounded enlightenment.

The paper contains chapters called: Postmodern Thinking - a Short Tour, French Enlightenment and German Counter-Enlightenment, American Enlightenment and Grounded Action Theory, Deconstructing and Grounding Research, Deconstructing and Grounding the Postmodern, Deconstructing and Grounding Numbers, Deconstructing and Grounding Operations, Deconstructing and Grounding the Mathematics Curriculum, A Grounded Perspective on Pastoral Mathematics, and The Humboldt Occupation of Europe.

For the MADIF 7 conference in 2010 I wrote the paper ‘Discourse Protection in Mathematics Education’.

The abstract says that social theory describes two kinds of social systems. One uses education to enlighten its people so it can practice democracy. One uses education to force upon people open or hidden patronization. A number-language is a central part of education. Two number-languages exist. Mathematics from-below is a physical science investigating the natural fact Many in a ‘manyology’ presenting its concepts as abstractions from examples. Mathematics from-above is a meta-physical science claiming Many to be an example of ‘metamatics’ presenting its concepts as examples from abstractions. Foucault’s discourse theory explains why manyology is suppressed and why even enlightening education patronizes by presenting mathematics from-above instead of from-below.

The paper contains chapters called: Investigating the natural fact many, the absence of a manyology, Social theory, Discourse Protection and Hegemony, Moo Review and Tabloid Review, and an appendix: the case of equations.

For the MADIF 8 conference in 2012 I wrote the paper 'Post-Constructivism'.

The abstract says that even if constructivism has been its major paradigm for several decades the relevance paradoxes in mathematics education remain; and furthermore constructivism has created a mathematics war between primary and secondary school, and between parents and teachers. Constructivism believes that numbers are meaningful and that algorithms are meaningless thus allowing students to construct their own algorithms. But maybe it is the other way around? Maybe a two-digit number is a highly abstract concept that, if not introduced slowly through cup-writing, may be meaningless to students; whereas algorithms introduced as internal trade between two neighbour cups is meaningful.

The paper contains chapters called: Constructivism, Numbers, Algorithms, Hermeneutics, Hermeneutic Research, Sceptical Cinderella Research.

However, I was not able to attend the conference, so the paper remains unpublished.

For the MADIF 9 conference in 2014 I wrote the paper 'Golden Learning Opportunities in Preschool'.

The abstract says that preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of the natural fact Many, we can ask: How will mathematics look like if built as a natural science about Many? To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

The paper contains chapters called: Math in Preschool – a Great Idea, Postmodern Contingency Research, Building a Science about the Natural Fact Many, Comparing Manyology and the Tradition, The Traditional Preschool Mathematics, Micro-Curricula at the MATHeCADEMY.net, Five plus Two Learning Steps, Designing a Micro-Curriculum so Michael Learns to Count, Observing and Reflecting on Lesson 1.

For the MADIF 10 conference in 2016 I wrote the paper 'Calculators and IconCounting and CupWriting in PreSchool and in Special Needs Education'.

The abstract says that to improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting in bundles less than ten implying recounting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, a calculator can predict recounting results before being carried out manually. By allowing overloads and negative numbers when recounting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary school only allowing ten-counting and on-top addition to take place.

The paper contains chapters called: Decreasing PISA Performance in spite of Increasing Research, Institutional Skepticism, Mathematics as Essence, Mathematics as Existence, Re-counting in the Same Unit and in a Different Unit, Reversing Adding On-top and Next-to, Primary Schools use Ten-counting only, Tested with a Special Needs Learner, Conclusion and Recommendation.

For the MADIF 10 conference in 2016 I also wrote the paper 'Grounding Conflicting Theories - an invitation to a dialogue to solve the Nordic Math MeltDown Paradox, a Manuscript to a Debate on Mathematics Education and its Research. However, it was not handed in.

The abstract says with heavy funding of mathematics education research brilliant results in the PISA scores are to be expected in the Nordic countries. So it is a paradox that all Nordic counties are facing a melt-down in their PISA scores in 30 years if nothing is changed; except for Denmark that has not increased it funding significantly. This was predicted by Tarp in his MADIF papers formulating an irrelevance paradox for mathematics education: more research leads to more problems when basing research on ungrounded theories and discourse protection and moo-review.

For the MADIF 11 conference in 2018 I wrote the paper 'The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants'.

The abstract says that educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Meeting Many, Children use Block-numbers to Count and Share, Meeting Many Creates a Count&Multiply&Add Curriculum, Meeting Many in a STEM Context, The Electrical circuit, an Example, Difference-research Differing from Critical and Postmodern Thinking, Conclusion and Recommendation,

For the MADIF 11 conference in 2018 I also wrote the paper 'Math Competenc(i)es - Catholic or Protestant?'

The abstract says that, introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Defining Mathematics Competencies, Discussing Mathematics Competencies, What kind of mathematics, What kind of Mastering, Competence versus Capital, The Counter KomMod report, Quantitative Competence, Proportionality, an Example of Different Quantitative Competences, Conclusion, Recommendation: Expand the Existing Quantitative Competence,

For the MADIF 12 conference in 2020 I wrote the paper 'Sustainable Adaption to Quantity: From Number Sense to Many Sense'.

The abstract says that their biological capacity to adapt to their environment makes children develop a number-language based upon two-dimensional box- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research; Mathematics and its Education; Biology Looks at Education; Psychology Looks at Education;

Meeting Many, Children Bundle to Count and Share; Discussing Number Sense and Number Nonsense; Conclusion and Recommendation.

For the MADIF 12 conference in 2020 I also wrote the paper ‘Per-numbers connect Fractions and Proportionality and Calculus and Equations.’

The abstract says that in middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word ‘percent’ to also talk about other ‘per-numbers’ as e.g. ‘per-five’ thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

The paper contains chapters called: Mathematics is Hard, or is it; The ICMT3 Conference; Different Ways of Seeing Fractions; Different Ways of Seeing Fractions; Ratios and Rates; Per-numbers Occur when Double-counting a Total in two Units; Fractions as Per-numbers Expanding and Shortening Fractions Taking Fractions of Fractions, the Per-number Way; Direct and Inverse Proportionality Adding Fractions, the Per-number Way; Solving Proportionality Equations by Recounting; Seven Ways to Solve the two Proportionality Questions; A Case: Peter, about to Peter Out of Teaching; Discussion and Recommendation.

For the MADIF 12 conference in 2020 I also wrote the paper ‘Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations.’

The abstract says that their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

The paper contains chapters called: A need for curricula for all students; A Traditional Precalculus Curriculum; A Different Precalculus Curriculum; Precalculus, building on or rebuilding?; Using Sociological Imagination to Create a Paradigm Shift; A Grounded Outside-Inside Fresh-start Precalculus from Scratch ; Solving Equations by Moving to Opposite Side with Opposite Sign; Recounting Grounds Proportionality; Double-counting Grounds Per-numbers and Fractions; The Change Formulas; Precalculus Deals with Uniting Constant Per-Numbers as Factors; Calculus Deals with Uniting Changing Per-Numbers as Areas; Modeling in Precalculus Exemplifies Quantitative Literature; A Literature Based Compendium; An Example of a Fresh/start Precalculus Curriculum; An Example of an Exam Question; Discussion and Conclusion.

For the MADIF 12 conference in 2020 I also wrote the workshop proposal ‘A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations.’

The workshop proposal contains chapters called Can Sociological Imagination Improve Mathematics Education; Time Table for the Workshop; Consensus and Dissension on Early Childhood Numbers & Operations.

For the MADIF 12 conference in 2020 I also wrote the workshop proposal ‘Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis?’

The workshop proposal contains chapters called Does Mathematics Education Research have an Irrelevance Paradox; The Replication Crisis in Science; Time Table for the Workshop.

19. The Swedish Mathematics Biennale

Proposals for the Mathematics Biennale 2016

All were rejected

Modern and postmodern mathematics

Modern mathematics presents concepts from above as examples of abstractions: a function is an example of a mediation relation. Postmodern mathematics presents concepts from below as abstractions from examples: A function is a name for calculations containing variable numbers. Postmodern mathematics reinvents mathematics as a science that explores the natural fact MANY. Everything will be different. The order of the types of account shall be reversed. Natural numbers become decimal numbers with units. Proportionality and integration are introduced in the first class in order to be able to add 3.1 4s and 2.3 5s.

Proportionality: linearity or per-numbers

Proportionality is a key subject in middle school, and in math in general. Proportionality can be introduced in two different ways. From above as an example of an abstraction: proportionality is an example of the linarity f(x+y) = f(x)+f(y). Or from below as an abstraction from examples of unit shifts: a beam can be measured in meters and kilograms. The change from a metre number to a kilogram number then occurs at a pertal, e.g. 3m/2kg. If the recount formula T=(T/b)*b is introduced in primary school, proportionality becomes simple. If 12m is to be converted into kg, 12 is recounted in 3s: T = 12m = (12/3)*3m = (12/3)*2kg = 8kg.

Math is a Science

Mathematics is a science that explores the natural fact of many, by bundling and stacking. From this everything becomes different: the natural numbers are decimal numbers with units: 3.5 tens or 4.3 8s. 10 is no longer the successor to nine, because at 7-bundling, 10 is sequel to six, and nine has 13 as successor. The natural order of the arithmetic species is inverse: /, *, -, +. Merging stacks such as 3, 4s, and 2.5s, leads to both proportionality and integral calculus. Multi-digit numbers become cognitive bombs. The core subject of mathematics can be learned with 1-digit numbers alone: 1DigitMath.

Model with or create math

The relationship between mathematics and its surroundings can be seen from above or below. Modern mathematics sees the relationship from above. Mathematics must be created before it can be applied - and must therefore also be learned before it can be applied. Postmodern mathematics sees the relationship from below. Mathematics is created by its concrete surroundings through abstractions: geometry is Greek for earth-measurement - and algebra is Arabic for the reunion of numbers, both equal and dissimilar piece-numbers and per-numbers. Modern mathematics protects its discourse by conceptual compulsion: what should be called the creation of mathematics is instead called the application of mathematics.

Mathematics: enlightenment or bildung

Knowledge can be perceived in two different ways, from above and below. The dispute over the nature of knowledge began in ancient Greece, when two groups spoke out about knowledge, sophia in Greek. The sophists believed that should the people practice democracy, they must be informed of the difference between nature and statute in order to avoid paternalism by statute presented as nature. The philosophers believed that statute is illusion, since everything physical is examples of metaphysical forms that can only be seen by philosophers trained at Plato's academy. The Christian Church continued Plato's paternalism in the form of pastoral tutelage, transforming the academies into monasteries, which were later converted into scholastic universities with the Reformation.

CATS: Math as Science

Math is a science that explores MANY by bundling and stacking. Everything becomes different: The natural numbers become decimal numbers with units: 3.5 tens or 4.3 8s. At 7-bundling, 10 sequels become six, and nine have 13 as successors. The natural order of the types of account will be the other way around: /, *, -, +. Merging stacks like 3, 4s, and 2, 5s, leads to both proportionality and first-class integral calculus. The core subject of mathematics can be learned with 1-digit numbers alone, 1DigitMath. CATS I is for primary school, and CATS II for secondary school. Materials and instruction are web-based..

Better PISA numbers in preschool

Sweden's PISA numbers in math has fallen from 509 to 478 in nine years, well below the OECD average of 494. The OECD report 'Improving Schools in Sweden' recommends in-service teacher training. 'Math from the outside-in' builds mathematics as a science of Many as an alternative to the prevailing 'had to be inside-out', which sees the outside world as adaptations of a self-referential mathematics. In preschool, children gain competence in proportionality and integration by counting and recounting in icons, such as 2 3s = ? 4s.

Better PISA numbers in the first school year

Sweden PISA numbers in math have fallen from 509 to 478 in nine years, well below the OECD average of 494. The OECD report 'Improving Schools in Sweden' recommends in-service teacher training. 'Math from the outside-in' builds mathematics as a science of Many as an alternative to the prevailing 'mat from the inside-out', which sees the outside world as adaptations of mathematics. In the first year of school, children gain competence in proportionality and integration by counting and recounting in icons before counting and adding in 10s.

Better PISA numbers in middle school years

Sweden PISA numbers in math have fallen from 509 to 478 in nine years, well below the OECD average of 494. The OECD report 'Improving Schools in Sweden' recommends in-service teacher training. 'Math from the outside-in' builds mathematics as a science of Many as an alternative to the prevailing 'mat from the inside-out', which sees the outside world as adaptations of mathematics. Double counting gives per-numbers such as 3kr/4kg, which indicates an easier path to proportionality, and at the same time it provides an introduction to calculus.

Better PISA numbers in high school

Sweden PISA numbers in math have fallen from 509 to 478 in nine years, well below the OECD average of 494. The OECD report 'Improving Schools in Sweden' recommends in-service teacher training. 'Math from the outside-in' builds mathematics as a science of Many as an alternative to the prevailing 'mat from the inside-out', which sees the outside world as adaptations of mathematics. Calculus is treated as addition and division into piecewise and locally constant per-numbers, and integration is treated before differentiation.

Better PISA numbers through teacher training

Sweden PISA numbers in math have fallen from 509 to 478 in nine years, well below the OECD average of 494. The OECD report 'Improving Schools in Sweden' recommends in-service teacher training. 'Math from the outside-in' builds mathematics as a science of Many as an alternative to the prevailing 'mat from the inside-out', which sees the outside world as adaptations of mathematics. With knowledge of both, teachers can conduct action research in their own classrooms to investigate which most effectively conveys math skills.

Proposals for the Mathematics Biennale 2018 and 2020

All were rejected

01) Start-math for children and migrants: Bundle**-count and re-count before adding**

Assembling 4 fingers 2 and 2, a 3-year-old will protest: "It is not 4, but two 2s". The child counts in bundle- and block-numbers just like we: 456 = 4 Bundle Bundles + 5 bundles + 6 unbundled. And recounts 3 4s to 5s, which leads to proportionality. And recounts 42 to 7s, which leads to equations. And adds 2 3s and 4 5s to 3Bundle2 8s that leads to calculus. The child is directed directly to core mathematics if allowed to keep its 2D bundle- and block-numbers, and to count and recount before adding.

A 3year old sees 4 fingers held together 2 by 2: "It is not 4, it is 2 2s". So a child counts in the block-numbers as we do: 456 = 4 bundle-bundles + 5 bundles + 6 unbundled. The child's block-numbers lead directly to proportionality and equations. So we should count before we add.

Digits units many sticks in one icon: Five sticks in the 5-icon etc.; up to ten = 1bundle0 = 1B0 = 10.

With a cup for bundles, a total T of 7 sticks is cup-counted in icon-bundles as T = 7 = 2 3s & 1 = 2]1 3s. Next, the total can be re-counted in the same unit to create overload or underload: T = 7 = 2]1 3s = 1]4 3s = 3]-2 3s.

A total can also be re-counted in a new unit (proportionality), e.g. 2 4s = ? 5s, predicted by a calculator as 2*4/5 = 1 and 2*4 – 1*5 = 3, so T = 2 4s = 1]3 5s.

We count by bundling and stacking predicted by operations, also being icons: Counting a total 8 in the 2s, 8/2 shows the broom that from 8 sweeps 2s away. Multiplication 4x2 shows the lift that stacks the 4 2s, and subtraction 8 – 2 shows the trace created by from 8 dragging 2 away. The result may therefore be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Re-counting from icon-bundles to 10s leads to the multiplication table: T = 3 4s = 3*4 = 12 = 1ten2 = 1]2 10s.

Reversing by re-counting from 10s to icons creates equations to be solved by re-counting: 'How many 5s give 40' leads to the equation ‘x*5 = 40’ solved by recounting 40 in 5s: 40 = (40/5)*5, giving x = 40/5. So an equation is solved by moving to the opposite side with the opposite sign.

02) Counting **before addi**ng will **strengthen the number sense in children and migrants**

We master many with a number-language sentences, formulas, e.g. T = 4 5ere = 4*5. Which shows that we enumerate totals T by bundling and stacking. So, 4*5 is 4 5s that can be recounted to another unit, e.g. 7s.

Math issues are prevented by bundle-numbers that can be trained as counting ‘6,..., 10’ also as ‘bundle less 4, B-3, B-2, B-1, Bundle’. And ‘10,..., 15’ as ‘Bundle, 1left, 2left, 3left, 4left, 5left’ to show that ‘eleven’ and ‘twelve’ originate from Viking counting.

We master Many using a number-language with number-language sentences, formulas, e.g. T = 4 5s = 4*5, showing how we master Many by bundling and stacking. So 4*5 is 4 5s that may be re-counted in another unit, e.g. in 7s. Or in tens, the international bundle-size.

Viewing numbers as bundle-formulas makes math easy and prevents math-problems and dyscalculia; therefore, to be practiced with various counting rhymes where '5, 6, 7, 8, 9, 10' is counted also as ' 5, bundle less 4, B-3, B-2, B-1, bundle'; and as ' ½bundle, ½bundle & 1, ½, ½bB&2, ½B &3, ½B &4, bundle. Likewise, '10, 11, 12, 13, 14, 15 ' can be counted as 'bundle, 1bundle & 1, 1B & 2, 1B & 3, 1B & 4, 1B & 5 ', and as 'Bundle, 1left, 2left, 3left, 4left, 5left ' in order to show that 'eleven' and 'twelve' is derived from the Viking age.

Digits unite many sticks into one icon: Five sticks in the 5-icon, etc., up to ten = 1bundle0 = 1B0 = 10. With a cup for bundles, a total T of 7 sticks is cup-counted in icon-bundles as T = 7 = 2 3s & 1 = 2]1 3s. Next, the total can be re-counted in the same unit to create overload or underload: T = 7 = 2]1 3s = 1]4 3s = 3]-2 3s. Likewise with totals counted in tens, T = 68 = 6]8 = 5]18 = 7]-2.

Before adding, the number sense is trained by the multiplication table, reduced to a 5 x 5 table by rewriting number above 5, e.g. 6 = ½bundle & 1 = bundle-4. First doubling, e.g. T = 2*7 = 2*(½ bundle & 2) = bundle & 4 = 14, or T = 2*7 = 2*(bundle-3) = 20-6 = 14. Then with Cup-counting, e.g. T = 2*38 = 2*3]8 = 6]16 = 7]6 = 76. Then halving, e.g. ½*38 = ½*3]8 = ½*4]-2 = 2]-1 = 19.

Multiplying with 5 means multiplying with half-bundles, 5*7 = ½bundle*7 = ½*70 = ½* 6]10 = 3]5 = 35.

03) Di**vision** d**islike cured with 5 sticks and 1 cup and** bundle**-writing**

A class dislikes division, e.g. 336/7. The solution is to see 336/7, not as 336 divided between 7, but as 336 counted in 7s; and to use bundle-writing 336 = 33B6 = 28B56, since a total can be recounted in three ways: normal and with overload or underload. Now, with T = 336 = 33B6 = 28B56, we have T/7 = 4B8 = 48.

Recounting may be trained with bundle-counting 5 sticks in 2s.

Normal: T = 5 = 2B1 2s. With overload: T = 5 = 1B3 2s. With underload: T = 5 = 3B-1 2s.

Likewise, with: T = 74 = 7B4 = 6B14 = 8B-6.

A class has problems with division, e.g. 336/7. The solution is to see 336/7, not as 336 divided among 7, but as 336 counted in 7s; and to use cup-writing 336 = 33]6, where the cup splits the total in bundled within the cup and unbundled outside.

And to cup-count totals in three ways: normal and with overload or underload.

First with 5 sticks cup-counted in 2s with a cup to the bundles.

Normal: T = IIIII = II II I = 2]1 2s. With overload: T = IIIII = II III = 1]3 2s. With underload: T = IIIII = II II II = 3]-1 2s.

In the same way we count in 10s: T = 74 = 7]4 = 6]14 = 8]-6.

So, with a total of 336 (i.e. 33.6 tens) there are 33 bundles inside the cup and 6 unbundled outside. But we prefer 28 within, so 5 bundles move outside as 50 giving 56 outside that divided by 7 gives 4 inside and 8 outside:

T = 336 = 33]6 = 28]56, and T/7 = 4]8 = 48.

Cup-writing can be used by all operations.

T = 65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92

T = 65-27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38

T = 7 * 48 = 7 * 4]8 = 28]56 = 33]6 = 336

T = 7 * 48 = 7 * 5]-2 = 35]-14 = 33]6 = 336

T = 336/7 = 33]6/7 = 28]56/7 = 4]8 = 48

T = 338/7 = 33]8/7 = 28]58/7 = 4]8 + 2/7 = 48 2/7

Cup-writing can also be used with the multiplication table:

T = 4*8 = 4*1]-2 = 4]-8 = 32 and 7*8 = 7*2 = 1]-7]-14 = 6]-4 = 5]6 = 56

04) Fractions and percentages as per-numbers

Fractions dislike disappear if viewing a fraction as a per-number obtained from double- counting in the same unit, 3/5 = 3$ per 5$; or as percent 2% = 2/100 = 2$ per 100$.

Recounting and double-counting uses the ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away.’ To find 2/3 of 12 means finding 2$ per 3$ of 12$. Here 12 recounts in 3s as 12$ = (12/3)*3$, giving (12/3)*2kr = 8$. So 2/3 of 12 is 8.

A class has trouble with fractions. Both to find a fraction of a total, and to expand and shorten fractions, where many add and subtract instead of multiplying and dividing.

The solution is to see a fraction as a per-number obtained by a double-counting in the same unit, 2/3 = 2$ per 3$ or as percentage 2% = 2/100 = 2$ per 100$.

Investment is expected to give a return, which may be higher or lower, e.g. 7$ per 5$ or 3$ per 5$.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Now 2/3 of 12 is found as 2$ per 3$ of 12$. So we re-count 12 in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$. So 2/3 of 12 is 8.

The task 'what is 3 in percent of 5? ' is solved by re-counting 100 in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3/5 is the same as 60 per 100, or 3/5 = 60%.

To expand or shorten a fractions is done by inserting or removing the same unit above and below the fraction bar: T = 2/3 = 2 4s/3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

Fractions and decimal numbers should be introduced in grade 1 relating to counting in icons under ten. 7 counted in 3s gives a stack on the 2 3s & 1. The unbundled 1 can be placed next-to as its own stack, a decimal number, T = 7 = 2.1 3s. Or it can be placed on-top counted as 3s, i.e. as a fraction: T = 7 = 2 1/3 3s.

05) Fractions and per-numbers add as integration

A class have problems with adding fractions. Many adds the numerators and the denominators separately.

The solution is to view a fraction as a per-number obtained from double-counting in the same unit, 3/5 = 3$ per 5$, or as the percentage of 3% = 3/100 = 3$ per 100$. As well as to begin with adding fractions with units, such as ½ of 2 + 2/3 of 3, that just gives 1 + 2 of 2 + 3, so 3/5 of 5. Here, then, ½ + 2/3 = 3/5, which is obtained by adding the numerators and denominators separately.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg. So here is 3$/kg + 5$/kg = 4.33 $/kg, called the weighted average.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph. Corresponding with fractions.

Adding two fractions a/b and c/d without units is meaningless, but can be given meaning if taken of the same total, b*d: a/b of b*d + c/d of b*d gives a total on a*d + c*b of b*d. So a/b + c/d = (a*d + c*b)/b*d.

Adding fractions and per-numbers with units provides a good introduction to calculus. As shown, multiplication before addition is the same as integration. And inverted integration is the same as differentiation:

The task ‘2kg at 3$/kg + 4kg at ? $/kg = 6kg á 5$/kg’ leads to the equation 6 + 4*x = 30 or T1 + 4*x = T2, solved with subtraction before division, called differentiation: x = (T2-T1)/4 = □T/4.

06) Proportionality as double-counting, with per-numbers

Proportionality dislike disappears by renaming it to ‘unit-shift by double-counting’, which leads to ‘per-numbers’ such as e.g. 2$ per 3kg or 2$/3kg or 2/3 $/kg. Recounting uses the ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away.’

Thus, a total of 16$ is recounted kg as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg can be recounted in $ as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

A class has a problem with proportionality. The price is 2$/3 kg. All will find the $-number for 12 kg, but only a few will find kg-number for 16$. The solution is to rename proportionality to ‘shifting units’ by 'double-counting', leading to 'per-numbers' such as 2$ per 3 kg or 2$/3kg or 2/3 $/kg. The units are connected by re-counting the known part of the per-number.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

This allows re-counting 16$ in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24kg. Likewise, the 12kg re-counts in 3s as T = 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$. Will this difference make a difference? In theory, yes, since proportionality is associated with counting, a basic physical activity.

In fact, proportionality takes place in grade 1 when counting totals in icon-bundles different from the standard bundle ten and by afterwards re-counting the total in a different unit. This leads directly to the re-count formula, which has the same shape as y = k*x.

Thus, a total of 8 re-counts in 2s as T = (8/2)*2 = 4*2 = 4 2s.

And a total of 3 4s re-counts in 5s as T = (3*4/5)*5 = 2*5 + 2.

And per-numbers lead directly on to the fractions, obtained by double-counting in the same unit, e.g. 2$ per 3$ = 2$/3$ = 2/3 = 2 per 3.

Getting 2/3 of 15 means getting 2$ per 3$ of 15$ found by re-counting 15 in 3s and take 215 thereof: T = 15$ = (15/3)*3$ giving (15/3)*2$ = 10$. So 2/3 of 15 is 10.

Likewise, 20% of 15 is found by re-counting 15 in 100s: T = 15 = (15/100)*100 giving (15/100)*20 = 3.

07) Equations solved by moving across**, revers**ed calculation, **or recounting**

Equations such as 2+3u = 14 and 25–u = 14 and 40/u = 5 are easily solved by the rule for reverse operations: ‘Move to opposite side with opposite calculation sign’.

The equation u+3 = 8 asks for a number u that added to 3 gives 8, which by definition is u = 8-3; thus +3 moves to the opposite side with the opposite calculation sign. Similarly with u*2 = 8 solved by u = 8/2; and with u^3 = 12 solved by u = 3√12, where the root is a ‘factor-finder’; and with 3^u = 12 solved by u = log3(12), where the logarithm is a ‘factor counter’.

A class has problems with the equations 2+3*u = 14 and 25 – u = 14 and 40/d = 5, where the equation is composite or where the unknown has a inverse sign. The solution is to use the definitions of the inverse operations to create the basic solution rule: 'move to the opposite side with the opposite sign'.

In u+3 = 8 we seek a numbers u that added to 3 gives 8, which is u = 8-3 by definition; so + 3 moves to the opposite side with the opposite sign. Corresponding with u*2 = 8, solved by u = 8/2; and with u^3 = 12, solved by u = 3√12, where the root is a factor-finder; and with 3^u = 12, solved by u = log3 (12), where the logarithm is a factor-counter.

The equation 2+3*u = 14 can be seen as a double calculation that is reduced to a single by a bracket around the stronger operation, 2+(3*u). Moving 2 to the opposite side with the opposite sign gives 3*u = 14–2. Then 3 moves to the opposite side with opposite sign, but first a bracket is placed around what first must be calculated: u = (14–2)/3 = 12/3 = 4.

Equations can also be solved by walking forward and backward: Forward we first multiply with 3 and then add 2. Backwards, we first subtract 2 and then divide by 3, so u = (14–2)/3 = 4.

In the equation 25 – u = 14, u has opposite sign and therefore moves to the opposite side to get a normal sign before 14 moves to the opposite side with opposite sign: 25 = 14 + u; 25–14 = u; 11 = u.

Corresponding with 40/u = 5 giving 40 = 5*u and 40/5 = u or 8 = u.

Having learned re-counting this can also be used: 40 = (40/u)*u = 5*u and 40 = (40/5)*5, giving u = 40/5.

08) Calculus: Addition and splitting **into locally constant per-numbers**

Calculus is made easy by beginning with integral calculus for adding piecewise or locally constant per-numbers by their areas. Adding ‘2kg á 3$/kg + 4kg á 5$/kg’, the unit numbers 2 and 4 add directly to 6, while the per-numbers 3 and 5 must be multiplied to unit-numbers before they can add: 3*2 + 5*4 = 26. Thus, the answer is 6 kg á 26/6 $/kg. However, multiplication creates areas, so per-numbers add by the area under the piecewise constant per-number graph.

A class has problems with calculus. The solution is to postpone differential calculus until after integral calculus is taught as a means of adding piecewise or locally constant per-numbers by their areas.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph, i.e. by adding a few area strips, S = Σp*Δx.

A non-constant per-number is locally constant (continuous), meaning adding of countless many area strips, S = ∫p*dx. Unless we can rewrite the strips as changes, p*dx = dy or dy/dx = p. For when adding changes, all middle terms disappear leaving just the total change from the start to the end point.

This motivates differential calculus: If the strip 2*x*dx can be rewritten as a change, d (x^2), then the sum ∫ 2*x*dx is the change of x^2 from the start to the end point.

Change-formulas can be observed in a rectangle, where changes Δb and Δh in the base b and height h gives the total change of the area ΔT as the sum of a vertical strip, Δb*h and a horizontal strip, b*Δh; and a corner, Δb*Δh that can be neglected with small changes.

Therefore, d(b*h) = db*h + b*dh, or, if counted in Ts: dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h.

So with (x^2)'/x^2 = x'/x + x'/x = 2*x'/x, (x^2)' = 2*x since x' = dx/dx = 1.

So differentiation is a smart way to add many numbers; but also useful to describe growth and decay and optimization.

09) Calculus in primary, middle and high school

Counted and recounted, blocks may be added, but on-top or next-to? Next-to addition of 2 3s and 4 5s as 8s means integrating their areas, called integral calculus where multiplication precedes addition. Asked oppositely ‘2 3s +? 5s gives 3 8s’, the answer is obtained by letting subtraction precede division, called differential calculus. So, with block numbers, calculus occurs already in primary school.

In middle school calculus occurs when adding per-numbers in blending tasks as ‘2kg á 3$/kg + 4kg á 5$/kg = ?’

Mathematics means knowledge in Greek, who chose the word as a common name for their four areas of knowledge, arithmetic and geometry and music and astronomy, which they saw as the study of many by itself, in space, in time and in space and time.

With music and astronomy gone, today mathematics is just a common name for algebra and geometry, both rooted in Many as evidenced by their meaning in Arabic and Greek: to reunite numbers and to measure the earth. Meeting Many we ask 'How many in total?' The answer we get by counting, before we add. Counting is done by bundling and stacking,

predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

Once counted, stacks can be added, but on-top or next-to?

Next-to addition of the stacks 2 3s and 4 5s as 8s means adding their areas, i.e. by integration, where multiplication comes before addition.

Reversed, we ask '2 3s +? 5s gives 3 8s', now letting subtraction come before division, called differentiation.

So in primary school, calculus occurs with next-to addition of stacks.

In middle school calculus occurs with blending and average tasks:

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration.

Thus per-numbers add by the area under the piecewise constant per-number graph, i.e. by adding area strips, $S = \Sigma p*\Delta x$, or $S = \int p*dx$ in high school, where per-numbers are locally constant (continuous), and where per-numbers are added before they can be subtracted by differentiation.

10) Stem-based core-math makes migrants pre-engineers

Our word- and number-language assign words and numbers to the world with sentences and formulas that contain a subject, a verb, and a predicate. With a number-language, young migrants can access core-math directly: Recounting in a new unit is predicted by a ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away’, e.g., T = 3 4s = (3*4)/5*5 = 2 5s & 2. After recounting and recounting, blocks may add next-to as areas (integration) or on-top if the units are made equal by recounting (proportionality).

We master the outside world by a word-language and a number-language, describing it by sentences and formulas containing a subject, a verb, and a predicate: 'the table is yellow' and 'the total is 3 4s'. The two languages both have a meta-language, a grammar and a mathematics, that should be learned after the language, otherwise causing dyslexia and dyscalculia.

Young migrants get direct access to the number-language with core-math curriculum:

A) Digits are the icons with the number of sticks, it represents.

B) Operations are icons for counting by bundling and stacking: division removes bundles, multiplication stack bundles, subtraction removes a stack to look for unbundled, addition unites stacks on-top or next-to.

C) Cup-counting and cup-writing shows the bundles inside the cup and the un-bundled outside, e.g. T = 4]5 = 4.5 tens = 45.

D) Totals must be cup-counted and re-counted and double-counted before they can add.

E) Re-counted in the same unit, a total can be written in 3 ways: normal, with overload or with underload, e.g. T = 46 = 4]6 = 3]16 = 5]-4.

F) Re-counting in a new unit (proportionality) be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

G) Re-counting from tens to icons creates equations, e.g. x*5 = 40 = (40/5)*5 with solution x = 40/5. Double-counting gives per-numbers and proportionality with re-counting in the per-number: with 2$ per 3kg, 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$.

H) After counting comes addition, on-top and next-to, leading to proportionality and integration.

I) Reverse addition leads to equations and differentiation.

J) Per-numbers lead to fractions, both operators needing to be multiplied to become numbers, and therefore added by their areas, i.e. by integration.

K) Calculus means adding and splitting into locally constant per-numbers.

L) Core STEM-areas become applications under the theme 'water in movement'.

Details: ' A STEM-based Core Math Curriculum for Outsiders and Migrants ', http://mathecademy.net/papers/miscellaneous/

11) **Math blocks for the block-organized secondary school**

In the EU, secondary school is a line-organized with compulsory classes forcing the teenagers to follow the year-group despite boys being two years behind in maturity. For economic reasons, low achievers are forced to stay in the class which cannot be repeated.

In the United States, secondary school supports the teenager's identity work by welcoming them with esteem: 'Inside, you carry a talent that together, we will now uncover and develop through daily homework in self-selected half-year blocks of a practical or theoretical nature together with teachers who have one subject only.'

12) The teacher as a difference-researcher

Difference research finds hidden differences that make a difference. It is used to solve problems in class. Or by teacher-researchers with shared time between academic work at a university and intervention research in a class. Or by full-time researchers working together with teachers: The teacher observes problems, the researcher identifies differences. A mutual micro-curriculum is created, tested by the teacher and reported by the researcher in a pretest-posttest study.

When traditions give problems, difference research uncovers hidden differences that make a difference. For example, the tradition says that 'a function is an example of a set relation where first component identity implies second component identity', which the learner hears as 'bublibub is an example as bablibab ', which nobody finds meaningful. A difference is to use Euler's original definition accepted by all: 'a function is a common name for calculations with both known and unknown numbers'.

Difference research can be used by teachers to solve problems in class, or by teacher-researchers sharing their time between academic work at a university and intervention research in a class. Or by full-time researchers, working with teachers to apply difference research: the teacher observes the problems, the researcher identifies the differences. Together they establish a micro-curriculum to be tested by the teacher and reported by the researcher after a pretest-posttest study. A typical difference researcher begins as an ordinary teacher who reflects on whether alternatives can solve learning problems observed.

A difference-researcher combines elements of action learning and action research and intervention research and design research. First a difference is identified, then a micro-curriculum is designed to be tested to see what kind of difference it makes. The effect will be reported internally and discussed with colleagues. After repeating this cycle of design, teaching and internal reporting, it is time for an external reporting of the difference and its effect in magazines or journals or at conferences.

Research should provide knowledge to explain nature and to improve social conditions. But as an institution in runs the danger of the what the sociologist Bauman calls a goal displacement, so research will be self-referencing instead of finding differences. Hargreaves, write for example: 'What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads' (Hargreaves, 1996, p. 7).

Hargreaves, D.H. (1996). *Teaching as a Research-based Profession: Possibilities and Prospects.* Cambridge: Teacher Training Agency Lecture.

MATHeCADEMY.net: Math as a Natural Science about Many, a Booth Exhibit

MATHeCADEMY.net provides online courses for teachers who want to teach mathematics as 'Many-Math', i.e. as a natural science about the physical fact many; as well as wanting to see mathematics as a number-language in family with the word-language, both using full sentences with a subject, a verb and a predicate, and where two competencies are sufficient: counting and adding in space and time. Many-Math respects the child's own number-language with flexible 2D bundle-numbers like T = 2*3 = 2 3s; and counting before addition.

A Case: Peter, stuck in division and fractions

Being a mathematics teacher in a class of ordinary students and repeaters flunking division and fractions, Peter is about to give up teaching when he learns about the '1cup & 5sticks' method to cure mathematics dislike by watching 'CupCount and ReCount before you Add' (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Here 5 sticks are CupCounted in 2s using a cup for bundles. He sees that a total can be recounted in the same unit in 3 different forms: overload, standard and underload:

T = 5 = I I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = II II II ~~I~~ = 3B-1 2s

So counted in bundles, a total has an inside number of bundles and an outside number of singles; and moving a stick out or in creates an over-load or an under-load.

When multiplying, 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 inside and 56 outside as an overload that can be recounted: T = 7 x 4B8 = 28B56 = 33B6 = 336.

And when dividing, 336/7 is bundle-written as 33B6 /7 recounted to 28 inside and 56 outside according to the multiplication table. So 33B6 /7 = 28B56 /7 = 4B8 = 48.

To try it himself, Peter downloads the 'CupCount & ReCount Booklet'. He gives a copy to his colleagues and they decide to arrange a free 1day Skype seminar.

In the morning they watch the PowerPoint presentation 'Curing Math Dislike', and discuss six issues: first the problems of modern mathematics, MetaMatism; next the potentials of postmodern mathematics, ManyMath; then the difference between the two; then a proposal for a ManyMath curriculum in primary and middle and high school; then theoretical aspects; and finally where to learn about ManyMath.

Here MetaMatism is a mixture of MatheMatism, true inside a classroom but rarely outside where '2+3 = 5' is contradicted by 2weeks+3days = 17days; and MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.

In the afternoon the group works with an extended version of the CupCount & ReCount Booklet where Peter assists newcomers. At the seminar there are two Skype sessions with an external instructor, one at noon and one in the afternoon.

Bringing ManyMath to his classroom, Peter sees that many difficulties disappear, so he takes a 1year distance learning education at the MATHeCADEMY.net teaching teachers to teach MatheMatics as ManyMath, a natural science about Many. Peter and 7 others experience PYRAMIDeDUCATION where they are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

At the academy, the 2x4 sections are called CATS for primary and secondary school inspired by the fact that to deal with Many, we Count & Add in Time & Space.

At the academy, primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something new about something I already knew.

Fifty years of research without improving mathematics education, why?

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report 'Improving Schools in Sweden, an OECD Perspective'.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is "Only one: to improve, mathematics education should ask, not what to do, but what to do differently."

Thus, to be successful, research should not study problems but look for differences that make a difference. Research that is skeptical towards institutionalized traditions could be called difference research. In France, Lyotard calls it 'paralogy' inventing dissension to the reigning consensus. Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory.

To elaborate, maybe mathematics education research is sterile because its three words are not that well defined?

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation's need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying existing theories, or at a research level questioning them. Just following theories is problematic in the case of conflicting theories as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for alternatives. So, if institutionalized education should help children and youngsters to master outside phenomena we must ask: What outside phenomena roots mathematics?

We master the outside world with two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair', and 'T = 3*4'. Both languages have a meta-language, a grammar, describing the language, describing the world. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the 'New Math' that turned mathematics upside down to become a 'meta-matics' presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as 'many-matics', a natural science about Many.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear.

Thus, one alternative immediately presents itself: Re-root mathematics in its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many by meaning 'reuniting numbers' and 'measuring earth' in Arabic and Greek respectively.

Consequently, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

For details, see 'Difference-Research Powering PISA Performance', Fifty Years of Research without Improving Mathematics Education', and 'A STEM-based Core Math Curriculum for Outsiders and Migrants' at http://mathecademy.net/papers/miscellaneous/.

Proposals for the Mathematics Biennale 2024

01 was accepted, the rest was rejected.

01. Booth: With Woke math, the child uses his own flexible bundle numbers with units to master Many before Math

Abstract: Woke-Math will not 'force fixed forms on flexible totals'. Woke-math respects that MANY is described by the child's own flexible bundle numbers with units, rather than being forced into a false identity with line numbers without units, which becomes 'mathematics' by claiming that 2 + 1 = 3 always, despite the fact that 2 weeks + 1 day = 15 days. And respects that five fingers can exist as different totals, all with units: as 5 1s, as 1 5s, as 1Bundt3 2s, as 2Bundt1 2s, as 3Bundt-1 2s, etc.

02. Lecture: Woke-math protects against AI.

Abstract: 350 famous persons warned in May 2023 that AI is developing into an independent alien intelligence that can hack and infiltrate all text-based knowledge. But math can protect itself by making itself text-free. For example, by is practiced as 'woke math', which respects that MANY is described with the child's own flexible bundle numbers with units, instead of being forced into a false identity with line numbers without units, which becomes 'mathematics' by claiming that 2 + 1 = 3 always, despite the fact that 2 weeks + 1 day = 15 days.

03. Lecture: AI had prevented massive math abuse from creating a corona scandal in Denmark

Abstract: Massive math abuse created the Corona scandal, which AI had prevented. Jones's infection pressure formula says that the contagion happens in space, but especially in time. Weak bodies must then avoid prolonged tightness, while strong ones burn off the infection. So football matches like Milan should be cancelled, but not skiing that burns infection. But it only warns against distance, makes unreliable averages for dissimilar bodies, and is based on unreliable infection rates, not on credible hospitalization figures. All examples of Danish math abuse are presented.

Content: Around the first of February 2020, it became clear that infection from a corona virus had spread from China with New Year's flights to Europe, and from there to the ski resorts of the Alps. And that the contagion here was harmless to strong bodies. The European Centre for Disease Prevention (ECDC) should therefore have used its own or artificial intelligence to inform about the precautions that follow from Jones' infection pressure formula, which states that the infection occurs in space, but especially in time. Under the assumption that the infection limit is contact at a distance of 1 meter for 1 quarter, infection will not occur at a greater distance, only at a longer duration, where the infection pressure on the other hand doubles over time, so a 2-hour contact will 8-fold increase the infection pressure from 2.5 to 20 persons per infected person; or to a 20 time multi-contagion for each individual.

An intelligent action would therefore be to take into account weak bodies by banning prolonged gatherings such as football matches and festivals during the three months it would take the strong bodies to burn the infection. But at the same time, weak bodies allow short-term get-togethers at a distance of over a metre in less than a quarter of an hour.

And at the same time state that the infection does not grow exponentially with a constant growth percentage beyond all limits, as this is impossible in a limited population, where there are eventually no more people to infect. So the infection dies out on its own when herd immunity has been achieved with about 70% infected. The growth rate will therefore be decreasing, so the number of newly infected people will grow, peak and decrease in a hill curve. This logistical form of growth can easily be demonstrated at school, as infection spreads like a rumor. In the beginning, everyone listens, but once 60% have heard it, no one wants to listen anymore. Then herd immunity has arisen.

As well as state that the burning period should not be extended so that the virus can mutate into another form. Mna should therefore refrain from disinfecting and extra washing, from carrying out lockdowns and from developing vaccines.

And state that the authorities will follow the spread of the infection by looking at the number of daily hospitalized patients, and of course not at the number of positive tested, as this infection number is random and depends on who wants to be tested, which unaffected strong bodies obviously do not have.

So if you had let your own or an artificial intelligence determine your behavior around the first of February 2020, the corona time would only have corresponded to a flu period.

But nothing happened, and the result became apparent soon after the football match on 19.2. in Milan with 80,000 spectators. The following week, Bergamo's hospitals and cemeteries were filled with sick and dead due to the extreme infection pressure that exposed weak overweight multi-medicated bodies to multi-infection:

First 2 hours at a bar in Bergamo, then 1 hour on the train to Milan, then 2 hours at a bar in Milan, then 2 hours in a queue to get in and out of the stadium, then 2 hours as shouting herring in barrel, then 2 hours at a bar in Milan, then 1 hour on the train home to Bergamo, then 2 hours at a bar in Bergamo. That is, approximately 14 hours at a distance of 25 cm, which gives an infection pressure of 14 hours times 4 quarters of an hour times 4 for density times 2.5, i.e. 560. Including low hygiene and lack of face masks, the figure is rather 1000 newly infected per infected. And it only takes 40 skiers to infect everyone from Bergamo.

This incident should have led to formulating a Bergamo hypothesis: "Weak obese multi-medicated bodies should avoid prolonged tightness while the strong bodies burn off the infection."

But the government opted instead for a lockdown based on three infection numbers, which mainly came from infected skiers from Austria. Despite the fact that the credible hospitalization figures showed that the infection would be over after 3 months.

For infection control, the government set up an expert group to calculate the unreliable infection figures.

The group chose a chain model: more reopening leads to more infections, which leads to more sickness, which leads to more hospitalizations, which leads to more deaths. The population was seen as homogeneous, though divided by age above and below 60 years, but without taking into account the Bergamo hypothesis.

The model's predictions were therefore far from what was observed. Even so, the experts defended their model with the George Box quote "All models are wrong, but some are useful". And their model was useful because it provides politicians with numbers. In doing so, they conceal the fact that models such as literature come in two forms, facts that must be checked for correct units, and fiction that must be supplemented with scenarios based on alternative assumptions.

Throughout the period, the media uncritically relayed the government's many irrelevant figures.

First and foremost, the infection rate, which is not made more credible by being stated as a percentage of the number of people tested or as an incidence number of the number of residents in the area. And which does not take into account the size of the geographical area, whereby smaller tourist towns and islands are closed when only a few have been infected.

Next is the load on the hospitals, which is stated as the number of beds, but is described with the word 'number of inpatients', which can also mean the number of daily admissions. As forties, although it is the only relevant number, as the ratio between two numbers four days apart indicates the infection pressure. Unfortunately, it was renamed to the contact number, although it otherwise nicely illustrates the difference between overpressure and underpressure above and below 1, where infection is pumped into or out of the population.

Finally, there are repeated examples of incorrect use of cross-tabulations, by changing relative description without going over the absolute numbers. An example will show the problem: I have 5 fingers, 2 H fingers on the right and 3 V fingers on the left. I now assume that 1H finger and 2 V fingers are sick. Claim: 1/3 of the H-fingers are diseased, since they make up 1/3 of the sick. No, because half of the H-fingers are sick.

The Corona situation could have shown how using mathematics can solve problems. Instead, it has shown how misuse of mathematics can create serious social problems.

20. The MES Conferences

The Mathematics Education and Society Conference

MES 1 1998, What if Mathematics is a Social Construction

MES 2 2001, Searching for Hidden Contingency

MES 4 2007, Modern or Postmodern Critical Research

MES 6 2011, Anti-bildung Enlightenment Education in Berlin

MES 8 2015, Invitation to a Duel on how to Improve Math Education

MES 9 2017, Paper, project, symposium & poster

What if Mathematics is a Social Construction

Why do post-modern reflexive students avoid modern ritualised mathematics

Today many students avoid math-based educations within science and engineering. This enrolment problem creates an opportunity to reflect on mathematics and its traditions. Is mathematics part of nature or is mathematics part of culture, a social construction? If mathematics is part of nature, it can be taken for granted, and only mathematics education can be discussed. If however mathematics is part of culture, also mathematics itself can be discussed since an alternative mathematics might be constructed. Asking "What if mathematics is socially constructed", this paper looks at different theories about social constructivism and their consequences for mathematics and mathematics education. The traditional abstract Platonic mathematics is confronted with an alternative contextual nominalistic mathematics.

Mathematics - constructed from below or from above

Man is living in two different worlds, a world of things, actions and practice and a world of concepts, reflection and communication. And man has been doing so ever since he stood up transforming his forelegs to tools for manipulating the world of things, and developed on top on his reptile brain and mammal brain an additional human brain to house a language. But is there also a meta-world above the concept world? Are concepts abstractions from practice or examples of universal ideas from the meta-world? Do concepts come from below of from above? In antiquity Plato pointed upwards, Aristotle downwards. In the medieval struggle of universals the realists pointed upwards, the nominalists downwards. The church pointed upwards, science downwards. Within mathematics the Bourbakists pointed upwards saying: a function is an example of a relation, Euler downwards saying: a function is an common name for calculations with variable quantities.

Pointing upwards modern mathematics creates problems to postmodern Wittgensteinean students looking downwards for a meaning in the use, thus constructing today's enrolment problem: students evading math-based educations within science and technology. But can mathematics be explained from below to revive the interest of postmodern students? Can mathematics be understood as growing and constructed from the world of things and actions? Can geometry and algebra be presented as abstractions instead of as examples? Geometry points down to measuring of earth, but what dose algebra point to? Can a contextually mediated nominalistic postmodern algebra be constructed and how would it affect postmodern students?

The nominalists of today

A nominalistic position would state that concepts come from below as names created by a culture to differentiate between things and thus dependent on the culture. This nominalistic tradition is today represented by different positions such as e.g. the "History and Philosophy of Science", the "Sociology of Scientific Knowledge" and the "Actor Network Theory" (se Pickering 1992 for an introduction).

In the "History and Philosophy of Science" T. S. Kuhn introduces the notion of paradigms as achievements attracting adherents from competing activities and providing problems for the adherents to solve (Kuhn 1962). The paradigms of science reproduce themselves through an authoritarian, dogmatic education, as Kuhn points out: "Even a cursory inspection of scientific ped-agogy suggests that it is far more likely to induce professional rigidity than education in other fields, excepting perhaps, systematic theology" (Kuhn 1963).

"Sociology of Scientific Knowledge" or "social constructivism" originates from Great Britain with B. Barnes, D. Bloor and H. Collins as prominent figures. A social construction is a "closure", i.e. a victorious solution to a social need. The winner rewrites history so that its competitors are forgotten and the closure appears as the only natural and rational solution. The passing of time may however see a "disclosure" of the closures ability to deliver, setting the stage for a "reclosure" through a reformulation of the social need and a new competition among alternative solutions. The SSK social constructivism should not be mistaken for the psychological social constructivism inspired by

Vygotsky. The former is about externalising and the latter about internalising concepts (Berger, Luckman 1966).

Actor network theory originates from France with B. Latour and M. Callon as prominent figures. Science is texts, or rather sets of allied texts referring to each other thus creating a network of discourse fighting other networks with military inspired techniques. The important thing is to maintain and expand the network and to take the power in the network by occupying "necessary passage points". Peace occurs when one network has become superior. This victorious network then has won the right to possess knowledge (Latour 1987).

Theories of modernity

Stressing the culture dependence of concepts, theories of modernity are essential to a nominalistic position. Modernity began with the enlightenment period around year 1800 but theorists disagree as to whether today's society should be labeled late modernity, postmodernity or other similar terms.

J. F. Lyotard describes postmodernity as incredulity toward meta-narratives (Lyotard 1985). A. Pickering describes questioning taken-for-granted distinctions as a trademark of postmodern thoughts (Pickering 1992). A. Giddens describes today's society as one of high or late modernity differing from preceding societies in respect to detraditionalisation, globalisation and individualisation. The dynamism of modernity is provided by a separation of time and space making social relations global; by abstract systems disembedding interaction form the particularities of locales, deskilling daily activities and mediating experiences; and by reflexivity imbedding knowledge at institutional and personal levels. In such a post-traditional order self identity becomes a reflexive project making individuals construct and sustain their own biographical narrative. Meaninglessness becomes a fundamental psychic problem, and authenticity becomes a pre-eminent value and a framework for self-actualisation (Giddens 1991). Traditions involve rituals, guardians, formulaic truth and a normative content. Rituals are the practical means of ensuring preservation and the continual reconstruction of the past. The guardians are the people believed to be the agents or mediators of the tradition, being not experts but dealers in mystery. The formulaic truth, to which only the guardians have full access, are often formulated in words or practices that the speakers or listeners can barely understand thus reducing the possibility of dissent. The normative content gives tradition a binding character. If tradition is not discursively articulated and defended in dialogues with its alternatives, it becomes fundamentalism (Beck et al 1994).

Some questions inspired by the new nominalists

Inspired by T. S. Kuhn we can ask: What are the paradigms of mathematics and mathematics education? Why does mathematics pedagogy remind of systematic theology?

Inspired by A. Giddens we can ask: What are the traditions of mathematics, i.e. its rituals, guardians, formulaic truth and normative content? Is the tradition involved in dialogues with its alternatives or has it become fundamentalistic? Is mathematics a tradition ready for de-traditionalization? What are the alternatives to current mathematics? What does a de-traditionalization of mathematics look like?

Inspired by J. F. Lyotard we can ask: Is mathematics a meta narrative suppressing local contextual narratives? What can be a possible content of mathematics as contextual narratives, of postmodern mathematics? Inspired by B. Latour we can ask: Who are the allies, and which are the necessary passage points that lead to the victory of the current mathematics discourse?

Inspired by the social constructivists we can ask: Can today's problems within mathematics such as math anxiety, innumeracy and decreasing enrolment to math-based educations be considered a disclosure of the present math closure? What is the social need that originated mathematics? To what social question is mathematics an answer?

Possible answers

Asking "To what social question is mathematics an answer" one possible answer could be: Mathematics is the grammar of our number language. In any culture there is a need to differentiate between and communicate about things. To this end a “word-language” is constructed assigning words to things by means of sentences: "This table is high". If the culture wants to differentiate between degrees of many a “number-language” is constructed assigning numbers to things also by means of sentences: "The height is ninety cm". Some numbers can only be or are quicker determined through a calculation: "The area is length times width". In abbreviated form these sentences are called equations: $h = 90$, $a = l*w$.

A language can be considered a house with two floors. A lower context floor housing discourses and literature using sentences and equations to describe the world of things and actions. An upper abstract floor housing meta-discourses describing sentences and equations. The two floors are called “language” and “meta-language” in the word language, and “applications of mathematics” and “mathematics” in the number language. These choices of names construct two contradictory myths: “Language before meta-language” and “Mathematics before application of mathematics” providing the world with literacy and innumeracy. Innumeracy may be turned into numeracy by treating the two languages alike an by substituting the term “application of mathematics” with “reckoning”, “calculating”, “number language” or a similar self-supporting name.

Today’s mathematics consists of concepts and statements. Concepts are defined from other concepts thus constituting a concept pyramid with the set concept as the mother concept at the top. Statements about the concepts have to be proven or taken for granted. In the former case the statements are called theorems, in the latter axioms. Since theorems have to be proved they are not socially constructed. The concepts however are not proven, they are chosen. The concepts of mathematics are examples of paradigms, guardians, subjects of a grand narrative, necessary passage points and socially constructed closures.

In his book "Introductio" from 1748, Euler creates a closure of the function concept as a name for a calculation with variable quantities. In the following century this closure was disclosed producing many competing alternatives. In this century a reclosure took place, creating the modern abstract definition: "A function is an example of a relation, which is an example of a set product, which is an example of a set". The Euler definition of a function was nominalistic pointing down to calculations. The abstract definition is Platonic pointing up to the more abstract concept of a relation.

Concepts construct textbooks containing definitions and theorems. Pointing upwards the textbooks of abstract mathematics can only tell its narrative in one way. The textbook thus constitutes a grand narrative, a victorious discourse, a ritual, a guardian, a formulaic truth and a normative prescription, a "bible". Alternatively contextual textbooks localise and situate mathematics in a context of essential problems from the world of things and actions. Abstract mathematics consider contextual mathematics as just applications of itself and not as another form of mathematics. By avoiding a dialogue with contextual mathematics abstract mathematics becomes fundamentalistic.

The textbooks ensure their own reproduction through echo teaching where teachers just repeat the book, and through written and oral exams controlled by external examiners. Exams and examiners are perhaps the strongest guardians of the tradition.

The concepts also create a community of practitioners also being guardians. The mathematicians working at the universities producing new theorems, new mathematicians and new mathematics teachers through an authoritarian, dogmatic education. The graduates who cannot produce new theorems themselves will be teachers of mathematics in the schools thus becoming the interface between mathematics and the public. Socialised into a Platonic paradigm many teachers become not narrators but echoes of the grand narrative unable to provide authenticity, only inhibiting authority (Fromm 1941). Unable to prepare a defence in a dialogue with e.g. contextual mathematics many are trapped in a “après nous le deluge” fundamentalism.

The mathematics teachers carry the ritual to the classroom, where the students have to learn or memorise definitions, theorems and proofs. Questioning the theorems is considered good manners. Questioning the concepts or the use or asking “What is mathematics” is considered questioning the paradigm. And questioning a taboo might lead to social excommunication.

Also the politicians act as guardians accepting abstract mathematics as the only way to a number language by formulating goal statements as: “The purpose of mathematics education is to learn mathematics”. Thus placing abstract mathematics at the necessary passage point of the educational discourse politicians create the evasion of post-modern reflexive students and the following enrolment problem. Changing the goal statement to “The purpose of mathematics education is to develop the student's number language” will allow a free competition between mathematics from above and below. This rehumanisation of mathematics could create a quite different situation with happy students making the enrolment problem disappear.

Numbers as social constructions

Numbers are an answer to the social need to differentiate between degrees of many. Not all cultures have this need. Many hunters and gatherers count one/two/many and may even lack an abstract standard name for many: flock, bundle, bunch, collection, stack, bouquet etc. Agricultural, industrial and information cultures however share a social need for differentiating between degrees of many. They are immediately confronted with a problem: infinitely many degrees of many suggest the creation of infinitely many names. This problem is solved by bundling, reusing the names when counting pieces, bundles, bundles of bundles, sub bundles etc. (Scriba 1968)

An international closure has taken place using ten as the bundling number, but some cultures had different bundling numbers. In French ninety is called “four-twenty-ten” and in Danish ninety is called “half-five” with “times twenty” as a silent understanding. Different subcultures and trades used different bundling practices. The farmers used 20, the merchants 12, both numbers being present in the former British monetary system. Although place, matter and money have become standardised, time has still kept its own bundling routine.

Operations as social constructions

Where numbers are an answer to the social question: “How many?”, operations are an answer to the social question: “How many in total?”. The Arabs used the name “Algebra” for the use of operations. The meaning of this word has since been lost and is today unknown to most mathematicians and math teachers. Algebra means reunite, i.e. uniting what might have been separated.

Operations are answers to the social need for reuniting. Addition was an answer to the farmers need to unite unequal numbers. Multiplication was an answer to the merchants need to unite equal numbers. The merchants of renaissance Italy reopening the old trade route to the far east, this time by sea, made so much money they could lend it out. Bankers thus needed to unite interest percentages, to which power was an answer. The British wanting to open a trade route to India sailing without the sight of land, where hostile Portuguese were waiting for an easy catch, had to depend on a way of determining altitude and latitude, which led to a need to unite varying "per numbers" (e.g. meter per second), to which integrating was an answer. And for the reversed calculations inverse operations as minus, division, root, logarithm and differentiation were constructed.

Mathematics as a social construction

Once socially constructed as closures to the social questions of how many and how many in total, numbers and operations themselves became the objects of narratives, meta narratives, mathematics, just like the sentences of the word-language become the object of meta narratives, grammar.

Within mathematics a closure took place in this century resulting in the modern abstract mathematics and its myth: “Good crated the numbers, man created the rest”. From now on numbers, operations and calculations were to be understood not as social constructions but as examples of sets, relations, functions and compositions.

Today this modern mathematics from above has been disclosed by problems as math anxiety, innumeracy and decreasing enrolment to math-based educations. So the time has come to reformulate the human need for a number language and look for alternative solutions. A natural thing is to look for a nominalistic mathematics from below. One possibility could be contextual totalling narratives (Tarp 1998).

The teaching of mathematics from below i.e. as contextual totalling narratives differs from that of traditional mathematics in many fundamental ways. The central message of the number language, that totals can be reached both by counting and by calculating, should be made clear from the first day in school. Also number language sentences, equations, should be introduced from first grade as totalling narratives to show why the Arabs chose the name Algebra for calculating. The total equation T = a*b+c*d and the split equation T = T/b*b shows the need to change before adding: the total of 3 meter and 5 cm cannot be calculated as 3+5. The total equation introduces multiplication before addition which is now possible with pocket calculators, and makes addition of fractions superfluous changing fractions to numbers before adding. In calculus the adding of varying per numbers through integration are introduced before differentiating, substituting the pluses with an integral in the total equation.

Presenting modern mathematics to post-modern students

Mathematics can be changed from a “function discourse” pointing upwards to an abstract concept pyramid into a “calculation discourse” pointing downwards to totalling problems from the world of things and actions. This change affects the learning of reflexive students. Telling a student: “A function is an example of a relation” is presenting him with a statement where both concepts are outside his narrative and thus only accessible through learning by heart making the student an echo of the function discourse. Echo learning produces encapsulated knowledge, easy to reproduce but hard to apply. Echo learning might keep the ritual alive, but does not extend neither the function narrative nor the students own narrative.

Echo learning was no problem in modernity, where students were socialised into arenas of high authority: the home, the school, the army, the workplace etc. through modernity’s grand story of success: "Just follow the authorities ". Becoming an echo of the grand abstract narrative of mathematics brought you through the exams and into a nice position as a math teacher where you job was to echo the same grand narrative.

In late or post-modernity robots are taking over production being easily programmed to follow the authorities. In late modernity the story of success says: "Build your own biographical narrative "(Giddens 1991). A post-modern narrative must contain the subject of a sentence in order to be expanded by the rest of the sentence. To a postmodern student the grand abstract narrative of modern mathematics is pure poison, which he quite naturally must try to avoid, thus enforcing the enrolment problems. Thus late or post-modernity calls for a de-traditionalization of mathematics i.e. of the narrative about numbers and operations, opening a contest between the grand abstract narrative and its competing alternatives. One alternative mathematics is the above mentioned contextual narrative about numbers as social constructions to the social need for differentiating between degrees of many, and operations as social constructions to the social need for reuniting.

The learning lives of Ruth, Peter and Ditte

The purpose of eating food is to integrate it into your system. However two abnormal eating habits exist. You don’t eat at all (anorexia). You eat it for later vomiting (bulimia). In both cases the food evade your system. When mental food as mathematics is served, another evading habit is added: You don’t mind eating it but failing to see any meaning in it you don’t let it into your system. Ruth, Peter and Ditte are three Danish high school graduates all evading math-based further education. Ruth met postmodern algebra the first year making her want more and echo-teaching the second year making her stop “eating” in class and fail the exam. Peter and Ditte both got top results in mathematics, but Peter cheated the system through rote learning and Ditte can’t see any meaning in math. Asked to tell about their learning life within mathematics the following narratives emerged.

> Ruth: In grade seven we were making graphs with negative and positive scales, how to draw them, and so when we asked why we made them, what purpose it kind of had, well you just had to make them, that’s how it was. You didn’t get any explanation as to the reality behind this math. Our number two

teacher, we had two different teachers that year, came in and was drunk as a lord, so we didn't learn very much. In the high school, where I had math the first year, and I must say this was just what suited my head, at any case the teaching method was different, one I think should be spread out, for the teacher had a quite different way to explain, one you could understand. You really felt you learned something, even if it was difficult for you, you still learned it along the road. Even if you were a little behind, because first of all, you had a good relationship to the teacher, you felt the teacher was part of the class, not a separate part of the class thinking he has a higher authority. We really felt, the teacher was on the same level, as to authority any way, of course as to mathematics he was at a higher level. I do not know what I can explain about that method, anyway there was something about it that was incredibly attractive. I can compare with mathematics the second year. The method, the teacher used the second year is simply one I find unsuitable and I know that many from the class agree. You felt precisely the opposite, the teacher was not so to speak a part of the class, you felt he was very authoritarian, he used his authority and taught directly from the book, and that helped us very little, when you go home and read the book and prepare your homework and then go back to school and say, that you don't understand it, the teacher explains it and mostly it helps only little for he explains it directly as it is in the book. He could have turned it, but he didn't.

Peter: In the primary school I was not very critical, I swallowed everything that came in any subject. I had no priorities, mathematics was as good as anything else. Languages interested me most but all of a sudden I found out that I wanted a career and an education and I was convinced that mathematics was necessary. So I chose to major in mathematics in the high school. In the first lesson we got a little chock, we were presented to complex numbers just for fun. But the problem was, for there were problems, I did not find mathematics appealing, the problem was I could simply not see where we were heading with the different areas from the first and second year. And we really were not told where we were heading with polynomials etc. I was always asking: why are we doing this? I was not caught by it, it was no sport to me. And then I started to find out what really interested me. And here I really needed these explanations- Okay! It was not until the third and last year I realised what we had been doing the first years. So in the beginning I was just disappointed over what went on, but I participated since it was a subject. In the final year it was a different kind of disappointment but also a different kind of insight, now I had a kind of aesthetic viewpoint: Okay now we know a hole lot of mathematics, and probably it is part of a classical education to know this also. But still I find it incredibly outdistanced, outdistanced from the realities. I can't find anything, I can - of course it is was everybody says, but sincerely I have great difficulties seeing any meaning in most of what we have done. Except that I can now say that I am a mathematician like Leibniz and Descartes. I have not had a mathematical thought since the oral exam, and probably I am about to loose what I have learned or learned by heart. I guess I have to admit that this is how I treated science. The important thing was to learn methods for homework, that is what matters when you don't feel for math, to cope in the best possible way. So I learned how to cheat them by learning by heart.

Ditte: It is clear that in high school mathematics gets to a higher level, it becomes more abstract, and often you think, what is the use of this. But the two first years I had a good teacher who could give examples. We followed the book, but often he gave a different explanation, so when we read the book at home we had two explanations. In the last year we followed the book strictly, nothing beyond, very quietly and slowly. And boring, I would say. Often I thought: Really, this is not - this I don't feel like, there is nothing to it, this is far away from reality. I have lost the interest for mathematics, but it has not been difficult. Often in mathematics you see it, you hear about it, you get an explanation. Then you think, this I can't grasp because it is so intangible. But then it comes slowly when it is repeated and you try it on some examples. But I think it is to a great extent just a lot of things you just use without understanding it. You learn it but you don't understand it, it does not really get in. It is only a lot of formulas you use a lot of times. And that is what I find a little stupid about mathematics: you just learn how to do it and then you get the answers right and then there is no more to it. *What do you need?* To get a real understanding. You don't get that by just applying your formula list and then just reel off. *What do you mean by a real understanding?* Where it comes from, what the connection is. Why the formula is correct. Of course we learn a lot of proofs, but I still think, that - well you just have to accept it is like that. *When you say where it comes from are you thinking about the daily life and the real world?* Yes, of course it is somewhat difficult to apply calculus to daily life, but yes, what it is used for. But there is no room and no time for that. *What do you mean with no time?* When you have finished an area, then you have just learned it and then you can use it for your homework. And then we continue with the next area. Preparing for the exam you get a little deeper into it, but still I don't know what use it has, you cant find that out just like that.

Ditte and Peter was both asked to explain the enrolment problem.

> Ditte: I think that, and to me it is also like that, mathematics is somewhat to narrow, in a sense. It is, and now I say it again, too boring. *How can mathematics be made more attractive?* But it can't - mathematics is mathematics! I really don't know.

> Peter: An important explanation is the way mathematics has outdistanced itself. I have often thought we might as well have played backgammon in the math class. It appears to be a vicious spiral, the science project. In the high school we get trained for university where we are trained to become teachers that can train new students etc. etc. etc. Applications simply have too little room. They don't give their own subject a chance, it is not allowed to fascinate people, on the contrary it scares people.

Conclusion

To make it "eatable" for toady's students, mathematics should respect its origin and present itself as it emerged, i.e. as abstractions from activities and not as examples of abstractions. The original meaning of the word "school" is time-out: Stop the activity for a moment of reflection and communication before returning to the activity. Classrooms could be transformed into a workplaces for counting and calculating activities reported by equations illustrated and solved by using information technology. And just temporarily stopped for some moments of "school". That learning can take place under such conditions is argued by several theorists. J. Lave talks about situated learning. D. Schön talks about the reflective practitioner and M. Polanyi talks about tacit knowledge. Most people are very competent in their mother language but unable to formulate grammatical knowledge, this knowledge is tacit. If we hold that both a word language and a number language is a human right, maybe we should treat the two languages alike and allow both grammar and mathematics to be to some extent and to some students' tacit knowledge.

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21. CERME Conferences

CERME: Congress of the European Society for Research in Mathematics Education

CERME 10 Irland: Cup-counting and Calculus in Preschool and in Special Needs Education

To improve PISA results, institutional skepticism rethinks mathematics education to search for hidden alternatives to choices institutionalized as nature. Rethinking preschool and primary school

mathematics uncovers cup-counting in bundles less than ten; as well as re-counting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, information and communications technology, a calculator can predict re-counting results before being carried out manually. By allowing overloads and underloads when re-counting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary classes only allowing ten-counting and on-top addition to take place.

Keywords: Numbers, numeracy, addition, calculus, elementary school mathematics.

Decreased PISA performance despite increased research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

Created to help students cope with the outside world, schools institutionalize subjects as inside means to outside goals. To each goal there are many means, to be replaced if not leading to the goal; unless a means becomes a goal itself, thus preventing looking for alternative means that could lead to the real goal if difficult to access. So we can ask: Does mathematics education have a goal-means exchange seeing inside mathematics as the goal and the outside world as a means?

Once created as a means to solve an outside problem, not solving the problem easily becomes a means to necessitate the institution. So to avoid a goal/means exchange, an institution must be reminded constantly about its outside goal. Institutional skepticism is created to do precisely that.

Institutional skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by Plato philosophy presenting choices as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser & Strauss, 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget, 1970). In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnoses, and education all presenting patronizing choices as nature (Lyotard, 1984; Tarp, 2004).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino, 2004, p. 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino, 2004, pp. 186–187). Inspired by Heidegger, Arendt divided human activity into labor and work aiming at survival and reproduction, and action focusing on politics, creating institutions to be treated with utmost care to avoid the banality of evil by turning totalitarian (Arendt, 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction between existence and essence offers a perspective to distinguish between one goal and many means.

Mathematics as essence

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite Many’ in Arabic.

Then the invention of the concept SET allowed mathematics to be a self-referring collection of ‘well-proven’ statements about ‘well-defined’ concepts, i.e. as ‘MetaMatics’, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. Thus SET transformed grounded mathematics into a self-referring ‘MetaMatism’, a mixture of MetaMatics and ‘MatheMatism’ true inside a classroom but not outside where claims as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days. And, as expected, teaching numbers without units and meaningless self-reference creates learning problems.

Mathematics as existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry. Algebra contains four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reversing splitting operation. So, with a human need to describe the physical fact Many, algebra was created as a natural science about Many. To deal with Many, we count by bundling and stacking. But first we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written less sloppy. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc.

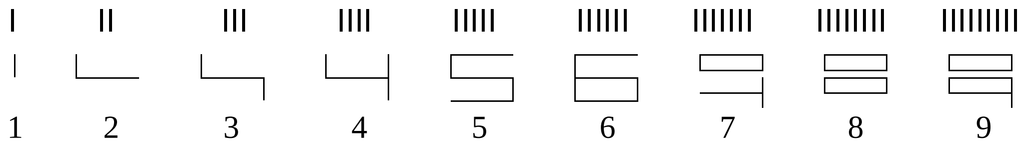
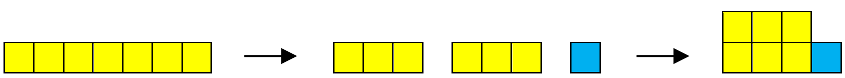


Figure 1: Digits as icons containing as many sticks as they represent

Holding 4 fingers together 2 by 2, a 3year old child will say ‘That is not 4, that is 2 2s. This inspires ‘cup-counting’ bundling a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1 where the bundles are placed in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then we describe by icons, first using ‘cup-writing’, T = 2)1, then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Moving a stick outside or inside the cup changes the normal form to overload or underload form. Also, we can use plastic letters as B and C for the bundles.

T = 7 = IIIIIII → III III I → II) I → 2)1 3s = 1)4 3s = 3)-2 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a cup-number or a decimal number, T = 7 = 2 3s & 1 = 2)1 3s = 2.1 3s.



We live in space and in time. To include both when counting, we introduce two different ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.

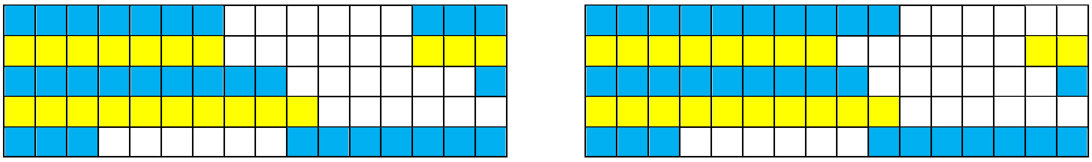


Figure 2: 7 counted in 3s on an abacus in geometry mode and in algebra mode

To predict the result we use a calculator. A stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once.

Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2)1 3s. Likewise, showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2)1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a ‘recount-formula‘, T = (T/B)*B, saying that ‘from T, T/b times Bs can be taken away’; and a ‘restack-formula’, T = (T–B)+B, saying that ‘from T, T–B is left if B is taken away and placed next-to’. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3)2 2s with an outside overload; or as 5)-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T =	Calculator	
B B B B	II II II II	4)0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3)2 2s	4*2 – 3*2	2
B B B B B <u>B</u>	II II II II II <u>II</u>	5)-2 2s	4*2 – 5*2	-2

Figure 3: Recounting 4 2s in the same unit creates overload or underload

To recount in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2)2 5s.

IIII IIII IIII → IIIII IIIII I I → 2)2 5s. With letters, C = BI so that BBB → BB IIII → CC II

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s we take away 5s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be recounted as 2 5s and 2, or as 2)2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to recount 6 in 2s: 6$ = (6/2)*2$ = 3*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = 4*2$ = 8$.

Once counted, totals can be added on-top or next-to.

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s as 1)1 5s that added to the 3 5s gives a total of 4)1 5s.

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4)1 5s. With letters: 3B + 2C = 3B III III = 4BI.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus we get 4)1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2)5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2)5 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2)5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2)6 8s?’, using sticks will give the answer 2)1 3s:

IIIII IIIII IIIII III III I ← IIIII III) IIIII III) IIIIII ← 2)6 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as shown in the gradient formula y’ = Δy/t = (y2 – y1)/t.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Primary schools use ten-counting only

In primary school numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as ‘3 4s = ? tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it is programmed to give the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the short form.

Recounting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without recounting, primary school labels the problem ‘38 = ? 7s’ as an example of a division, 38/7, which is hard to many, or as an equation ’38 = x*7’ to be postponed to secondary school.

Designing a micro-curriculum

With curriculum architecture as one of its core activities, the MATHeCADEMY.net was asked to design a micro-curriculum understandable and attractive to teachers stuck with division problems; and allowing special need students to return to their ordinary class. Two were designed.

In the ‘1 cup and 5 sticks’ micro-curriculum, 5 is cup-counted in 2s as 1)3 2s or 2)1 2s or 3)-1 2s to show that a total can be counted in 3 ways: overload, normal or underload with an inside and an outside for the bundles and singles. So to divide 336 by 7, 5 bundles are moved outside as 50 singles to recount 336 with an overload: 336 = 33)6 = 28)56, which divided by 7 gives 4)8 = 48.

Besides the ‘Cure Math Dislike by 1 cup and 5 sticks’, 8 extra micro-curricula were designed (mathecademy.net/preschool/) where cup-counting involves division, multiplication, subtraction and later next-to and on-top addition, in contrast to primary school that turns this order around and only allows on-top addition using carrying instead of overloads. Thus, if using cup-writing with overloads or underload instead of carrying, the order of operations can be turned around to respect that totals must be counted before being added.

	Carry	Cup-writing	Words
Add	1 4 5 1 7 6 2	4)5 1)7 5)12 6)2 = 62	4 ten 5 1 ten 7 5 ten 12 5 ten 1 ten 2 6 ten 2 = 62
Subtract	1 4 5 1 7 2 8	4)5 1)7 3)-2 2)10-2 2)8 = 28	4 ten 5 1 ten 7 3 ten less2 2 ten 8 = 28
Multiply	4 2 6 * 7 1 8 2	7 * 2)6 14)42 18) 2 = 182	7 times 2 ten 6 14 ten 42 14 ten 4 ten 2 18 ten 2 = 182
Divide	2 4 rest 1 3 | 7 3 6 1 3 1 2 1	7)3 counted in 3s 6)13 6)12 + 1 2 3s)4 3s + 1 24 3s + 1 73 = 24*3 + 1	7ten3 6ten 13 6ten12 + 1 3 times 2ten4 + 1 3 times 24 + 1

Figure 4: Cup-writing with overloads and underloads instead of carrying

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reporting first with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and underloads. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other. In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2)1 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	‘2.7 5s is also how many 5s?’ IIIII IIIII IIIIII = V V V II = V V V V III 2)7 = 2+1)7–5 = 3)2 = 3+1)2–5 = 4)-3 So 2.7 5s = 3.2 5s = 4.-3 5s	(2*5+7)/5 (2*5+7) – 3*5 (2*5+7) – 4*5	3.some 2 -3
M4	2 5s is how many 4s?’ IIIII IIIII = IIII I IIII I = IIII IIII I I	2*5 / 4 2*5 – 2*4	2.some 2

	So 2 5s = 2.2 4s	
M5	‘2 5s and 4 3s total how many 5s?’ IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 4.some (2*5+4*3) – 4*5 2
M6	‘2 5s and 4 3s total how many 8s?’ IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 2.some (2*5+4*3) – 2*8 6
M7	‘2 5s and ? 3s total 4 5s?’ IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 3.some (4*5 – 2*5) – 3*5 1
M8	‘2 5s and ? 3s total how 2.1 8s?’ IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(2*8+1 – 2*5)/3 2.some (2*8+1 – 2*5) – 2*8 1

Figure 5: A calculator predicts counting and adding results

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. For further details watch www.youtube.com/watch?v=IE5nk2YEQIA.

After these micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus cup-counting and a calculator for predicting recounting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal-means exchange had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

Literature on cup-counting

No research literature on cup-counting was found. Likewise, it is not mentioned by Dienes (1964).

Conclusion and recommendation

Institutionalized education sees mathematics as a goal in itself that, once learned, can be applied to outside tasks as word problems and modeling; and reaching its outside goal, mastering Many, may be possible if retaking its original meaning as a mere label for two activities, algebra and geometry going hand in hand at all levels. However, SET transformed grounded mathematics into self-referring and meaningless MetaMatism hiding the natural ways to mastering Many by forcing learners to add without units instead of counting with units; by insisting that only ten-counting is allowed; by using the word natural for numbers with misplaced decimal point and the unit left out; by reversing the natural order of the basic operations division, multiplication, subtraction and addition; and by neglecting activities as creating or removing overloads and double-counting.

To see how mathematics looks like if built as a natural science about its root, the physical fact Many, institutional skepticism has used the existentialist distinction between existence and essence to uncover ‘ManyMatics’ as a hidden alternative to the ruling tradition. Dealing with Many means bundling and cup-counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition. Strong primary school traditions might be softened by introducing ManyMatics micro-curricula in preschool and in special needs education. To avoid goal-means exchanges: Think things, teach existence, do not preach essence.

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CERME 11 Holland: Addition-free Migrant-Math Rooted in STEM Re-Counting Formulas

A curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to quickly become STEM pre-teachers and pre-engineers. Typical multiplication formulas expressing re-counting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

Keywords: STEM, migrant, elementary school mathematics, curriculum, PISA.

Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise has funding, see e.g. Swedish Centre for Mathematics Education. Yet, despite extra research and funding, decreasing Swedish PISA result caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change" since "more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (p. 3).

To find an unorthodox solution we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM based core curriculum allowing young migrants to return as STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to help migrants and how to improve schools in Sweden and elsewhere.

Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). As to institutions, of which mathematics education is an example, he talks about rational action "in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement (p. 84)."

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Difference research looks at mathematics education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' (Tarp, 2017) is searching for hidden differences making a difference. An additional inspiration comes from existentialist philosophy described by Sartre (2007, p. 20) as holding that "Existences precedes essence". So, to avoid a goal displacement in math education, difference-research asks: How will math look like if grounded in its outside root, Many?

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Meeting Many creates a 'count-before-adding' curriculum

Meeting Many, we ask "How many in Total?" To answer, we total by counting to create number-language sentences as e.g. T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence; and connecting the outside total T with its inside predicate 2 3s (Tarp, 2018b).

Rearranging many 1s into one symbol with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

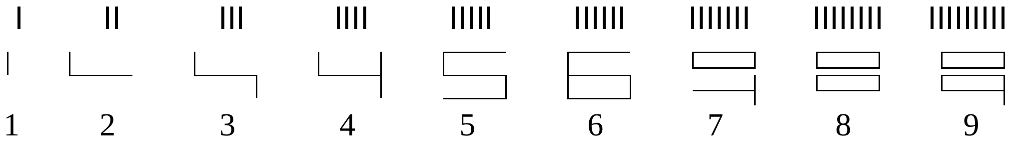

Figure 1. A digit contains as many sticks as it represents if written less sloppy

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Re-counting in the same unit or in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2B1 2s as 1B3 2s with an outside 'overload'; or as 3B-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; or 336 = 35B-14, so 336/7 = 5B-2 = 48. Re-counting in a different unit means changing unit, also called proportionality. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer, '2.some', predicts that the singles come from taking away 2 5s, now asking '3*4 – 2*5'. The answer, '2', predicts that 3 4s can be re-counted in 5s as 2B2 5s or 2.2 5s.

Re-counting to and from tens

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer directly as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Re-counting from tens to icons by asking '35 = ? 7s' is called solving an equation x*7 = 35. It is easily solved by re-counting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we re-count 6 in the per-number 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

A short curriculum in addition-free mathematics

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc.; or 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

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Meeting Many in a STEM context

OECD (2015b) says: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.' STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities.

Nature as heavy things in motion

To meet, we must specify space and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time. A falling ball introduces nature's three main factors, matter and force and motion, like the three social factors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

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Warming and boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam. If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving material in water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so re-counting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

Building batteries with water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H_2O, and carbon dioxide CO_2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH_4, the energy comes from using 4 Os to burn it.

Technology and engineering: letting steam and electrons produce and distribute energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units. Thus a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, p*V = n*R*T, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

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An electrical circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit. Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere. To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, because of proportionality the latter needs twice the current, and consequently half the resistance of the former.

How high up and how far out

An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total sped with the same factor sin45 = cos45 = 0,707.

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Adding addition to the curriculum

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = ΔT/4. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. The number formula T = 456 = 4*B^2 + 5*B + 6*1 shows there are four ways to unite numbers: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b).

Conclusion and recommendation

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and re-counting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a). Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing line-numbers with block-numbers; and allows learning core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017). Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

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Workshop in Addition-free STEM-based Mathematics

Meeting many

Meeting many inspires 'bundle-counting', recounting a total in icon-bundles. Thus, a total T of 5 1s is recounted in 2s as T = 2 2s & 1; and described by 'bundle-writing', T = 2B1 2s, or 'decimal-writing', T = 2.1 2s, where a decimal point separates the inside bundles from the unbundled outside the bundle-cup.

A calculator thus uses a 'recount formula', T = (T/B)*B, to predict that 'from T, T/B times, Bs can be taken away'. This recount formula occurs all over mathematics: when relating proportional quantities as y = c*x; in trigonometry as sine and cosine and tangent, e.g. a = (a/c)*c = sinA*c; in coordinate geometry as line gradients, Δy = (Δy/ Δx)*Δx = c*Δx; and in calculus as the derivative, dy = (dy/dx)*dx = y'*dx.

Recounting in the same unit and in a different unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2B1 2s as 1B3 2s with an outside 'overload'; or as 3B-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48.

Recounting in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, 2.some, predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer, 2, predicts that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Recounting to and from tens

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation x*7 = 35. It is easily solved by recounting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

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CERME 12 Italy: Bundle-Numbers Bring Back Brains from Special Education

Outside, addition folds, but multiplication holds since factors are units while addition presupposes like units. This creates two paradigms in mathematics, an outside 'unit' paradigm, and an inside 'no-unit' paradigm making mathematics a semi-greenhouse. To make mathematics a true science with valid knowledge, we ask what mathematics can grow from the bundle-numbers with units that children use before school, being areas instead of points on a number line. Recounting 8 in 2s creates a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. By changing units, it occurs as proportionality formulas all over STEM. So, outside mathematics with units is the same as inside mathematics with units, only the order is different. Its direct links to outside things provides a usability that brings back brains from special education. And, with multiplication preceding it, addition occurs as integral calculus only, unless taking place inside brackets with like units outside.

Keywords: Arithmetic, numeracy, STEM education, mathematics anxiety, special education

The Two Paradigms in Mathematics Education

The necessity of numbers and calculations as social and individual tools makes them educational tasks in school. Writing the book 'Mathematics as an Educational Task', Freudenthal (1973) succeeded in giving university mathematics monopoly by claiming that mastering mathematics is the only way to later mastering Many. So, from day one in school, or even in preschool, children are forced to learn its foundation, the one-dimensional number line were 1+1 = 2, despite the fact that, with units, this is seldom the case.

But, numerous international tests together with more than 50 years of mathematics education research following the first International Congress on Mathematics Education, ICME, in 1969 have shown that mathematics is hard to learn. Consequently, more learning must come from more research mediated by more facilitators to more educated teachers so they can be more successful with more hard-working students helped by more advanced technology. In short, send more money.

Difference research (Tarp, 2018) instead asks the Cinderella question: are there hidden alternative ways to master Many that evades the hard-to-learn 'no-unit' mathematics? What if children already learn to master Many from adapting to it, can school then develop this into mastering a mathematics that may be different from the school version? So, we ask: What mathematics can grow from the mastery of Many children develop before school?

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists in space and time: bundles of 2s, and 2 of them.

So, children adapt to Many by using 2-dimensional bundle-numbers with units. And they use full sentences as in the word-language with a subject, 'that', and a verb, 'is', and a predicate, '2 2s', which shortened transforms a number-language sentence into a formula '*T* = 2 2s'.

However, the number line does not include units. Instead, school teaches a mathematics that is built upon the assumption that 1+1 = 2 unconditionally. And that thereby fails to meet the basic condition of a science: its statements must not be falsified outside. Which '1+1 = 2' typically is when including units: 1 week + 1 day = 8 days, 1 km + 1mm = 1 km, etc.

Thus, where school works with one-dimensional line numbers without units, children work with two-dimensional area numbers with units. So, there seems to be two paradigms in mathematics. The first is a ruling 'no-unit' paradigm that makes mathematics a semi-greenhouse since outside, addition folds and multiplication holds: 2x3 = 6 simply states that 2 3s can be recounted to 6 1s. The other paradigm is an opposite 'unit' paradigm' where numbers always carry units.

Inside the ‘no-unit’ paradigm, outside sentences as *T* = 2 3s are shortened to 2, thus leaving out the subject, and the unit. This is called an inside modeling of the outside world. The reverse then is called an outside ‘de-modeling’ or reifying of an inside statement (Tarp, 2020).

As described by Kuhn (1962), to have a career, a researcher must work inside the ruling paradigm or ‘truth regime’ (Foucault, 1995) where it is more important to include library references than to perform laboratory testing. A paradigm shift first comes when somebody ignores the existing literature, and use outside validation as the only criteria for quality.

One answer to the above question is given by Tarp (2018, 2020). This answer will now be brought to a more detailed level to design micro-curricula to be tested in design research (Bakker, 2018). Because of space limitations, micro-curricula designs are left out here.

Counting in Time with Sequences

Many occurs in time and space. Repetitions in time may be represented in space by a row of sticks. In space, a lot may be rearranged in a row that is transformed into a total by repeating pushing away one item at a time, and at the same time wording the actual total. Counting outside the ‘no-units’ paradigm we always include units when describing what exists. So, we say ‘0 bundle 1’ instead of just ‘1’.

Number-names become flexible when allowing ‘overloads’ as ‘twenty-nine, twenty-ten, twenty-eleven’. And, when allowing ‘underloads’ counting ‘bundle-less2, bundle-less1, bundle’ instead of ‘8, 9, ten’.

…

Counting in Space with Icons for Digits and Bundling

In space, four ones may be rearranged as one four-icon called a 4-digit. And that the same is almost the case with the other digits also. So basically, a digit is an icon with as many sticks or strokes as it represents if written less sloppy (Tarp, 2018). Bundling in tens, ten has no icon, since ten 1s is 1 ten-bundle and no unbundled: *T* = ten = 1Bundle0 tens = 1*B*0 tens, modeled inside the ‘no-unit’ paradigm as 10.

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Recounting in and from Tens, we Solve Equations and meet Algebra

Recounting from tens to icons means asking, e.g., “How many 2s in 8?”. Using *u* for the unknown number, this may be written as an equation “*u* x 2 = 8”. But, since 8 can be recounted in 2s as 8 = (8/2) x 2, we see that *u* = 8/2. So, the equation *u* x 2 = 8 is solved by *u* = 8/2, i.e., by moving the known number to the opposite side with the opposite calculation sign. After solving an equation, the answer must be tested in the original equation: With *u* = 8/2 = 4, *u* x 2 = 4 x 2 = 8 as expected. Later we will see that it resonates with the formal definition for other operations, so basic equations are solved by moving to opposite side with opposite calculation sign:

u + 2 = 8	*u* x 2 = 8	*u* ^2 = 8	2^*u* = 8
u = 8 – 2	*u* = 8/2	$u = \sqrt[2]{8}$	*u* = log28

Figure 1. Solving equations by moving to opposite side with opposite calculation sign

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Recounting in Physical Units Creates Per-numbers

Inside the ‘no-unit’ paradigm, shifting physical units is seen as an outside application of proportionality where division allows finding the unit cost to answer the two question types: “With 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?” Outside the ‘no-unit’ paradigm, recounting a total that is already counted in one physical unit creates a ‘per-number’ as 3$/2kg, used to easily shift units by recounting: *T* = 6 kg = (6/2) x 2 kg = (6/2) x 3 $ = 9$; and *T* = 15$ = (15/3) x 3$ = (15/3) x 2kg = 10 kg.

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Trigonometry Recounts the Sides in a Box Halved by its Diagonal

Inside the 'no-unit' paradigm, trigonometry is taught after plane and coordinate geometry, and with heavy emphasis on trigonometric identities. Outside, trigonometry are per-numbers coming from recounting the sides in a height-x-base box halved by a diagonal cut, which produces two like right triangles with horizontal base-side, *b*, a vertical height-side, *h*, and a slant cut-side, *c*. Recounting the height in the base we get height = (height / base) x base = tangent *A* x base, shortened to *h* = (*h* / *b*) x *b* = tan *A* x *b* = tan *A b*s, giving the formula tangent *A* = height / base, or tan *A* = *h*/*b*.

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Once Counted and Recounted, Total can Add Next-to or On-top

Inside the 'no-unit' paradigm, numbers are seen as placed along a one-dimensional number line with addition defined as counting on. Outside, numbers carry units and become 2-dimensional stacks with areas that add next-to or on-top. Adding 2 3s and 4 5s next-to as 8s means adding or integrating the areas, also called integral calculus. Adding 2 3s and 4 5s on-top, the units must be the same by squeezing one or both stacks, i.e., by recounting one or both. So, when adding stacks, multiplication comes before addition. The recount formula predicts the result on a calculator by entering (2 x 3 + 4 x 5)/*B*, where *B* can be 3 or 5 or 8.

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Adding Per-Numbers and Fractions

Inside the 'no-unit' paradigm, the concept 'per-number' is not accepted; and fractions are added without units, e.g., 1/2 + 2/3 is 7/6. Outside, this hold only if the fractions are of the same total. It does not hold with like units since 1/2 of 1 pie plus 2/3 of 1 pie total 7/12 of 2 pies; and since 1/2 of 2 apples plus 2/3 of 3 apples total 1+2 of 2+3 apples, i.e., 3/5 of the 5 apples, and not 7 apples of 6 as the 'no-unit' paradigm teaches.

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Subtracting and Adding Numbers with like Units

Inside the 'no-unit' paradigm, 1digit numbers are drawn serial next-to to find the result by counting on from 6 or 9. Outside, they are drawn parallel on-top showing that *T* = 6 + 9 = 2*B*3 6s = 2*B*-3 9s, that both recounts as 2*B*-5 tens, or 1*B*5 tens. Or, directly with flexible bundle-numbers, *T* = 6 + 9 = *B*-4 + *B*-1 = 2*B*-5 = 15. However, subtraction before addition will allow using the hands to show the result after predicting it with a calculator. Fingers or snap-cubes show that 5 – 3 = 2. We also observe that 5 = 2 + 3, so that 5 = (5 – 3) + 3. With unspecified numbers this gives a 'restack formula' *T* = (*T*–*B*) + *B* saying "From a total *T*, *T*–*B* is left when pulling away *B* to place next-to". This formula solves addition-equations.

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Change by Adding or by Multiplying

Inside the 'no-unit' paradigm, change by adding or multiplying is called arithmetic and geometric sequences that are added as series. Outside, we see that adding 3$/day to 2$ gives a total of *T* = 2 + 3 x *n* after *n* days. The general formula, *T* = *b* + *a* x *n*, is called change by adding, or linear change. Also, we see that adding 3%/day to 2$ gives a total of *T* = 2 x 103%^*n* after *n* days since adding 3% means multiplying with 103%. The general formula, *T* = *b* x *a*^*n* = *b* x (1+*r*)^*n*, is called change by multiplying or exponential change. Combining the two gives a simple formula for saving, *A*/*a* = *R*/*r*, in an installment plan racing with a Debt-formula, *D* = *D*o x (1+*r*)^*n*. Here *a* and *r* is the per-day input, and *A* and *R* is the final output, where 1+*R* = (1+*r*)^*n*.

Conclusion. Finally, a Communicative Turn in Mathematics Education

It goes without saying that a total must be counted before being added. Where small totals may be glanced directly in space, larger totals need to be counted in time by pushing away 1s, i.e., by dividing by 1. Here the recount formula gives, e.g., 8 = (8/1) x 1 = 8 x 1, which shows that the total's space-number is the same as its counted time-number.

But when counting in 1s, we never meet the bundles-of-bundles since the square, 1 1s, is still 1, whereas the square, 2 2s, is 4. And we may not see that in reality we bundle in tens, and ten-tens, etc., since we may never push away tens, the real unit in our number system as expressed by the full number-formula, the polynomial, T = 345 = 3 Bundle-bundles + 4 Bundles + 5 = 3B^2+4B+5.

Whereas, if counting in 3s and using the name '1 Bundle 0', for 3, we meet 9 as the bundle-of-bundles square, which may inspire us to use the same name for hundred when counting in tens. So, we need to count in at least 2s to see the nature of outside bundle-counting. 1 is not a prime unit, as the other prime units that cannot split into new prime units.

The original question may now be answered in the following way: Children count in bundles, which gives the recount formula, $T = (T/B)\text{x}B$, as a direct key to core mathematics.

By changing units, it creates per-numbers that add by integrating their areas as in calculus, and that opens up to proportionality and linearity, and thus to countless STEM formulas, and to solving their equations. It makes trigonometry precede plane and coordinate geometry. It shows that per-numbers and fractions are not numbers, but operators needing a number to become a number. It shows that counting before adding makes the two basic counting operations, push and stack, division and multiplication, precede subtraction and addition that is ambiguous with its two options, next-to and on-top. It shows our four basic operations as icons for the outside counting actions: push, lift, pull and unite. It shows that negative numbers, decimals and fractions are different ways to count the unbundled. But, most importantly, it shows the power of formulas as predicting number-language sentences making us master many in nature and in society, and in time and space.

So, building on the mastery of Many developed when adapting to many before school, a 'counting before adding' curriculum allows children to outside master the same mathematics as is taught with great difficulties inside the 'no-units' paradigm's paradigm seeing its very foundation, 1+1=2, fold when meeting units. So once tested, flexible bundle-numbers with units may also fold the myth "math is hard, and needs more funding." Meaning that we can finally have a communicative turn in number-language education as the foreign language education had in the 1970's (Widdowson, 1978). Feynman famously asked: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words?" Certainly, the recount formula is a candidate. So why not enlighten humans about it instead of forcing them inside the darkness of a paradigm without units.

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Recounting before Adding leads Directly to STEM

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Adding Per-Numbers and Fractions

Inside the 'no-unit' paradigm, the concept 'per-number' is not accepted; and fractions are added without units, e.g., 1/2 + 2/3 is 7/6. Outside, this hold only if the fractions are of the same total. It does not hold with like units since 1/2 of 1 pie plus 2/3 of 1 pie total 7/12 of 2 pies; and since 1/2 of 2 apples plus 2/3 of 3 apples total 1+2 of 2+3 apples, i.e., 3/5 of the 5 apples, and not 7 apples of 6 as the 'no-unit' paradigm teaches.

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Subtracting and Adding Numbers with like Units

Inside the 'no-unit' paradigm, 1digit numbers are drawn serial next-to to find the result by counting on from 6 or 9. Outside, they are drawn parallel on-top showing that *T* = 6 + 9 = 2*B*3 6s = 2*B*-3 9s, that both recounts as 2*B*-5 tens, or 1*B*5 tens. Or, directly with flexible bundle-numbers, *T* = 6 + 9 = *B*-4 + *B*-1 = 2*B*-5 = 15. However, subtraction before addition will allow using the hands to show the result after predicting it with a calculator. Fingers or snap-cubes show that 5 – 3 = 2. We also observe that 5 = 2 + 3, so that 5 = (5 – 3) + 3. With unspecified numbers this gives a 'restack formula' *T* = (*T*–*B*) + *B* saying "From a total *T*, *T*–*B* is left when pulling away *B* to place next-to". This formula solves addition-equations.

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Change by Adding or by Multiplying

Inside the 'no-unit' paradigm, change by adding or multiplying is called arithmetic and geometric sequences that are added as series. Outside, we see that adding 3$/day to 2$ gives a total of *T* = 2 + 3 x *n* after *n* days. The general formula, *T* = *b* + *a* x *n*, is called change by adding, or linear change. Also, we see that adding 3%/day to 2$ gives a total of *T* = 2 x 103%^*n* after *n* days since adding 3% means multiplying with 103%. The general formula, *T* = *b* x *a*^*n* = *b* x (1+*r*)^*n*, is called change by multiplying or exponential change. Combining the two gives a simple formula for saving, *A*/*a* = *R*/*r*, in an installment plan racing with a Debt-formula, *D* = *D*o x (1+*r*)^*n*. Here *a* and *r* is the per-day input, and *A* and *R* is the final output, where 1+*R* = (1+*r*)^*n*.

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Conclusion. Finally, a Communicative Turn in Mathematics Education

It goes without saying that a total must be counted before being added. Where small totals may be glanced directly in space, larger totals need to be counted in time by pushing away 1s, i.e., by dividing by 1. Here the recount formula gives, e.g., 8 = (8/1) x 1 = 8 x 1, which shows that the total's space-number is the same as its counted time-number.

But when counting in 1s, we never meet the bundles-of-bundles since the square, 1 1s, is still 1, whereas the square, 2 2s, is 4. And we may not see that in reality we bundle in tens, and ten-tens, etc., since we may never push away tens, the real unit in our number system as expressed by the full number-formula, the polynomial, *T* = 345 = 3 Bundle-bundles + 4 Bundles + 5 = 3*B*^2+4*B*+5.

Whereas, if counting in 3s and using the name '1 Bundle 0', for 3, we meet 9 as the bundle-of-bundles square, which may inspire us to use the same name for hundred when counting in tens. So, we need to count in at least 2s to see the nature of outside bundle-counting. 1 is not a prime unit, as the other prime units that cannot split into new prime units.

The original question may now be answered in the following way: Children count in bundles, which gives the recount formula, *T* = (*T*/*B*)x*B*, as a direct key to core mathematics.

By changing units, it creates per-numbers that add by integrating their areas as in calculus, and that opens up to proportionality and linearity, and thus to countless STEM formulas, and to solving their equations. It makes trigonometry precede plane and coordinate geometry. It shows that per-numbers and fractions are not numbers, but operators needing a number to become a number. It shows that counting before adding makes the two basic counting operations, push and stack, division and multiplication, precede subtraction and addition that is ambiguous with its two options, next-to and on-top. It shows our four basic operations as icons for the outside counting actions: push, lift, pull and unite. It shows that negative numbers, decimals and fractions are different ways to count the unbundled. But, most importantly, it shows the power of formulas as predicting number-language sentences making us master many in nature and in society, and in time and space.

So, building on the mastery of Many developed when adapting to many before school, a 'counting before adding' curriculum allows children to outside master the same mathematics as is taught with great difficulties inside the 'no-units' paradigm's paradigm seeing its very foundation, 1+1=2, fold when meeting units. So once tested, flexible bundle-numbers with units may also fold the myth "math is hard, and needs more funding." Meaning that we can finally have a communicative turn in number-language education as the foreign language education had in the 1970's (Widdowson, 1978). Feynman famously asked: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words?" Certainly, the recount formula is a candidate. So why not enlighten humans about it instead of forcing them inside the darkness of a paradigm without units.

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De-modeling Arithmetic Uncovers a Different Long Division

Outside, addition folds, but multiplication holds since factors are units while addition presupposes like units. This creates two paradigms in mathematics, an outside 'unit' paradigm, and an inside 'no-unit' paradigm making mathematics a semi-greenhouse. Mathematics becomes a true science with valid knowledge by accepting the bundle-numbers with units that children use before school, being areas instead of points on a number line. And that come from recounting, e.g., 8 in 2s thus creating a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. So, where inside, division is defined by sharing, outside it is 'de-modeled as bundle-counting. This puts a new perspective on *long division, often seen as the hardest part of arithmetic even if two different methods exit. A non-representative pilot-test indicates that using bundle-numbers in long division as a third alternative, this difference makes a difference that deserve being tested more thoroughly.*

Keywords: Arithmetic, long division, modeling, numeracy, interaction

The Two Paradigms in Mathematics Education

The necessity of numbers and calculations as social and individual tools makes them educational tasks in school. Writing the book 'Mathematics as an Educational Task', Freudenthal (1973) succeeded in giving university mathematics monopoly by claiming that mastering mathematics is the only way to later mastering Many. So, from day one in school, or even in preschool, children are forced to learn its foundation, the one-dimensional number line were 1+1 = 2, despite the fact that, with units, this is seldom the case.

But, numerous international tests together with more than 50 years of mathematics education research following the first International Congress on Mathematics Education, ICME, in 1969 have shown that mathematics is hard to learn. Consequently, more learning must come from more research mediated by more facilitators to more educated teachers so they can be more successful with more hard-working students helped by more advanced technology. In short, send more money.

Difference research (Tarp, 2018) instead asks the Cinderella question: are there hidden alternative ways to master Many that evades the hard-to-learn 'no-unit' mathematics? What if children already learn to master Many from adapting to it, can school then develop this into mastering a mathematics that may be different from the school version?

Mathematics is perhaps the most important subject in education because of its many outside applications. And it seems hard to deny, that of course mathematics must be learned before it can be applied, unless it is possible to learn kts through its use. Normally mathematics is seen as a tool for modeling outside situations, but here we will raise the inverse question: Will more learn more if de-modeling inside mathematics outside?

Traditionally, mathematics teaches addition first defined as counting on, then subtraction defined as the reverse of addition, then multiplication defined as repeated addition, and finally division defined as reverse multiplication, i.e., as sharing. But, de-modeled, the situation may be different.

What mastery of Many children develop before school?

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists in space and time: bundles of 2s, and 2 of them. So, children adapt to Many by using 2-dimensional bundle-numbers with units. And they use full sentences as in the word-language with a subject, 'that', and a verb, 'is', and a predicate, '2 2s', which shortened transforms a number-language sentence into a formula 'T = 2 2s'.

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Counting in Time with Sequences

Before designing, we reflect on how a row with many sticks is worded by a counting sequence with different names until we reach the bundle, after which a reuse typically takes place when multi-counting the singles, the bundles, the bundles-of-bundles, etc., In the end we get a final total as T = 345 = 3 bundles-of-bundles & 4 bundles & 5 unbundled. Many occurs in time and space. Repetitions in time may be represented in space by a row of sticks. In space, a lot may be rearranged in a row that is transformed into a total by repeating pushing away one item at a time, and at the same time wording the actual total.

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Micro Curriculum, Different Counting Sequences Using Flexible Bundle-numbers

The goal is to see inside numbers as short models for what exists outside, totals of unbundled singles, bundles, bundle-of-bundles, etc. The means is to count a total in flexible bundle-numbers using different bundle-sizes; and to outside de-model inside shortened numbers.

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Counting in Space with Icons for Digits and Bundling

Before designing, we reflect on how in space, four ones may be rearranged as one four-icon called a 4-digit. And that the same is almost the case with the other digits also. So basically, a digit is an icon with as many sticks or strokes as it represents if written less sloppy (Tarp, 2018). Bundling in tens, ten has no icon, since ten 1s is 1 ten-bundle and no unbundled: T = ten = 1Bundle0 tens = 1B0 tens, modeled inside the 'no-unit' paradigm as 10.

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Micro Curriculum, Iconizing Digits and Recounting

The goal is to see digits as number-icons with as many sticks as they represent; and to see the operations division, multiplication, and subtraction as operation-icons for pushing, lifting and pulling away bundles; and to see that when changing units, the result may be predicted by a recount formula saying that a total T contains T/B units B. The means is to rearrange many sticks, cars or dolls into 1 icon; and to recount a row of snap-cubes using a playing card to push away bundles to be lifted into a stack, that is pulled away by a rope or a rubber band; and to let a calculator predict the recounting result before carrying it out.

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Recounting in and from Tens, we Solve Equations and meet Algebra

Recounting from tens to icons means asking, e.g., "How many 2s in 8?". Using u for the unknown number, this may be written as an equation "u x 2 = 8". But, since 8 can be recounted in 2s as 8 = (8/2) x 2, we see that u = 8/2. So, the equation u x 2 = 8 is solved by u = 8/2, i.e., by moving the known number to the opposite side with the opposite calculation sign. After solving an equation, the answer must be tested in the original equation: With u = 8/2 = 4, u x 2 = 4 x 2 = 8 as expected. The 'opposite side & sign' method resonates with the formal definition for division inside the 'no-unit' paradigm. Here 8/2 is what multiplied with 2 gives 8: if 8/2 = u then u x 2 = 8. Later we will see that it resonates with the formal definition for other operations, so basic equations are solved by moving to opposite side with opposite calculation sign:

$u + 2 = 8$	$u \times 2 = 8$	u ^2 = 8	2^u = 8
$u = 8 - 2$	$u = 8/2$	$u = \sqrt[2]{8}$	u = log28

Figure 1. Solving equations by moving to opposite side with opposite calculation sign

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De-modeling Arithmetic

So iconizing numbers and operations allows the inside to be de-modeled outside and obtain new meanings. Where inside, division is defined as sharing, outside it is de-modeled as counting by pushing away bundles, so 8/2 is de-modeled as 8 counted in 2s. And, where inside, multiplication is defined as repeated addition, outside it is de-modeled as a stack of bundles, so 3x4 is de-modeled as 3 4s.

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De-modeling Short Division

The inside division task 56/2 may be de-modeled outside by asking "How many 2s in 56?" We get an answer by re-writing 56 as a flexible bundle number, 56 = 5B6 = 4B16, allowing both the bundles and the unbundled to be counted in 2s as 4B16 / 2 = 2B8 outside, or 28 inside.

De-modeling Long Division

The inside task 384/16 may be de-modeled outside by asking “How many 16s in 384?” We get an answer by re-writing 384 as a flexible bundle number, 384 = 38B4= 32B64, allowing both the bundles and the unbundled to be counted in 16s as 32B64 / 16 = 2B4 outside, or 24 inside. The inside task 395/16 may be de-modeled outside by asking “How many 16s in 395?” We get an answer by writing 395 as a flexible bundle number, 395 = 39B5= 32B75 = 32B64 + 11, allowing both the bundles, and the unbundled, and the surplus to be counted in 16s as 32B64 / 16 + 11/16 = 2B4 + 11/16, or 24 11/16 inside.

De-modeling Multiplication Tables

However, before counting how many 16s there are in a given total, we should make a list of the different 16s, i.e., set up a multiplication table. First, the inside task 2x16 may be de-modeled outside by asking “How many tens are 2 16s?” We get an answer by re-writing 16 as a flexible bundle number, 16 = 1B6, allowing recounting in tens with an overload, as 2 x 1B6 = 2B12 = 3B2 outside, or 32 inside.

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Testing to see if the Difference Makes a Difference

At a school three teachers accepted to try the three different methods in three classes. Class 1 used the traditional DMSB method, Divide, Multiply, Subtract, Bring Down. Class 2 used the ‘Canadian Method; and class 3 used the de-modeling method. To be acquainted with de-modeling in flexible bundle-numbers, class 3 began be realizing that 5 may be de-modeled in 2s in two ways: with an overload as T = 5 = 1B3 2s, or normal as T = 5 = 2B1 2s.

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Once Counted and Recounted, Total can Add Next-to or On-top

Inside the ‘no-unit’ paradigm, numbers are seen as placed along a one-dimensional number line with addition defined as counting on. Outside, numbers carry units and become 2-dimensional stacks with areas that add next-to or on-top. Adding 2 3s and 4 5s next-to as 8s means adding or integrating the areas, also called integral calculus. Adding 2 3s and 4 5s on-top, the units must be the same by squeezing one or both stacks, i.e., by recounting one or both. So, when adding stacks, multiplication comes before addition. The recount formula predicts the result on a calculator by entering (2 x 3 + 4 x 5)/*B*, where *B* can be 3 or 5 or 8.

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Conclusion. De-modeling Creates a Communicative Turn in Math Education

It goes without saying that a total must be counted before being added. Where small totals may be glanced directly in space, larger totals need to be counted in time by pushing away 1s, i.e., by dividing by 1. Here the recount formula gives, e.g., 8 = (8/1) x 1 = 8 x 1, which shows that the total’s space-number is the same as its counted time-number.

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Whereas, if counting in 3s and using the name ‘1 Bundle 0’, for 3, we meet 9 as the bundle-of-bundles square, which may inspire us to use the same name for hundred when counting in tens. So, we need to count in at least 2s to see the nature of outside bundle-counting. 1 is not a prime unit, as the other prime units that cannot split into new prime units.

The original question may now be answered in the following way: Children count in bundles, which gives the recount formula, *T* = (*T*/*B*)x*B*, as a direct key to core mathematics.

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So, building on the mastery of Many developed when adapting to many before school, a 'counting before adding' curriculum allows children to outside master the same mathematics as is taught with great difficulties inside the 'no-units' paradigm's paradigm seeing its very foundation, 1+1=2, fold when meeting units. So once tested, flexible bundle-numbers with units may also fold the myth "math is hard, and needs more funding." Meaning that we can finally have a communicative turn in number-language education as the foreign language education had in the 1970's (Widdowson, 1978). Feynman famously asked: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words?" Certainly, the recount formula is a candidate. So why not enlighten humans about it instead of forcing them inside the darkness of a paradigm without units.

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CERME 13 Hungary: To Master or not to Master Math Before Many, that is the Question

It seems evident that the goal of mathematics education must be to learn mathematics, which once learned inside has many important applications outside by allowing humans to master Many. It is only a shame that mathematics is so hard to learn that it produces many 'slow learners'. However, grand theory asks the opposite question: with 'mastery of math' as an inside means to 'mastery of Many' outside, are there different ways to this end goal, that might even show a different way to later master math if wanted? To get an answer we observe that the mastery children develop when adapting to Many before school leads to 11 micro-curricula so completely different that they create a Kuhnian paradigm-shift within mathematics education as radical as the change from a flat to a round earth.

Keywords: Elementary school mathematics, mathematics curriculum, numeracy, division, calculus

Inside, children adapt smoothly to their outside world

It is glad to see how children vividly communicate about Many before school. And it is sad to see how they then stop doing so, and how more and more are excluded from traditional education and sent to special education. A day inside a classroom tells you why. The students no more communicate about Many, instead a textbook mediated by a teacher teaches them about what they need in order to communicate: multidigit numbers obeying a place value system, first to be added then subtracted with no respect to their units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers.

In a class for 'slow learners', the same is taught but at a slower pace. Which makes you wonder: With their preschool foundation, can children learn number-language through communication as with the word-language (Widdowson, 1978)? And can 'mastery of Many' lead to 'mastery of math' later if needed? In which case, inside mastery of math will not be the only way to master Many outside.

Looking for differences, Difference Research (Tarp, 2018) searching for differences making a difference may inspire micro curricula to be tested using, e.g., Design Research (Bakker, 2018).

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that existence precedes essence so that in a sentence, the subject is more important than any chosen predicate (Marino, 2004). Many therefore should be seen ontologically, what it is in itself, instead of epistemologically, how some may perceive and verbalize it. The tradition sees Many as an example of cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is seen as a union of stacks coming from multi-counting singles, bundles, bundles of bundles, etc., e.g., $T = 345 = 3*BB + 4*B + 5*1$.

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Meeting many, children count with bundles as units

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old typically will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, "That is not four, that is two twos", thus describing what exists in space and in time: bundles of 2s in space, and 2 of them when counted in time. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And they also use full sentences as in the word-language with a subject 'that', and a verb 'is', and a predicate '2 2s', which abbreviated shows a formula as a number-language sentence 'T = 2 2s'.

Children's own flexible bundle-number curriculum

Listening to 'slow learning' students helps understanding when and why they fall behind. Inspired by this we may design question guided micro curricula, MC, to further develop the number-language and mastery of Many children get before school, allowing them a comeback to traditional education.

MC01. Digits

The tradition presents both digits and letters as symbols. A difference is letting students experience themselves digits as not symbols, but icons with as many sticks or strokes as they represent if written less sloppy (Tarp, 2018). In this way students see that inside icons link directly to outside degrees of Many. And that ten has no icon since, as a bundle it becomes the unit, so that two-digit numbers really are two numberings of bundles and of unbundled singles. A guiding question can be "There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?" Outside material could be sticks, a folding ruler, cars, dolls, spoons, etc. Discussing why numbers after ten has no icon leads on to bundle-counting.

MC02. Bundle-counting sequences

Using a place value system, the tradition counts without bundles. A difference is to practice bundle-counting in tens, fives, and threes. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names 'eleven' and 'twelve' are Viking names meaning 'one left' and 'two left', and that the name 'twenty' has stayed unchanged since the Vikings said 'tvende ti'; and that English roughly is a mixture of Viking words labelling concrete things and actions, and French words labelling abstract ideas. The Viking tradition saying 'three-and-twenty' instead of 'twenty-three' was used in English for many years. Now it stops after 20. The Vikings also counted in scores: 80 = 4 scores, 90 = half-5 scores.

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MC03. Bundle-counting with underloads and overloads

Strictly following the place value system, the tradition silences the units when writing 'two hundred and fifty-seven' as plain 257. A difference may be inspired by the Romans using 'underloads' when writing four as "five less one", IV; and by overloads when small children use 'past-counting': "twenty-nine, twenty-ten, twenty-eleven". A guiding question can be. "Let us count with underloads missing for the next bundle. And with overloads as children saying 'twenty-eleven'." Outside material could be sticks, cubes, and an abacus.

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MC04. Doing math with flexible bundle-numbers with units

The tradition uses carrying when adding and multiplying and borrowing when subtracting and dividing. Here, a difference is to use flexible bundle-numbers instead. A guiding question can be. "Let us do inside math with flexible bundle-numbers."

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 1: Doing math with flexible bundle-numbers with units

MC05. Talking math with formulas

In a number-language sentence as "The total is 3 4s", the tradition silences all but the calculation 3*4 recounted to 12. A difference is to use full sentences with an outside subject, a verb, and an inside predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence “T = 5*6 = 30” thus inside predicts that outside 5 6s can be re-counted as 3 tens. A guiding question can be. “Let us talk math with full sentences about what we count and how.”

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MC06. Naming the unbundled singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question can be “How to see the unbundled singles?”. Outside materials can be cubes or an abacus. Before outside recounting 9 in 2s, inside we let a calculator predict the result: Entering 9/2 gives ‘4.some’ predicting that “9 contains 4 2s and then some”. To find those, we outside pull away the 4-by-2 stack, and inside predict the result by entering ‘9 – 4 * 2’ giving 1. So, inside the calculator predicts that 9 recounts as 4B1 2s, which is also observed outside.

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MC07. Changing number units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon-unit to another, from icons to tens, or from tens to icons, or into a square. A guiding question can be “How to change number units?”. Outside materials can be an abacus.

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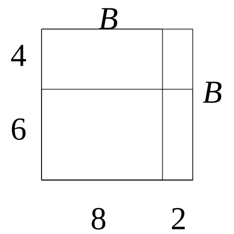	T = 6 * 8 = (B-4) * (B-2) = BB – 2B – 4B + 4*2 = 4B8 = 48	$T = \begin{pmatrix} 1B & -4 \\ 1B & -2 \end{pmatrix}$ = 1BB – 2B – 4B + 4*2 = 10B – 6B + 8 = 4B8 = 48	$T = \begin{pmatrix} 6B & +4 \\ 8B & +2 \end{pmatrix}$ = 48BB+12B+32B+8 = 48BB+44B+8 = 52BB 4B 8 = 5248

Figure 2: Multiplying 6*8 and 64*82 as binomials

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MC08. Changing physical units with per-numbers

The tradition sees shifting physical units as an application of proportionality. Typically, finding the unit cost will answer questions as “with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?” A difference is to use ‘per-numbers’ (Tarp, 2018) coming from double-counting the same quantity in the two units, e.g., T = 3$ = (3$/2kg) * 2kg = p * 2kg, with the per-number p = 3$/2kg, or 3/2 $/kg. A guiding question can be “How to change physical units?”. Outside materials can be coloured cubes.

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MC09. Recounting the sides in a stack halved by its diagonal gives trigonometry

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, recounting the sides in a stack halved by its diagonal. A guiding question can be “How to recount the sides in a stack halved by its diagonal?”. Outside materials can be tiles, cards, peg boards, and books.

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MC10. Adding next-to and on-top

The tradition sees numbers as 1-dimensional line-numbers with addition defined as counting on. A difference is to accept children’s 2-dimensional bundle-numbers that add next-to and on-top. A guiding question can be “How to add 2 3s and 4 5s on-top and next-to?”. Material: cubes and abacus. To add 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing or pulling, inside by recounting to change units. The recount formula predicts the result when entering (2*3+4*5)/B, where B can be 3 or 5 or 8. Added next-to by areas is called integral calculus.

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MC11. Adding Per-Numbers and Fractions

Adding numbers without units may be called 'mathematism', true inside but seldom outside where, e.g., 2m + 3cm = 203cm. A difference respects that the recount-formula shows that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding. A guiding question can be "What is 2kg at 3$/kg plus 4kg at 5$/kg?" Outside materials can be a peg board with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

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We now know 3 of 4 ways to unite parts into a total, and to split a total in parts, the 'Algebra-square' respecting that the Arabic meaning of the word algebra is to re-unite:

Operations unite/split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$T = a + n$ $T – n = a$	$T = a*n$ $T / n = a$
Per-numbers m/s, $/kg, $/100$ = %	$T = \int a*dn$ $dT/dn = a$	$T = a^n$ $\sqrt[n]{T} = a$ $n = \log_a T$

Figure 3: The 4 ways to unite parts into a total, and the 5 ways to split a total into parts

Conclusion

In the mastery that children develop when adapting to Many before school, numbers and operations and functions apparently have different meanings. Numbers are 2D flexible bundle-numbers with units allowing both overloads and underloads. The operation order is the opposite. Division means counting that by pushing away bundles creates a recount-formula with a per-number used to change units, thus used all over STEM. And making functions occur as full number-language sentences from day one. And allowing calculators to predict results. Multiplication means stacking bundles to be pulled away by subtraction to find the unbundled that are presented by decimals, fractions, or negative numbers. So 6*8 is 6 8s, a stack that if recounted in tens changes both width and height outside and introduces early algebra when written with underloads: 6*8 = (B-4)*(B-2) = (10-4-2)*B + 4*2 = 4B8. When counted and recounted, totals may add on-top, or next-to by areas as in integral calculus.

These differences are so radical that they can only be tested with 'slow learners'. Children's own flexible bundle-number curriculum thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre- and in-service teacher education asking the subject instead of the instructor as exemplified on the MATHeCADEMY.net website.

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' We could ask if this is possible if an educational tradition rejects the child's own flexible bundle-numbers with units, and replaces them with inflexible line-numbers without units? So maybe the time has come where mathematics education should stop teaching 'mathematism' to children and instead begin learning from them how to master Many as a means to later master mathematics, if wanted?

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22. The Catania Trilogy 2015: Diagnosing Poor PISA Performance

The Mathematics Education for the Future Project

Decreased PISA Performance in spite of Increased Research

Being highly useful to the outside world, math is a core part of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed by e.g. the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA levels in mathematics decrease from 509 in 2003 to 478 in 2012, far below the OECD average at 494. This made OECD write a report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Created to help students cope with the outside world, schools are divided into subjects that are described by goals and means with the outside world as the goal and the subjects as the means. However, a goal/means confusion might occur where the subjects become the goals and the outside world a means.

A goal/means confusion is problematic since while there is only one goal there are many means that can be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal.

So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? Or, how would mathematics look like if built as a means for proper real world actions?

The three papers below constitute a 'Catania Trilogy' written for 13th International Conference of The Mathematics Education for the Future Project in Catania, Sicily, September 2015.

Conclusion: Goal/Means Confusion leads to Poor PISA Performance

Increased research has let to decreasing PISA math results as In Sweden caused by a goal/means confusion. Grounded as a means to an outside goal, mathematics becomes a natural science about the physical fact Many. This ManyMatics differs from the school's MetaMatism, mixing MetaMatics, defining its concepts as examples from internal abstractions, with MatheMatism, true inside but not outside the classroom.

Content

Count in Icons before Tens, then Add NextTo before OnTop

Abstract

Preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of Many, we can ask: How will mathematics look like if built as a natural science about physical fact Many? To deal with Many we count and add. The school counts in tens, but preschool allows counting in icons also. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. Adding next-to means adding areas, also called integration. So icon-counting and next-to addition offers golden learning opportunities in preschool that is lost in ordinary school allowing only ten-counting to take place.

Math in Preschool – a Great Idea

Mathematics is considered one of the school's most important subjects. So it seems a good idea to introduce mathematics in preschool - provided we can agree upon what we mean by mathematics.

As to its etymology Wikipedia writes that the word mathematics comes from the Greek máthēma, which, in the ancient Greek language, means "that which is learnt". Later Wikipedia writes:

In Latin, and in English until around 1700, the term mathematics more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. (http://en.wikipedia.org/wiki/Mathematics)

This meaning resonates with Freudenthal writing:

Among Pythagoras' adepts there was a group that called themselves mathematicians, since they cultivated the four "mathemata", that is geometry, arithmetic, musical theory and astronomy. (Freudenthal 1973: 7)

Thus originally mathematics was a common word for knowledge present as separate disciplines as astronomy, music, geometry and arithmetic.

This again resonates with the educational system in the North American republics offering courses, not in mathematics, but in its separate disciplines algebra, geometry, etc.

In contrast to this, in Europe with its autocratic past the separate disciplines called Rechnung, Arithmetik und Geomtrie in German were integrated to mathematics from grade one with the arrival of the 'new math' wanting to revive the rigor of Greek geometry by defining mathematics as a collection of well-proven statements about well-defined concepts all being examples of the mother concept set.

Kline sees two golden periods, the Renaissance and the Enlightenment that both created and applied mathematics by disregarding Greek geometry:

Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

Furthermore, Gödel has proven that the concept of being well-proven is but a dream. And Russell's set-paradox questions the set-based definitions of modern mathematics by showing that talking about sets of sets will lead to self-reference and contradiction as in the classical liar-paradox ‘this sentence is false’ being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

Without an agreement as to what mathematics is and with the negative effects of imposing rigor, preschool math should disintegrate into its main ingredients, algebra meaning reuniting numbers in Arabic, and geometry meaning measuring earth in Greek; and both should be grounded in their common root, the physical fact Many. To see how, we turn to skeptical research.

Postmodern Contingency Research

Ancient Greece saw a knowledge controversy between the sophists and the philosophers. The sophists warned that in a republic people must be enlightened about choice and nature to prevent being patronized by choices presented as nature. In contrast to this skepticism philosophers saw the physical as examples of meta-physical forms only visible to the philosophers educated at Plato’s academy, who then should be allowed to patronize.

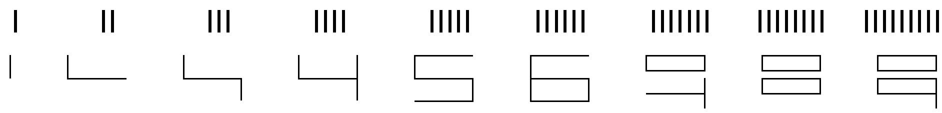
Enlightenment later had its own century, the 18^{th}, that created two republics, an American and a French. Today the sophist warning against hidden patronization is kept alive in the French republic in the postmodern skeptical thinking of Derrida, Lyotard, Foucault and Bourdieu warning against patronizing categories, discourses, institutions and education presenting their choices as nature (Tarp 2004).

Thus postmodern skeptical research discovers contingency, i.e. hidden alternatives to choices presented as nature. To make categories, discourses and institutions non patronizing they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment republic, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

To look for patronization hidden in the words, truths and discourses of math education we ask: How will mathematics look like if built, not as a self-referring science about its own invention Set, but as a natural science about the physical fact Many; and how can this affect early childhood education? The answers are presented in papers and in YouTube videos (2013).

Building a Natural Science about Many

To deal with the physical fact Many, first we iconize, then we count by bundling. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc..

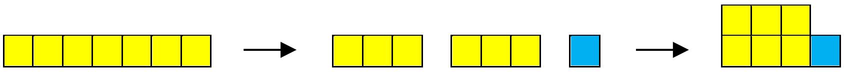


1 2 3 4 5 6 7 8 9

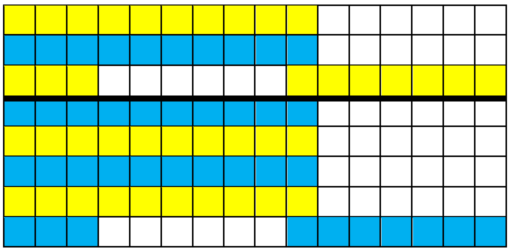
With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. The unbundled can be placed in a right single-cup, and in a left bundle-cup we trade the bundles, first with a thick stick representing a bundle glued together, then with a normal stick representing the bundle. The cup-contents is described by icons, first using ‘cup-writing’ 2)1, then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Alternatively, we can also use plastic letters as B, C or D for the bundles.

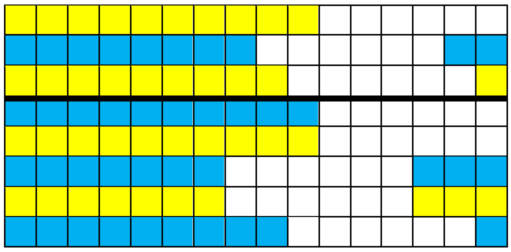
IIIIIII → III III I → ▌▌) I → II) I → 2)1 → 2.1 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number.



We live in space and in time. To include both when counting, we can introduce two different ways of counting: counting in space, geometry-counting, and counting in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.





To predict the counting result we can use a calculator. Building a stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. As to the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once. Thus by entering '7/3' we ask the calculator 'from 7 we can take away 3s how many times?' The answer is '2.some'. To find the leftovers we take away the 2 3s by asking '7 – 2x3'. From the answer '1' we conclude that 7 = 2.1 3s. Showing '7 – 2x3 = 1', a display indirectly predicts that 7 can be recounted as 2 3s and 1.

7 / 3	2.some
7 – 2 x 3	1

Re-counting in the Same Unit and in a Different Unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s, 2.4 2s. Likewise 4.2s can be recounted as 5 2s less or short of 2; or as 6 2s less 4 thus leading to negative numbers:

Letters	Sticks	Calculator		T =
B B B B	II II II II			4.0 2s
B B B I I	II II II I I	4x2 – 3x2	2	3.2 2s
B B I I I I	II II I I I I	4x2 – 2x2	4	2.4 2s
B B B B B	II II II II II	4x2 – 5x2	-2	$5.\underline{2}$ 2s
B B B B B B	II II II II II II	4x2 – 6x2	-4	$6.\underline{4}$ 2s

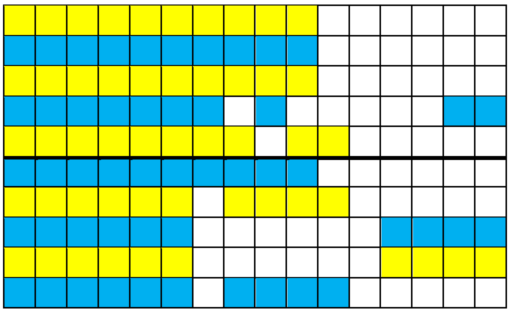
To recount in a different unit means changing unit, called proportionality or linearity also. Asking '3 4s is how many 5s?' we can use sticks or letters to see that 3 4s becomes 2.2 5s.

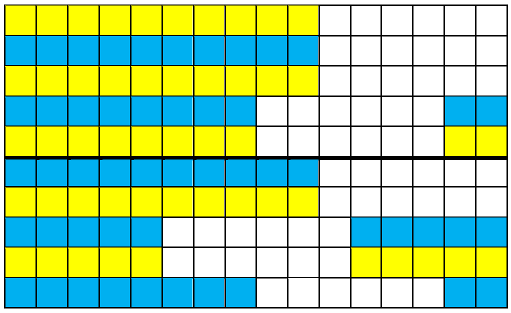
IIII IIII IIII → IIIII IIIII I I → 2) 2 5s → 2.2 5s

or with C = BI, BBB → BBIIII → CCII

Using geometry-counting on an abacus, reserving the bottom line for the single 1s, a stack of 3 4s is moved from left to right on an abacus. The top bundle is changed to 1s in the single line and twice a stick is removed to enlarge the two 4-bundles to 5-bundles. This shows that '3 4s can be recounted as 2.2 5s.'

Using algebra-counting, 3 beads are moved to the right on the bundle-line. Then one 4-bundle is changed to 4 1s on the single-line. Moving 2 beads to the left on the single-line allows enlarging the 4s to 5s thus showing that 3 4s = 2.2 5s





Using a calculator to predict the result we enter '3x4/5' to ask 'from 3 4s we take away 5s how many times?' The calculator gives the answer '2.some'. To find the leftovers we take away the 2 5s and ask '3x4 – 2x5'. With the answer '2' we conclude that T = 3 4s = 2.2 5s.

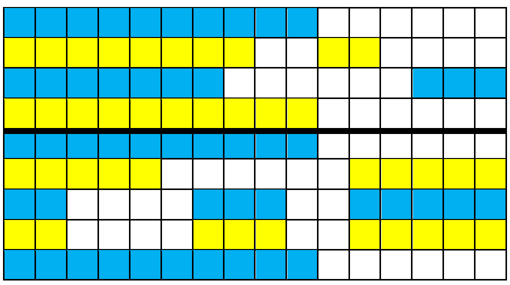
3 x 4 / 5	2.some
3 x 4 – 2 x 5	2

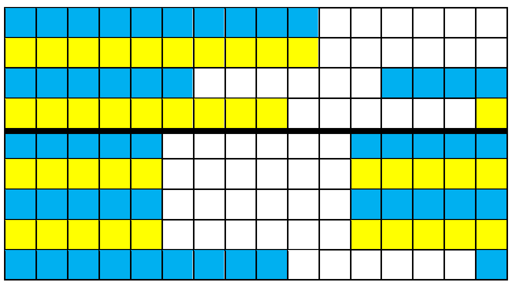
Adding On-top and Next-to

Once counted, totals can be added on-top or next-to. Asking '3 5s and 2 3s total how many 5s?' we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4BI. With sticks:

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1 5s → 4.1 5s,

On an abacus in geometry mode a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now, the 2 3s is changed to 6 1s on the bottom line allowing one additional 5s to be moved to the top of the stack of 5s to show the grand total is 4.1 5s. Using algebra mode, the 3 5s become 3 beads on the bundle line and the 2 3s become 2 beads on the line above. Again the 2 3s is changed to 6 1s on the bottom line allowing one additional bead to be added to the bundle-line to give the result 4.1 5s





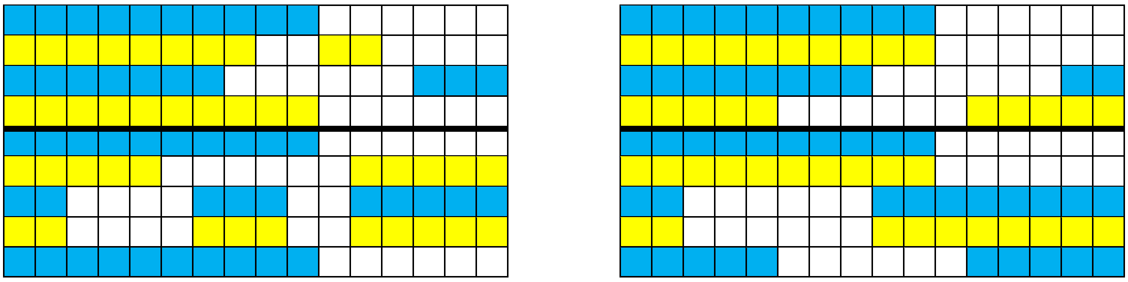
Using a calculator to predict the result we use a bracket before counting in 5s: Asking '(3x5 + 2x3)/5', the answer is 4.some. Taking away 4 5s leaves 1.

(3 x 5 + 2 x 3)/ 5	4.some
(3 x 5 + 2 x 3) – 4 x 5	1

To add next-to means adding areas called integration. Asking '3 5s and 2 3s total how many 8s?' we use sticks or letters to see that the answer is 2.5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5 8s → 2.5 8s

On an abacus in geometry mode a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now a 5-bundle is moved to the single line allowing the two stacks to be integrated as 8s, showing that the grand total is 2.5 8s. Likewise when using algebra mode.



Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking '(3x5 + 2x3)/8', the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2.5 8s.

(3 x 5 + 2 x 3)/ 8	2.some
(4 x 5 + 2 x 3) – 2 x 8	5

Reversing Adding On-top and Next-to

To reverse addition is called backward calculation or solving equations also. To reverse next-to addition is called reversed integration or differentiation. Asking '3 5s and how many 3s total 2.5 8s?' sticks will get the answer 2 3s:

IIIII IIIII IIIII III III ← IIIII III IIIII III IIIII ← 2) 5 8s ← 2.5 8s

On an abacus in geometry mode with 2 8s and 5 moved to the right, first 3 5s is moved to the left, then the remaining is recounted in 3s as 2 3s. Using a calculator to predict the result the remaining is bracketed before counted in 3s.

(2 x 8 + 5 – 3 x 5)/ 3	2
(2 x 8 + 5 – 3 x 5) – 2 x 3	0

Adding the two stacks 2 3s and 3 5s next-to each other means performing multiplication before adding. Reversing integration means performing subtraction before division, as in the gradient formula y' = dy/t = (y2 – y1)/t.

Conclusion

To find how mathematics looks like if built as a natural science about Many, and how this could affect early childhood education, postmodern contingency research has uncovered a 'ManyMatics' as a hidden alternative to the ruling tradition in mathematics. Dealing with Many means bundling and counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition. These golden learning opportunities must be realized in preschool since they are lost from grade one, where the monopoly of ten-counting and the opposite order of operations prevent both from happening. Furthermore, here grounded ManyMatics is replaced by 'MetaMatism', a mixture of 'MetaMatics' turning mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, and ‘MatheMatism’ true inside a classroom but not outside where claims as ‘1+2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days.

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Truth, Beauty, and Goodness in Mathematics Education

Abstract

In math education we can ask how education can lead to mathematics. But we could also ask how mathematics could lead to general educational goals as the three classical virtues: Truth, Beauty and Goodness. To do so, math must change from a self-referring MetaMatism true inside but not outside the classroom to a grounded ManyMatics, a natural science about Many, with numbers as blocks and with algebra as the art of reuniting numbers.

Goals and Means in Mathematics Education

Mathematics education is a core part of a school and is described by goals and means. Typically, mathematics is the goal with assessment focusing on the degree to which it has been learned. As means, different kinds of education are considered: Should the main emphasis be on teaching with high quality in teacher training and textbooks? Or should the main emphasis be on learning with focus on constructivism be it social or radical?

Once a means has been chosen education can begin, hopefully resulting in leading to the goals. However, PISA studies show that student performances are decreasing e.g. in the former model country Sweden seeing its mathematics levels decrease from 509 in 2003 to 478 in 2012 far below the OECD average at 494. This made OECD write a report describing the Swedish school system as being in need of urgent change (OECD, 2015).

Increased funding of mathematics education research in the period seems to have made the situation even worse. So to change the situation, unorthodox methods must be used by e.g. turning the goal and means discussion around and ask: How can mathematics contribute to general educational goals?

As to general educational goals Howard Gardner, known for his theory on MI, multiple intelligences, writes

> In my book *The Disciplined Mind*, published in 1999, I put forth a simple educational agenda: To help students understand, and act, on the basis of what is true, what is beautiful and what is good. I believed – and still believe – in that agenda. (Gardner 2001, xiv)

From this we can ask: how can mathematics be a means leading to the goal of implementing the three classical virtues Truth, Beauty and Goodness?

Truth in Mathematics

As to mathematics, its strength comes from including only well-defined concepts and well-proven statement, and from being highly applicable to the outside world. However, the declining PISA performance in many countries leads to ask: Is it mathe-matics that is taught or ‘meta-matism’, a mixture of ‘meta-matics’ and ‘mathe-matism’?

MetaMatics is mathematics that uses self-reference to define its concepts top-down as examples of abstractions instead of using its historically roots to define its concepts bottom-up as abstractions from examples. Originally Euler defined a function as a common name for calculations containing numbers and letters. The invention of the abstraction Set turned this upside down so that today a function is defined as an example of a many-to-one set relation.

MatheMatism is mathematics that is true inside but not necessarily outside the classroom. Thus the statement ‘2+3 = 5’ is not true with different units, e.g. 2 weeks + 3 days = 17 days. The statement ‘2x3 = 6’ is always true since here 3 is the unit. Likewise with fractions where 1 empty bottle of 2 added to 2 empty bottles of 3 totals 3 empty bottles of 5 and not 7 empty bottles of 6. So to teach mathematics instead of mathematism we must always include the units as shown when writing out numbers fully: 345 = 3x100 + 4x10 + 5x1.

In the questionnaire below, teacher-answers marked with xs differ from the correct answers marked with dots. This and textbook definitions of functions show that what schools teach is indeed metamatism, not mathematics.

This is true	Always	Never	Sometimes
2+3 = 5	x		●
2x3 = 6	x ●		
1/2+2/3 = 3/5		x	●
1/2+2/3 = 7/6	x		●

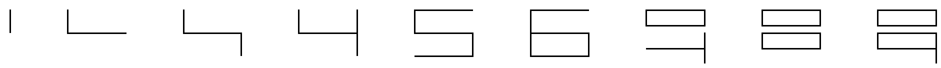
To bring back truth to mathematics it must be rebuilt from its original roots.

Building a Natural Science about Many

The core of mathematics is geometry and algebra, meaning to measure earth in Greek and to reunite numbers in Arabic. This shows that the root of mathematics is the physical fact ‘Many’ as it occurs in space and time.

To deal with Many, first we iconize, then we count by bundling. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. We create icons until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle one, one bundle two etc..

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. So we place 2 sticks in a left bundle-cup and the unbundled we place in a right single-cup.

Writing the total in ‘algebra-form’, the cup-content is described by an icon, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T= 2.1 3s.

Alternatively, we can use plastic letters as B, C or D for the bundles.

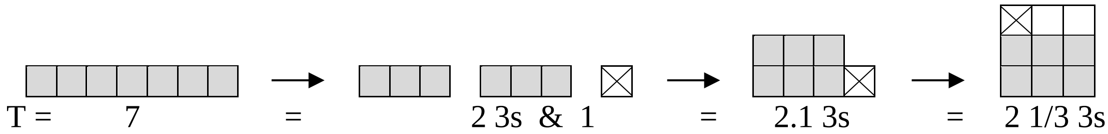
IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BBI → 2BI

A calculator can predict the counting result. To count in 3s we take away 3s, iconized as ‘/3’ showing the broom wiping away the 3s several times. Building a stack of 2 3s we iconize as 2x3 showing a jack used to lift the 3s. And the trace coming from taking away the stack of 2 3s to look for unbundled is iconized as ‘-2x3’. These three operations are called division, multiplication and subtraction respectively.

Entering ‘7/3’ the answer is ‘2.some’. To find the unbundled we take away the 2 3s by asking ‘7 – 2x3’. From the answer ‘1’ we conclude that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 x 3	1

Writing the total in ‘geometry-form we use squares or LEGO blocks or an abacus to stack the 2 3-bundles on-top of each other with an additional stack of unbundled 1s next-to or on-top, thus describing the total as a decimal number 2.1 3s, or as a fraction number 2 1/3 3s counting the unbundled in 3s.



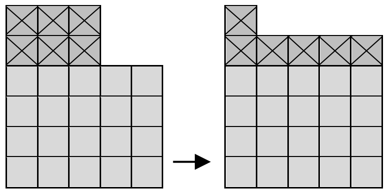
DoubleCounting creates PerNumbers Connecting Units

A physical quantity can be counted in different units. With 4kg = 5$ we have the 'per-number' 4kg/5$ = 4/5 kg/$. To shift from one physical unit to another we simply use the per-number to recount in a new number unit. To change unit is called proportionality, which is one of the core concepts in mathematics.

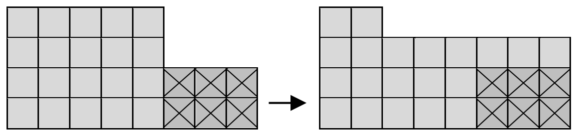
7 kg = ? $	8 $ = ? kg
7 kg = (7/4) x 4 kg = (7/4) x 5 $ = 8.75 $	8 $ = (8/5) x 5 $ = (8/5) x 4 kg = 6.4 kg

Adding Totals

Once Counted, totals can be added on-top or next-to. To add on-top, the units must be changed to be the same, typically by recounting one total in the other's unit. Adding next-to is called integrating areas.



OnTop: 4 5s + 2 3s = 5.1 5s



Next-to: 4 5s + 2 3s = 3.2 8s

NextTo addition is used when adding piecewise constant per-numbers:

4 kg at 5 $/kg + 2 kg at 3 $/kg = (4x5 + 2x3) $ = Σ (per-number x quantity)

Or when adding locally constant (continuous) per-numbers:

6 kg at 5 $/kg decreasing to 3 $/kg = $\int_0^6 (5 + \frac{3-5}{6}u)\,du$

Reversing Addition, or Solving Equations

Reversing addition we ask e.g. '2+? = 8'. With the restack-formula **T= (T-b)+b** we can restack 8 as (8-2)+2 to get the answer 8-2. Reversing multiplication we ask e.g. '2x? = 8'. With the re-count formula **T= (T/b)xb** we can recount 8 as (8/2)x2 to get the answer 8/2. We see that solving equations means moving numbers to the opposite side with opposite sign.

OnTop		NextTo
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3 8s**
? = 8-2	? = 8/2	? = (3 8s – 2 3s)/5

Reversing adding next-to we ask e.g.'2 3s + ? 5s = 3 8s'. To find what was added we take away the 2 3s and count the rest in 5s.

Combining subtraction and division in this way is called reversed integration or differentiation.

The Algebra Project: the Four Ways to Add

Meaning ‘to re-unite’ in Arabic, the ‘Algebra-square’ shows that with variable and constant unit-numbers and per-numbers there are four ways to unite numbers into a total, all present when writing 345 as 3x10^2 + 4x10 + 5x1; and that there are five ways to split up a united total.

Uniting/*splitting*	Variable	Constant
Unit-numbers	**T = a + n**, *T – a = n*	**T = a x n**, *T/n = a*
Per-numbers	**T = ∫ a dn**, *dT/dn = a*	**T = a^n**, *loga(T) = n, n√T = a*

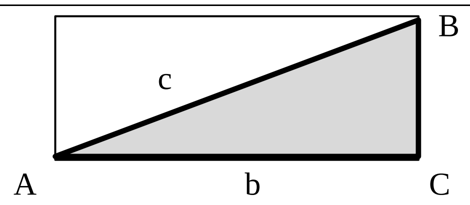
Geometry: Measuring Earth divided into HalfBlocks

Geometry means earth-measuring in Greek. The earth can be divided in triangles that can be divided in right triangles that can be seen as blocks halved by their diagonals thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras theorem a^2 + b^2 = c^2; and connected with the angles by formulas recounting the sides in diagonals:

a = (a/c) x c = sin A x c

b = (b/c) x c = cos A x b

a = (a/b) x b = tan A x b



Different answers to the same Questions

Asking the same questions Q, ‘ManyMatics’ and ‘MetaMatics’ gives different answers A1 and A2

Q: Digits? A1: ‘Icons, different from letters’. A2: ‘Symbols like letters’.

Q: Count? A1: ‘Count in icons before in tens’. A2: ‘Only count in tens’.

Q: Natural numbers? A1: ‘2.3 tens’. A2: ‘23’.

Q: Fractions? A1: ’Per-numbers needing a number to produce a number’. A2: ‘Rational numbers’.

Q: Per-numbers? A1: ‘Double-counting’. A2: ‘Not accepted’.

Q: Operations? A1: ‘Icons for the counting processes’. A2: ‘Mappings from a set-product to a set’.

Q: Order of operations? A1: ‘/, x, - , +’. A2: ‘+, - , x, /’.

Q: Addition? A1: ‘On-top and next-to’. A2: ‘On-top only’.

Q: Integration? A1: ‘Preschool: Next-to addition; Middle school: Adding piece-wise constant per-numbers. High school: Adding locally constant per-numbers’. A2: ‘Last year in high school, only for the few’.

Q: A formula? A1: ‘A stand-by calculation with numbers and letters’. A2: ‘An example of a function that is an example of a relation in a set-product where first component identity implies second component identity’.

Q: Algebra? A1: ‘Re-unite constant and variable unit-numbers and per-numbers’. A2: ‘A search for patterns’.

Q: The root of Mathematics? A1: ‘The physical fact Many’. A2: ‘The metaphysical invention Set’.

Q: Concepts? A1: ‘Abstraction from examples’. A2: ‘’Example of abstractions’.

Q: An equation? A1: ‘A reversed operation’. A2: ‘An example of an equivalence relation between two number-names’.

Can Education be Different?

From secondary school, continental Europe uses line-organized education with forced classes and forced schedules making teenagers stay together in age-groups even if girls are two years ahead in mental development.

The classroom belongs to the class. This forces teachers to change room and to teach several subjects outside their training in lower secondary school.

Tertiary education is also line-organized preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemployment, and it forces the youth to stay in education until close to 30 making reproduction fall to 1.5 child per family so the European population will be reduced to 10% in 200 years.

Alternatively, North America uses block-organized education saying to teenagers: “Welcome, you carry a talent! Together we will uncover and develop your personal talent through daily lessons in self-chosen half-year blocks.” If successful the school will say ‘good job, you have a talent, you need more’. If not, the school will say ‘good try, you have courage, now try something else’. The classroom belongs to the teacher teaching only one subject and helped by daily lessons to adapt quickly to learner differences.

Likewise, college is block-organized to be tested already in high school and easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children to secure reproduction: one for mother, one for father and one for the state.

So different education forms might not all lead to Truth, Beauty and Goodness.

Conclusion: Blocks in Mathematics Education, Please

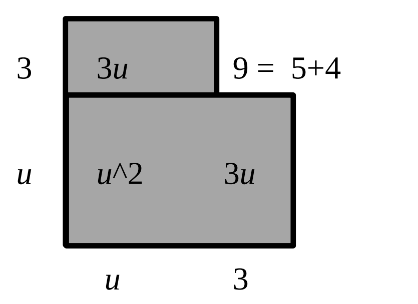
We asked: How can mathematics be a means leading to the goal of implementing the three classical virtues Truth, Beauty and Goodness? The answer is very simple: Blocks in mathematics and in education, please.

Blocks will bring Truth and Goodness back if mathematics will

- respect the nature of numbers as integrated blocks
- replace self-referring meta-matics and falsified mathe-matism with grounded many-matics presenting mathematics as a natural science about the physical fact Many
- make geometry grounded in blocks halved by their diagonals
- bring back algebra to its original Arabic meaning: to reunite constant and variable unit-numbers and per-numbers.

Blocks will bring Truth and Goodness to education that uncovers and develops a teenager’s individual talent through daily lessons in self-chosen half-years blocks made possible when replacing line- with block-organization.

Blocks will bring Beauty to the streets with Block-Art posters showing how algebra and geometry work nicely together:



2 cards solve quadratic equations

u^2 + 6u + 5 = 0

(u + 3)^2 = u^2 + 6u + 5 + 4 = 0 + 4 = 4

u + 3 = ± 2

u = -3 ± 2

u = -1 and u = -5

Recommendation: Mathematics, Unmask Yourself, Please

Mathematics, in Greek you mean ‘knowledge’ and you were chosen as a common label for 4 activities: Music, Astronomy, Geometry & Arithmetic (Freudenthal 1973: 7). Today only 2 activities remain: Geometry and Algebra. Then Set transformed you from a natural science about the physical fact Many to a self-referring metaphysical subject, MetaMatism, combining MetaMatics and MatheMatism (MrAlTarp YouTube videos 2013).

So please, unmask your true identity, and tell us how you would like to be presented in education: MetaMatism for the few - or ManyMatics for the many.

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PerNumbers replace Proportionality, Fractions & Calculus

Abstract

Increased research can lead to decreasing PISA math results as In Sweden. A goal/means confusion might be the cause. Grounded as a means to an outside goal, mathematics becomes a natural science about the physical fact Many. This ManyMatics differs from the school's MetaMatism, mixing MetaMatics, defining its concepts as examples from internal abstractions, with MatheMatism, true inside but not outside the class. Replacing proportionality, fractions and calculus with per-numbers will change math from goal to means.

Decreasing PISA Performance, a Result of a Goal/Means Confusion?

Being highly useful to the outside world has made mathematics a core of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA level in mathematics decrease from 509 in 2003 to 478 in 2012, far below the OECD average at 494. This has made OECD write a report describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

Created to enable students cope with the outside world, schools consist of subjects that are described by goals and means with the outside world as the goal and the subjects as the means. However, a goal/means confusion might occur where the subjects become the goals and the outside world a means.

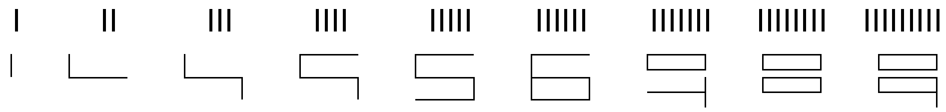
A goal/means confusion is problematic since while there is only one goal there are many means that can be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal.

So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? For a grounded answer (Glaser 1967) we reformulate the question: How will mathematics look like if built as a means for proper real world actions?

Mathematics is not an action word itself, but so are its two main activities, geometry and algebra, meaning to measure earth in Greek, and to reunite numbers in Arabic. Thus mathematics is an answer to the two basic questions of mankind: How to divide the earth we live on, and the many goods it produces? (Tarp 2012). So what we really ask is: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

Mathematics as a Natural Science about Many

To deal with Many we count and add. To count we stack icon-bundles. To iconize five we bundle five ones to one fives to be rearranged as one five-icon 5 with five sticks if written in a less sloppy way. We create icons until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle one, one bundle two etc.



1 2 3 4 5 6 7 8 9

With Icons we count by bundling a total in icon-bundles. Thus a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. Now we place two sticks in a left bundle-cup and one stick in a right single-cup to write the total in 'algebra-form'. Here the cup-content is described by an icon, first using 'cup-writing' 2)1), then using 'decimal-writing' with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T= 2.1 3s.

Alternatively, we can use the plastic letters, B for a bundle and C for a bundle of bundles.

IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BBI → 2BI

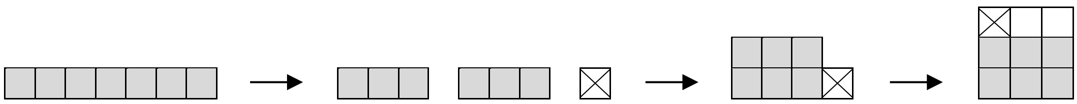
A calculator can predict the counting result. To count in 3s we take away 3s, iconized as '/3' showing the broom wiping away the 3s several times. Building a stack of 2 3s we iconize as '2x3' showing a jack used to lift the 3s. And '-2x3' iconizes the trace coming from taking away 2 3s to look for unbundled. These three operations are called division, multiplication and subtraction respectively.

Entering '7/3' the answer is '2.some'. To find the unbundled we take away the 2 3s by asking '7 – 2x3'. From the answer '1' we conclude that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 x 3	1

Thus a total T is counted in 3s by taking away 3 T/3 times. This can be written as a 're-count formula' T = (T/3)x3 or as T = (T/b)xb if re-counting T in bs. Taking away a stack b to be placed next-to the unbundled T-b can be written as a 're-stack formula' T= (T-b)+b.

To write the total in 'geometry-form' we use squares or LEGO blocks or an abacus to stack the 2 3-bundles on-top of each other with an extra stack of unbundled 1s next-to or on-top, thus describing the total as a decimal number 2.1 3s, or as a fraction number 2 1/3 3s counting the unbundled 1 in 3s.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

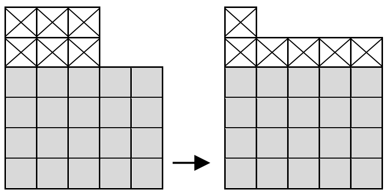
DoubleCounting creates PerNumbers Bridging Units

A physical quantity can be counted in different units, e.g. as 4kg or as 5$. This creates the 'per-number' 4kg/5$ = 4/5 kg/$. To shift from one unit to another we simply recount in the part of the per-number that has the same unit:

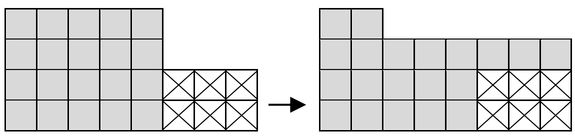
7 kg = ? $	8 $ = ? kg
7 kg = (7/4) x 4 kg = (7/4) x 5 $ = 8.75 $	8 $ = (8/5) x 5 $ = (8/5) x 4 kg = 6.4 kg

Adding Totals

Once Counted, totals can be added on-top or next-to. To add on-top, the units must be changed to be the same, typically by recounting one total in the other total's unit. Adding next-to is called integrating areas.



OnTop: 4 5s + 2 3s = 5.1 5s



Next-to: 4 5s + 2 3s = 3.2 8s

Next-to addition is also used when adding piecewise constant per-numbers:

4 kg at 5 $/kg + 2 kg at 3 $/kg = (4x5 + 2x3) $ = Σ (per-number x quantity)

Or when adding locally constant (continuous) per-numbers:

6 kg at 5 $/kg decreasing to 3 $/kg = $\int_0^6 (5 + \frac{3-5}{6} u)\, du$

Global, piecewise, and local constancy all express the fact that y is a constant k if the distance between the two can be made arbitrarily small:

- y is globally constant k if $\forall \varepsilon > 0$: $| y - k | < \varepsilon$.
- y is piecewise constant k if $\exists C$ so $\forall \varepsilon > 0$: $| y - k | < \varepsilon$ inside C.
- y is locally constant k if $\forall \varepsilon > 0$ $\exists$ C : $| y - k | < \varepsilon$ inside C.

Reversing Addition, or Solving Equations

Reversing addition, we ask e.g. '2+? = 8'. Restacking 8 as (8-2)+2 we get the answer 8-2. Reversing multiplication, we ask e.g. '2x? = 8'. Recounting 8 in 2s as (8/2)x2 we get the answer 8/2. We see that solving equations means moving numbers to the opposite side with the opposite sign.

OnTop		NextTo
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3 8s**
? = 8-2	? = 8/2	? = (3 8s – 2 3s)/5

Reversing adding next-to, we ask e.g. '2 3s + ? 5s = 3 8s'. To find what was added we take away the 2 3s and count the rest in 5s. Combining subtraction and division in this way is called reversed integration or differentiation.

The Algebra Project: the Four Ways to Add

Meaning 'to re-unite' in Arabic, the 'Algebra-square' shows that with variable and constant unit-numbers and per-numbers there are four ways to unite numbers into a total and five ways to split a total: addition/subtraction unites/splits-into variable unit-numbers, multiplication/division unites/splits-into constant unit-numbers, power/root&log unites/splits-into constant per-numbers and integration/differentiation unites/slits into variable per-numbers.

Uniting/*splitting*	Variable	Constant
Unit-numbers	**T = a + n**, *T – a = n*	**T = a x n**, *T/n = a*
Per-numbers	**T = ∫ a dn**, *dT/dn = a*	**T = a^n**, *loga(T) = n, n√T = a*

School only counts in tens writing 2.3 tens as 23 thus leaving out the unit and misplacing the decimal point. So icon-counting must take place in preschool.

Writing 345 as 3x10^2 + 4x10 + 5x1, i.e. as areas placed next-to each other, again shows that there are four ways to unite, and that all numbers have units.

ManyMatics versus MatheMatism and MetaMatics

Built as a natural science about the physical fact Many, mathematics becomes ManyMatics dealing with Many by counting and adding as shown by the Algebra-square and in accordance with the Arabic meaning of algebra.

With counting and adding Many as outside goal, a proper means would teach icon-counting and on-top and next-to addition in grade one. However, only ten-counting occurs. And addition takes place without including units claiming that 2+3 IS 5 in spite of counterexamples as 2 weeks + 3 days = 17 days. So what is taught in primary school is not ManyMatics leading to proper actions to deal with Many, but what could be called 'MatheMatism' true inside but not outside a class thus making itself a goal not caring about outside world falsifications.

A counting result can be predicted by a re-count and a re-stack formula. So formulas as means to real world number-prediction should be a core subject in secondary school. However, here a formula is presented as an example of a function, again being an example of a set-relation where first-component identity implies second-component identity. So what is taught in primary school is not ManyMatics leading to the ability to predict numbers, but what could be called ‘MetaMatics’ presenting concepts from the inside as examples of abstractions instead of from the outside as abstractions from real world examples; and becoming ‘MetaMatism’ when mixed with MatheMatism.

So yes, a goal-means confusion exits in mathematics education seeing MetaMatsm as the goal and real world as applications and means; and claiming that ‘of course mathematics must be learned before it can be applied’. To lift this confusion the outside world must again be the goal and ManyMatics the means. Testing examples will show if this can turnaround the PISA-results.

Proportionality or Linearity

Linearity is a core concept in mathematics, defined by MetaMatism as a function f obeying the criterion $f(x+y) = f(x)*f(y)$. The function $f(\mathrm{x}) = a*x$ is linear since

$f(x+y) = a*(x+y) = a*x + a*y = f(x) + f(y)$.

This ‘proportionality function’ is applied to the outside world when solving a ‘3&4&5-problem’: ‘If 3 kg cost 4 $ then 5 kg cost ? $’. Asking ‘5 kg = ? $’ shows that the ‘3&4&5-problem’ is an example of a more general ‘change-unit problem’ as e.g. ‘5 £ = ? $’.

Historically, the outside goal ‘to change-units’ has created different means. The Middle Ages taught ‘Regula Detri’, the rule of three: The middle number is multiplied with the last number and then divided by the first number.

The industrial age introduced a two-step rule: First go to the unit by dividing 4 by 3, then multiply by 5. Having learned how to solve equations in secondary school, a proportion can be set up equalizing two ratios: 3/4 = 5/u. Now cross-multiplication leads to the equation 3xu = 4x5 with the solution u = 4x5/3.

As shown above, the per-number 3$/4kg offers a fifth alternative finding the answer by recounting 5 in 3s: T = 5$ = (5/3)x3 $ = (5/3)x4 kg.

So the action ‘to change unit’ can be attained by five different means, all to be part of teacher education in order to create a turnaround in the PISA results.

Fractions

Defining everything as examples of sets, MetaMatism sees fractions as what is called ‘rational numbers’, defined as equivalence sets in the set-product of ordered pairs of integers created by an equivalence relation making (a,b) equivalent to (c,d) if cross multiplication holds: axd = bxc.

In primary school fractions come after division, the last of the basic operations. Unit fractions come in geometry as parts of pizzas or chocolate bars; and in algebra as parts of a total: 1/4 of the 12 apples is 12/4 apples.

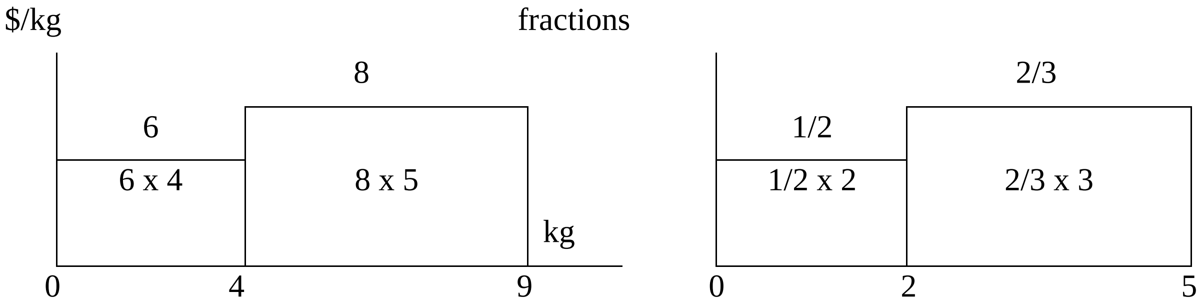
To find 4/5 of 20, first 1/5 of 20 is found by dividing with 5 and then the result is multiplied by 4. Then it is time for decimals as tenths, and percentages as hundredths. Then similar fractions occur when adding or removing common factors in the numerator and the denominator.

When including units, fractions respect the outside goal ‘to divide something’. Excluding units, adding fractions becomes MateMatism as shown by the ‘fraction paradox’: 1/2 + 2/3 is 7/6 inside a classroom, but can be 3/5 outside where 1 red of 2 apples plus 2 red of 3 total 3 red of 5 and certainly not 7 of 6.

From outside examples, per-numbers become fractions, 4kg/5$ = 4/5 kg/$. And, as per-numbers, fractions add by integrating the areas under their graph:

4kg at 6$/kg + to 5kg at 8$/kg = 9 kg at (6x4 + 8x5)/9 $/kg.

2 of which 1/2 + 3 of which 2/3 = 5 of which (1/2 x 2 + 2/3 x 3)



Integration

Adding variable per-numbers by integrating blocks, integration is one of the four ways to add as shown by the Algebra-square. So, integration should not be postponed to late secondary school but be part of primary school when adding icon-blocks next-to and when integrating areas under fraction graphs.

Also, integration should be taught before differentiation and before functions, since what we integrate (and differentiate into) is per-numbers, not functions.

Conclusion

To see if mathematics education has a goal/means confusion we asked: How will mathematics look like if built as a means for proper real-world actions?

Or more precisely: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

To deal with Many, first we count, then we add. But first rearranging Many create icons.

Counted in icon-bundles, a total transforms into a stack of unbundled, bundles, bundles of bundles etc., i.e. into a decimal number with a unit.

The basic operations, / and x and –, iconize the three counting operations: to take away bundles, to stack bundles and to take away a stack.

Double-counting in different units create per-numbers used to bridge the units.

Once counted, totals can be added on-top or next to; and addition can be reversed by inventing reverse operations as shown in the Algebra-square.

Constructed as abstractions from the physical fact Many, ManyMatics prevents a goal/means confusion in mathematics education seeing the outside world as applications of MetaMatism, a mixture of MetaMatics defining concepts as examples from abstractions instead of as abstractions from examples, and MatheMatism with statements that are true inside but not outside a classroom.

Recommendations

So, to improve PISA results, mathematics education must teach actions enabling students to deal with the physical fact Many. Making mathematics a means and the outside world the goal prevents a goal/means confusion to occur.

Consequently, mathematics education must teach ManyMatics abstracted from the outside world as a natural science about Many. And it must reject self-referring MateMatism containing concepts based internally instead of externally, and neglecting outside falsification of inside correctness.

In primary school, recounting in different icons should precede adding on-top and next-to. And double-counting create the per-numbers allowing the two units to be bridged without waiting for proportionality.

To avoid nonsense, fractions must be added as per-numbers by integrating areas thus introducing primary school calculus as the fourth way to unite numbers. In this way everybody will be able to deal with Many by applying the full Algebra-square.

The MATHeCADEMY.net is designed to teach teachers to teach mathematics as ManyMatics as illustrated by its many MrAlTarp videos on YouTube.

References

Glaser, B.G. & Strauss, A.L. (1967). *The Discovery of Grounded Theory*, New York: Aldine de Gruyter

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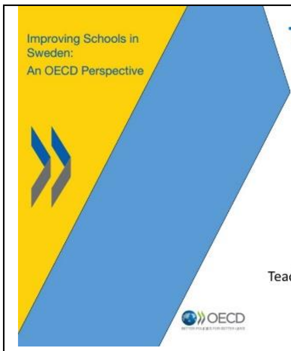
23. CTRAS Conferences

Classroom Teaching Research for All Students, 2016 in Germany



To improve Schools, chose

Good, Bad or Evil MatheMatics Education

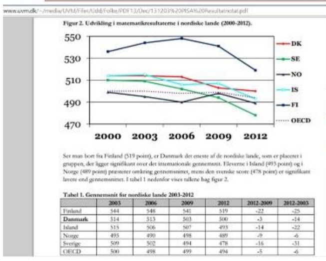
Allan.Tarp
@MATHeCADEMY.net

Teaches Teachers to Teach MatheMatics as ManyMatics, a Natural Science about the physical fact Many

The 2016 CTRAS Conference in Hamburg

MATHeCDMY

The MeltDown of Swedish PISA results in spite of Increased Funding – caused by Bad & Evil Math?



All melt down, but as to the OECD average, Finland & Denmark are significantly above, Iceland & Norway are on level, only Sweden is significantly below

MATHeCDMY

Schools Exclude 1 of 4 Socially

"PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (page 3)

http://www.oecd.org/sweden/sweden-should-urgently-reform-its-school-system-to-improve-quality-and-equity.htm

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MATHeCDMY

A Need for Urgent Reforms and Change

Part I: A school system in need of urgent change. (p. 11)

"Sweden has the highest percentage of students arriving late for school among all OECD countries, especially among socio-economically disadvantaged and immigrant students, and the lack of punctuality has increased between 2003 and 2012. There is also a higher-than-average percentage of students in Sweden who skip classes, in particular among disadvantaged and immigrant students. Arriving late for school and skipping classes are associated highly negatively with mathematics performance in PISA and can have serious adverse effects on the lives of young people, as they can cut into school learning and also distract other students." (p. 69)

MATHeCDMY

Serious Situation and Serious Deterioration

"If serious shortcomings are identified in a school, the Schools Inspectorate can determine that the deficient school should be closed for up to six months until the deficiencies are corrected. However, this is very much a last resort and has rarely been applied." (p. 51)

"The reforms of recent years are important, but evidence suggests they are also somewhat piecemeal, and simply too few, considering the serious situation of the Swedish school system." (p. 55)

"Sweden faces a serious deterioration in the quality and status of the teaching profession that requires immediate system-wide attention. This can only be accomplished by building capacity for teaching and learning through a long-term human resource strategy for the school sector." (p. 112)

MATHeCDMY

A Goal/Means Confusion in Math Education?

Occam's Razor principle: First look for a simple explanation.

An educational subject always has an outside goal to be reached by inside means. But, if seen as mandatory, an inside means becomes a new goal that might hinder learners reaching the original outside goal.

Not accepting its outside goal, Mastering of Many, Mathematics Education is an undiagnosed cure, forced upon patients showing natural resistance against an unwanted treatment.

So, to explain the meltdown in Swedish/Nordic PISA results we can ask:

Is there a Goal/Means Confusion in Swedish/Nordic Mathematics Education?

MATHeCDMY

Exemplifying Good & Bad & Evil

Evil if I hide alternatives and present my **choice** as nature

Bad if I force my own **choice** upon others

Good if I perform all 3 parts of a democratic **choice** process

- Tell nature from **choice** and uncover all alternatives
- Analyze and discuss the alternatives
- Make a flexible **choice** - to change if new information should appear

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MATHeCDMY

Defining MatheMatics

According to Freudenthal (1973) the Pythagoreans used the Greek word for mastering, mathematics, as a common label for their four master areas.

With astronomy and music as independent subjects, today only the two other activities remain, both rooted in the physical fact Many:

- Geometry, meaning to measure earth in Greek
- Algebra, meaning to reunite in Arabic

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publ. Comp.

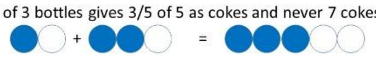
MATHeCDMY

Defining MatheMatism , Bad MatheMatics

MatheMatism is a statement that is correct inside, but seldom outside a classroom , thus being bad by forcing its choice upon others.

Ex1. Adding numbers without units as 2+3 = 5, where e.g. 2**w**+3**d**=17**d**. In contrast to 2x3 = 6 observing that 2 **3s** can be recounted as 6 **1s**.

Ex2. 'The fraction paradox': The textbook and the teacher insist that 1/2 + 2/3 **IS** 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles:



MATHeCDMY

Defining MetaMatics, Evil MatheMatics

MetaMatics is defining a concept, not as an abstraction from many examples but as an example of an abstraction, derived from the meta-physical abstraction SET, made meaningless by self-reference as shown by Russell's version of the liar paradox: If M does, it does not (and vice versa) belong to the set of sets not belonging to itself: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

By hiding its outside grounded alternative, self-referring MetaMatics is evil.

Ex. What is a **function**?

From inside, an example of a **set relation** where 1component identity implies 2component identity. From outside, a **name** for a formula with 2 unknowns.

MATHeCDMY

Teaching MatheMatics as MetaMatism Means Trouble

MetaMatism = MetaMatics + MatheMatism, so by its self-reference, MetaMatism only provides inside definitions and inside proofs of its concepts and statements, thus hiding outside roots and validity.

Two Claims

- The Swedish/Nordic PISA meltdown is caused by teaching MatheMatics as MetaMatism instead of as ManyMatics
- As Evil and Bad MatheMatics, MetaMatism creates DysCalCulia - that can be avoided by teaching ManyMatics instead.

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MATHeCDMY

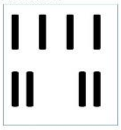
ManyMatics, created to Master Many

To tell nature from choice, we ask: How will math look like if built as a Natural Science about the physical fact Many, i.e., as a ManyMatics?

- Take 1: To master Many, we math! *Oops, math is a label, not an action word.*
- Take 2: To master Many, we act. Asking 'How Many?', we Bundle & Stack: 456 = 4 x BundleBundle + 5 x Bundle + 6 x 1 = three stacks of bundles.

All numbers have units - as recognized by children when showing 4 fingers held together 2 by 2 makes a 3-year-old child say: 'No, that is not 4, that is 2 **2s**.'

So natural numbers are 2D stacks, not placed on a 1D line.



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MATHeCDMY

ReCounting roots Algebra's 4 ways to ReUnite

Addition / *Subtraction* unites / *splits* into Variable Unit-numbers
Multiplication / *Division* unites / *splits* into constant Unit-numbers
Power / *Root&Log* unites / *splits* into constant Per-numbers
Integration / *Differentiation* unites / *splits* into variable Per-numbers

Operations **unite** / *split into*	Variable	Constant
Unit-numbers *m, s, $, kg*	**T = a + n** *T – a = n*	**T = a x n** *T/n = a*
Per-numbers *m/s, $/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** *log$_a$T = n, ^{n}√T = a*

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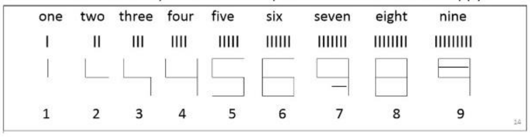
MATHeCDMY

Creating Icons: I I I I → IIII → 4

Counting in ones means naming the different degrees of Many. We stop at nine since when bundling in tens, ten becomes 1 Bundle, needing no icon of its own. Counting in icons means changing four ones to one fours rearranged as a 4-icon with four sticks or strokes. So, an icon contains as many strokes as it represents if written less sloppy.



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MATHeCDMY

Counting in Icons: 9 = ? **4s**

9 = I I I I I I I I I = IIII IIII I = **II**)I = 2)1 = 2.1 **4s**

To count, we bundle & use a bundle-cup with 1 stick per bundle. We report with **cup-writing** or **decimal-writing** where the decimal point separates the bundles from the singles. Shown on a western **ABACUS** in

Geometry/space mode or **A**lgebra/time mode

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MATHeCDMY

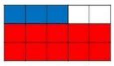
The UnBundled become Decimals or Fractions 0.3 **5s** or 3/5

When counting by bundling and stacking, the unbundled singles can be placed

NextTo the stack counted as a stack of **1s**

T = 2.3 **5s**
A decimal number

OnTop of the stack counted as a bundle



T = 2 3/5
A fraction

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MATHeCDMY

Counting creates Division & Multiplication & Subtraction - also as Icons

'From 9 take away **4s**' we write 9/4
<u>iconizing</u> the sweeping away by a broom, called division.

'2 times stack **4s**' we write 2x4
<u>iconizing</u> the lifting up by a jack called multiplication.

'From 9 take away 2 **4s**' to look for un-bundled we write 9 – 2x4
<u>iconizing</u> the dragging away by a stroke called subtraction.

The counting process includes division, multiplication and subtraction:
Finding the bundles: 9 = 9/4 **4s**. Finding the un-bundled: 9 – 2x4 = 1.

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MATHeCDMY

Counting creates Two Counting Formulas

Bundling & stacking create two counting formulas (re-bundle and re-stack):

T = (T/b) x b	from a total T, T/b times, bs is taken away and stacked
T = (T–b) + b	from a total T, T – b is left when b is taken away and placed next-to

The two counting formulas allow a calculator to predict the counting result

9/4	**2.some**
9 – 2x4	**1**

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MATHeCDMY

Counting Sequences

Being counted as Bundle, the Bundle number needs no icon.

	I	I	I	I	I	I	I	I	I	I	I	I
5s	0B1	0B2	0B3	0B4	**B**	1B1	1B2	1B3	1B4	2B	2B1	2B2
7s	0B1	0B2	0B3	0B4	0B5	0B6	**B**	1B1	1B2	1B3	1B4	1B5
tens	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	**B**	1B1	1B2

3 **4s** counted	in **5s** as	T = 2B2 = 2 x 5 + 2 = 2.2 **5s**
	in **7s** as	T = 1B5 = 1 x 7 + 5 = 1.5 **7s**
	in **tens** as	T = 1B2 = 1 x ten + 2 = 1.2 **tens**

As to number names, eleven and twelve come from 'one left' and 'two left' in Danish, (en / twe levnet), again showing that counting takes place by taking away bundles.

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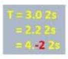
MATHeCDMY

ReCounting in the Same Unit creates Overloads & Negative Numbers



ReCounting 3 **2s** in **2s**:

Sticks	Calculator		Total T	Cup-Writing
IIII IIII IIII			3.0 **2s**	3) **2s**
IIII IIII I I	3x2 – 2x2	2	2.2 **2s**	2) 2 **2s**
IIII IIII IIII IIII	3x2 – 4x2	-2	4.-2 **2s**	4) -2 **2s**

(4.-2 = 4 less 2)

And **2digit** Numbers if using Bundles of Bundles:

IIIIII	=	IIII IIII IIII	=	IIII IIII	IIII			
6	=	3 **B**	=	1 **BB**	1 **B**			
6	=	3)	=	1)	1)	=	11.0 **2s** = 11 **2s**	

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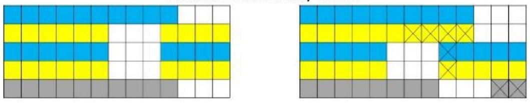
ReCounting in a Different Unit 3 **4s** = ? **5s**

3 **4s** = IIII IIII IIII = I I I I I I I I I I I I = IIIII IIIII II = 2.2 **5s**

CALCULATOR-prediction:

3x4/5	**2**.some
3x4 – 2x5	**2**

Abacus in Geometry mode



Change Unit = Proportionality, Core Math

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ReCounting in Tens 3 **7s** = ? **tens**

3 **7s** = IIIIIII IIIIIII IIIIIII = IIIIIIIIII I IIIIIIIIII = 2.1 **tens**

CALCULATOR-prediction: The calculator has no ten icon.

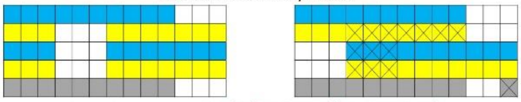
The calculator gives the answer directly

- but without unit and with misplaced decimal point

3x7	21

A Natural Number ???

Abacus in Geometry mode



ReCounting to tens = Multiplication Tables, Core Math

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ReCounting from Tens 29 = ? **6s**

29 = ? **6s** = IIIIIIIIII IIIIIIIIII IIIIIIIII = IIIIII IIIIII IIIIII IIIIII IIIII = 4.5 **6s**

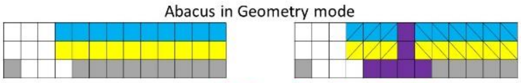
CALCULATOR-prediction:

29/6	**4**.some
29 – 4x6	**5**

Reversed calculation (Equation): ? x 6 = 29 = (29/6) x 6, so ? = 29/6 = 4 + 5/6

'Opposite side with opposite sign' method: if u x 6 = 29 then u = 29/6

Abacus in Geometry mode



ReCounting from tens = Solving an Equation, Core Math

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MATHeCADMY

ReCounting big Numbers in Tens (Multiplication)

Recounting 6 **47s**	Recounting 36 **47s**
T = 6 x 47 = 6 x 4) 7 = 24)42 = 28) 2 = 282	T = 36 x 47 = 36 x 4) 7 = 144)252 = 169) 2 = 1692
T = 6 x 47 = 6 x 5) -3 = 30)-18 = 28) 2 = 282	T = 36 x 47 = 36 x 5) -3 = 180)-108 = 169) 2 = 1692

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MATHeCADMY

ReCounting big Numbers in Icons (Division)

Recounting a total T of 478 in **7s**	Recounting a total T of 374 in **12s**
T = 478 = 47) 8 = 42) 58 = 42) 56 + 2 = 6x7) 8x7 + 2 = 68 x 7 + 2	T = 374 = 37) 4 = 36) 14 = 36) 12 + 2 = 3x12) 1x12 + 2 = 31 x 12 + 2

T = 478 = 68 x 7 + 2 T = 374 = 31 x 12 + 2

478 / 7 = 68 + 2/7 374 / 12 = 31 + 2/12

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MATHeCADMY

DoubleCounting creates PerNumbers (Proportionality)

With **4kg = 5$** we have 4kg/5$ = 4/5 kg/$ = a per-number

4$/100$ = 4/100 = 4%

Questions:

7kg = ?$	**8$ = ?kg**
7kg = (7/4)*4kg = (7/4)*5$ = 8.75$	8$ = (8/5)*5$ = (8/5)*4kg = 6.4kg

Answer: *Recount in the per-number*

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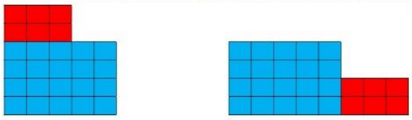
MATHeCADMY

Once Counted & ReCounted, Totals can be Added

OnTop	**NextTo**
2 **3s** + 4 **5s** = 1.1 **5s** + 4 **5s** = 5.1 **5s**	2 **3s** + 4 **5s** = 3.2 **8s**
The units are changed to be the same.	The areas are integrated.
Change unit = Proportionality	*Integrate areas = Integration*



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Adding PerNumbers as Areas (Integration)

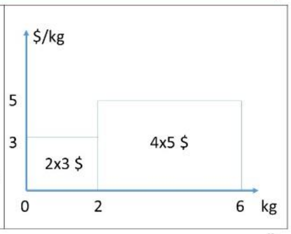
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

(2+4)kg at (2x3 + 4x5)/(2+4) $/kg

Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph.



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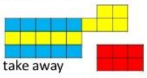
MATHeCADMY

Reversing Addition, or Solving Equations

OnTop	**OppoSIte SIde & SIgn**	**NextTo**
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3.2 8s**
? = 8-2	? = 8/2	? = (3.2 8s – 2 3s)/5
Solved by re-stacking	*Solved by re-bundling*	*Solved by differentiation*



Hymn to Equations

Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don't want to stop!

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MATHeCADMY

Geometry: Measuring HalfStacks

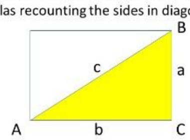
Geometry means to measure earth in Greek

The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras theorem a^2 + b^2 = c^2. And connected with the angles by formulas recounting the sides in diagonals:

a = (a/c) x c = sin A x c

b = (b/c) x c = cos A x b

a = (a/b) x b = tan A x b



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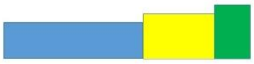
Defining ManyMatics: To master Many, we Recount in Stacks to add OnTop or NextTo

In ManyMatics,

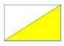
Numbers are 2D stacks,

- not on a 1D line



Algebra: to (re)unite stacks on-top or next-to

Geometry: to measure half-stacks



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Is ManyMatics Different from 'MatheMatics'

Same Questions	**ManyMatics**	**MatheMatics**
Digits	Icons, different from letters	Symbols like letters
Natural numbers	2.3 tens	23
Fractions	Operators (per-numbers) needing a number to produce a number	Rational numbers
Per-numbers	Double-counting	Not accepted
Operations	Icons for the counting process	Mappings from a set-product to a set
Order of operations	/, x, -, +	+, -, x, /

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Same Questions – Different Answers

Addition	On-top and next-to	Only on-top
Integration	Preschool: Next-to addition, for all Middle school: Adding piece-wise constant per-numbers, for all High school: Adding locally constant per-numbers, for almost all	Last year in high school, for the few
A formula	A stand-by calculation with numbers and letters	An example of a function that is an example of a relation in a set-product where first component identity implies second component identity
Algebra	Re-unite constant and variable unit-numbers and per-numbers	A search for patterns

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Yes, ManyMatics is Different

An equation	A name for a reversed operation	An example of an equivalence relation between two number-names
The root of Mathematics	The physical fact Many	The metaphysical invention SET
A concept	An abstraction from many examples	An example of an abstraction derived from SET (MetaMatics)
How true is 2+3 = 5 & 2x3 = 6	2x3 = 6 is true by nature since 2 3s can be recounted as 6 1s. 2+3 = 5 is true inside but seldom outside a class: 2**w**+3**d** = 17**d**, etc.	Both true by nature (MatheMatism)

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Yes, Math Ed has a Goal/Means Confusion

Chosen as a common label for its two remaining activities, Geometry & Algebra, mathematics has two outside goals: to measure Earth and to reunite Many.

Transformed to self-referring MetaMatism, it became its own goal blocking the way to the outside goals, reduced to applications of mathematics to be taught, 'of course', after mathematics itself has been taught and learned.

In this way 'mathematics education' becomes an undiagnosed cure against a self-referring need, unneeded by patients showing natural resistance.

So, to reach the outside goal, mastering of Many, we must look for a different alternative way, as e.g. ManyMatics, built as a grounded theory, a Natural Science, about the physical fact Many.

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MatheMatics: Unmask Yourself, Please

- In Greek you mean 'mastering'. You were chosen as a common label for 4 activities: Music, Astronomy, Geometry & Arithmetic. Later only 2 activities remained: Geometry and Algebra
- Then Set transformed you from a Natural Science about the physical fact Many to a metaphysical subject, MetaMatism, combining MetaMatics and MatheMatism
- So please, unmask your true identity, and tell us how you would like to be presented in education:
- MetaMatism for the few - or ManyMatics for the many

*But to Improve Schools: Don't preach **Essence**, teach **Existence***

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Creating or Avoiding DysCalCulia

Having problems learning mathematics has many names:

Difficulty, disability, deficiency, disorder, low attainment, low performance or DysCalCulia.

How to Create it	How to Avoid it
• Teach 1D LineNumbers as '8' • No Counting before Adding • Adding before Multiplying • Adding without Units: 2+3=5	• Teach 2D BlockNumbers as '2 **4s**' • ReCounting before Adding • Multiplying before Adding • Adding with Units: 2**w**+3**d**=17**d**

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ReCount - don't Add: Respect the Child's own 2D Numbers

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To Improve Schools, Chose

Good & ~~Bad~~ & ~~Evil~~

MatheMatics Education

ByeBye to LineOrganized MetaMatism
Welcome to BundleOrganized ManyMatics

Thank You for Your Time

Allan.Tarp@MATHeCADEMY.net
Free Uni Franchise

CTRAS 2017 in China

Twelve Proposals for 1day Skype Seminars

01) The Root of Mathematics, Many, dealt with by Block-Numbers, Cup-Counting and Preschool Calculus.

02) 12 Luther-like Theses about how ManyMath can Improve Math Education

03) Curing Math Dislike with one Cup and five Sticks

04) DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers

05) Algebraic Fractions made easy by Block-Numbers with Units

06) Algebra and Geometry, always Together, never Apart

07) Calculus in Middle School and High School

08) Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?

09) Quantitative Literature also has three Genres: Fact and Fiction and Fiddle

10) Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space

11) 50 years of Sterile Mathematics Education Research, Why?

12) Difference-Research, a more Successful Research Paradigm?

01) The Root of Math, Many, dealt with by Block-Numbers, Cup-Counting and Preschool Calculus.

"How old next time?" I asked the child. The answer was four with four fingers shown. But held together two by two created a protest: "That is not four, that is two twos!". That opened my eyes. Children come to school with two-dimensional block-numbers where all numbers have units. Instead, school teaches cardinality as a one-dimensional line with different number-names; thus disregarding the fact that numbers are two- dimensional blocks all having a unit as shown when writing out fully a total T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1. So, a number is blocks united (integrated) next-to each other, showing the four ways to unite numbers presented by Algebra, meaning reuniting in Arabic: Power and multiplication and 'on-top' and 'next-to' addition (integration).

Consequently, mathematics education should develop the two-dimensional block-numbers that children bring to school and allow them to practice counting before adding.

To master Many, we ask 'how many?' To answer, we cup-count using a cup for the bundles. So, a number always has some bundles inside and some unbundled outside the cup.

Recounting in the same unit creates overloads or underloads by moving in or out of the cup, T = 5 = 2]1 2s = 1]3 2s = 3]-1 2s. This makes calculation easy: T = 4 x 56 = 4 x 5]6 = 20]24 = 22]4 = 224.

Once counted and recounted, totals may de added. To have 2 3s and 4 5s added on-top as 5s, a unit must be changed, called proportionality. To add them next-to as 8s means adding their areas, called integration; which becomes differentiation when reversed by saying 2 3s + ? 5s = 3 8s', thus allowing calculus to take place in preschool.

02) 12 Luther-like Theses about how ManyMath can Improve Math Education

1. Digits are icons with as many sticks as they represent.

2. A total T can be 'cup-counted' in the normal way or with an overload or underload: T = 5 = 2]1 2s = 1]3 2s = 3]-1 2s.

3. 'Cup-writing' makes operations easy: T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T

= 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, '? 7s = 3 tens', or 'u*7 = 30 = (30/7)x7', the answer u = 30/7 is found by 'move to opposite side with opposite sign'.

9. Adding totals is ambiguous: OnTop using proportionality, or NextTo using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = ΔT/3.

11. Double-counting in different units gives 'per-numbers' as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a number to become a number, thus adding by their areas as integration.

03) Curing Math Dislike with one Cup and five Sticks

A class is stuck in division and gives up on 234/5. Having heard about '1cup & 5 sticks', the teacher says 'Time out. Next week, no division. Instead we do cup-counting'. Teacher: 'How many sticks?' Class: '5.' Teacher: 'Correct, 5 1s, how many 2s?' Class: '2 2s and 1 left over'. Teacher: 'Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?' Class: '1 inside and 3 outside; and 3 inside and 1 lacking outside.' Teacher: 'Correct. A total of 5 sticks can be counted in 3 ways. The normal way with 2 inside and 1 outside. With overload as 1 inside and 3 outside. With underload as 3 inside and less 1 outside.' Class: 'OK'. Teacher. 'Now 37 means 3 inside and 7 unbundled 1s outside. Try recounting 37 with overload and underload. Class: '2 inside and 17 outside; and 4 inside and less 3 outside.'

Teacher: 'Now let us multiply 37 by 2, how much inside and outside?' Class: 6 inside and 14 outside. Or 7 inside and 4 outside. Or 8 inside and less 6 outside.' Teacher: 'Now to divide 74 by 3 we recount 7 inside and 4 outside to 6 inside and 14 outside. Dividing by 3 we get 2 inside and 4 outside; plus 2 leftovers that still must be divided by 3. So 74/3 gives 24 and 2/3.' Class: 'So to divide 234 by 5 we recount 234 as 20 inside and 34 outside. Dividing by 5 we get 4 inside and 6 outside; plus and 4 leftovers that still must be divided by 5. Thus 234/5 gives 46 and 4/5?' Teacher: 'Precisely. Now try multiplication using cup-counting'.

04) DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers

A class is stuck in fractions and percentages and gives up on ¾ = 75%. Having heard about 'per-numbers', the teacher says: Time out. Next week, no fractions, no percentage. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$. To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

05) Algebraic Fractions made easy by Block-Numbers with Units

A class is stuck in algebraic fractions insisting that (2b+4)/2b is 4. Having heard about 'Block-Numbers with units, the teacher says: 'Time out. Next week, no algebraic fractions. Instead we count totals with units.' Teacher, showing six sticks: 'How many sticks?' Class: '6.' Teacher: 'Correct, 6 *1s*, how many *2s*?' Class: '3 *2s'*. Teacher: 'Correct, we count in 2s by taking away 2s, that is by dividing by 2, so T = 6 = (6/2) 2s = 3 *2s* = 3*2. So, factorizing 2b as 2*b, 2b is 2 *bs* or b *2s*. Can 4 be written with a unit?' Class: '4 is 2 *2s'*. Teacher: 'Correct, so 2b and 4 can be written as b 2s and 2 2s totalling b+2 2s or (b+2)*2.' Class: 'OK'. Teacher: 'Now, 6 *2s* divided by 3 *2s* gives 6 divided by 3 or 2. And c *2s* divided by 3 *2s* gives c divided by 3.' Class: 'OK'. Teacher: 'So, b+2 *2s* divided by b *2s* gives b+2 divided by b.' Class: 'OK, and that gives 2?' Teacher: 'Well, division means removing a common unit. So, with b as b *1s* and 2 as 2 *1s* we can remove the *1s*. But b+2 *1s* divided by b *1s* still gives b+2 divided by b, which is the result.' Class: 'OK'. Teacher: 'Now try (3c+9)/6c.' Class: 'We factorize to find a common unit 3: 3c is c *3s*, 9 is 3 *3s*, and 6c is 2x3xc or 2c *3s*. Removing the common unit we get (3c+9)/6c = (c+3)/2c.' Teacher: 'Correct. Now try (b^2c+ bd^3)/bc'. Class: 'We factorize to find a common unit b: b^2c is bxbxc or bc *bs* , bd^3 is d^3 *bs*, and bc is c *bs*. Removing the common unit, we get (b^2c+ bd^3)/bc = (bc + d^3)/c.'

06) Algebra and Geometry, always Together, never Apart

The ancient Greeks used mathematics as a common label for their four knowledge areas, arithmetic, geometry, music and astronomy, seen as many by itself, many in space, many in time and many in space and time. With music and astronomy gone, mathematics was a common label for algebra and geometry until the arrival of the 'New Math' that insisted that geometry must go and that algebra should be defined from above as examples of sets instead of from below as abstractions from examples. Looking at the set of sets not belonging to themselves, Russell showed that set-reference means self-reference as in the classical liar paradox 'this sentence is false' being true if false and vice versa. Still, the new set-based 'meta-matics' entered universities and schools as the only true mathematics; except for the US going 'back to basics', that by separating algebra and geometry crates learning problems that disappear if they are kept together as advocated by Descartes. Thus, in primary school, numbers should be two dimensional LEGO-blocks as 2 3s. And 3x6 should be a block of 3 6s that if recounted in tens must widen its width and shorten its height, so that 3 6s becomes 1.8 tens. And in secondary school bxc should mean b cs; and fractions should be operators needing a number to become a number thus by multiplication becoming areas that are added by integration. Likewise, Euclidean geometry should be introduced in a coordinate system allowing equations to predict the exact position of intersection points of lines in triangles before being constructed with ruler and compasses. And the quadratic equation x^2+bx+c=0 geometrically tells that since x^2+bx = -c, the four parts of a (x+b/2) square reduce to (b/2)^2-c = D, allowing x to be found easily as x = -b/2 ±√D.

07) Calculus in Middle School and High School

A class is stuck in differential calculus and gives up on d/dx(x^2) = 2x. Having heard about 'per-numbers', the teacher says 'Time out. Next week, no differentiation. Instead we go back to middle school and look at per-numbers.' Class: 'Per-numbers, what is that?' Teacher: 'Per-numbers are for example meter per second, dollar per kilo, or dollar per hour. Here is an example: What is the total of 2 kg at 3 $/kg + 4 kg at 5$/kg?'. Class: 'The kg-numbers add to 6, but how do we add per-numbers?' Teacher: 'Can we change $/kg-numbers to $-numbers?'. Class: 'We can multiply 2 and 3 to 6$, and 4 and 5 to 20$ that add up to 26$. But multiplication means adding areas?' Teacher: 'Precisely. Adding per-numbers by their areas is called integral calculus, also called finding the area under the per-number-graph.'

Class: 'But what if the per-number graph is not constant? Then there are too many strips to add!' Teacher: 'We use a trick. Adding 1000 numbers is difficult but adding 1000 differences is easy since the middle numbers cancel out, so we are left with the difference between the end and the start number.' Class: 'But how can we write area-strips as differences?' Teacher: 'Well, if p is the per-number, then the area-strip with width dx is close to p*dx; but it is also the difference between the end area A2 and the start area A1, so p*dx = A2-A1 = dA, or p = d/dx(A).' Class: 'But that is differentiation?' Teacher: 'Precisely, so if we know that d/dx(x^2) = 2x, then we know that the area under the 2x graph is A2-A1 with A = x^2. So to find a quick way to area-formulas we need to learn to differentiate.' Class: 'OK.'

08) Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?

Humans have two languages, a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and '3 chairs have a total of 3x4 legs', abbreviated to 'T = 3x4'.

Both languages have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3x4' leads to a meta-sentence ''x' is an operation'.

We master outside phenomena through actions, so learning a word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language.

Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language.

Saying 'the number-language is an application of mathematics' implies that then 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

Instead school should follow the word-language and use full sentences 'The total is 3 4s' or 'T = 3x4'. By saying '3x4' only, school removes both the subject and the verb from number-language sentence, thus depriving it of its language nature.

09) Quantitative Literature also has three Genres: Fact and Fiction and Fiddle

Humans communicate in languages: A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. 'Word stories' come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like 'This sentence is false'. 'Number stories' are often called mathematical models. They come in the same three genres.

Fact models can be called a 'since-then' models or 'room' models. Fact models quantify quantities and predict predictable quantities: "What is the area of the walls in this room?". Since the model's prediction is what is observed, they can be trusted. Algebra's 4 basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy.

Fiction models can be called 'if-then' models or 'rate' models. Fiction models quantify quantities but predict unpredictable quantities: "My debt is gone in 5 years at this rate!". Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves.

Fiddle models can be called 'then-what' models or 'risk' models. Fiddle models quantify qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" Fiddle models should be rejected asking for a word instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

10) Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space

The MATHeCADEMY.net teaches teachers teach mathematics as 'many-math', a natural science about Many. It is a virus academy saying: To learn mathematics, don't ask the instructor, ask Many. To deal with Many, we Count and Add in Time and Space. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How can we count possibilities? How can we predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? Quantitative Literature, what is that? Does it also have the 3 different genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn. The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.

The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair.

11) 50 years of Sterile Mathematics Education Research, Why?

PISA scores are still low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation's need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

As to research, academic articles can be at a master level, exemplifying existing theories, or at a research level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying 'teach as little and as much as possible'.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So, to answer the question 'How to improve mathematics education research', first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger, asking 'what is 'is'?'

It turns out that, instead of mathematics, schools teaches 'meta-matism' combining 'meta-matics', defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and 'mathe-matism' true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching ½+2/3 = 7/6.

So, instead of meta-matism, teach 'many-math' in self-chosen half-year blocks.

12) Difference-Research, a more Successful Research Paradigm?

Despite 50 years of research, many PISA studies show a continuing decline. Maybe, it is time for difference-research searching for hidden differences that make a difference:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: cup-counting in icon-bundles less than ten, recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2]1 2s = 1]3 2s = 3]-1 2s.

3. The tradition uses carrying. A difference is to use cup-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4]8 = 28]56 = 33]6 = 336. Likewise with division: T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)x4kg = (10/4)x5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

Difference-Research Powering PISA Performance: Count & Multiply before You Add

To explain 50 years of low performing mathematics education research, this paper asks: Can mathematics and education and research be different? Difference-research searching traditions for hidden differences provides an answer: Traditional mathematics, defining concepts from above as examples of abstractions, can be different by instead defining concepts from below as abstractions from examples. Also, traditional line-organized office-directed education can be different by uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks. And traditional research extending its volume of references can be different, either as grounded theory abstracting categories from observations, or as difference-research uncovering hidden differences to see if they make a difference. One such difference is: To improve PISA performance, Count and Multiply before you Add.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience low and declining PISA performance. And apparently research can do nothing about it. Which raises the question: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, it is time to seek guidance by difference-research.

Difference-research Searching for Hidden Differences

Difference-research asks two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences can also use difference-research; and since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.' Also, he talks about rationality as the base for social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to uncover choice presented as nature is the aim of difference research.

A philosophical Background for Difference Research

Difference research began with the Greek controversy between two attitudes towards knowledge, called ‘sophy’ in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose how the world behaves.

However, in the Renaissance difference research returns with Brahe, Kepler and Newton. Observations showed Brahe that planetary orbits are predictable in a way that did not falsify the church’s claim that the earth is the center of the universe. Kepler pointed to a different theory with the sun in the center. To falsify the Kepler theory a new planet had to be launched, which was impossible until Newton showed that planets and apples obey the same will, and a falling apple validates Kepler’s theory.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way home across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa’s cost forced them to take the open sea and navigate by the moon. But how does the moon move? The Church had one opinion, Newton had a different.

‘We believe, as is obvious for all, that the moon moves among the stars,’ said the Church; opposed by Newton saying: ‘No, I can prove that the moon falls to the earth as does the apple.’ ‘We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,’ said the Church; opposed by Newton saying: ‘No, I can prove they follow their own physical will, a force that is predictable because it follows a mathematical formula.’ ‘We believe, as Aristotle told us, that a force upholds a state,’ said the Church; opposed by Newton saying: ‘No, I can prove that a force changes a state. Multiplied with the time applied, the force’s impulse changes the motion’s momentum; and multiplied with the distance applied, the force’s work changes the motion’s energy.’ ‘We believe, as the Arabs have shown us, that to deal with formulas we use algebra,’ said the Church; opposed by Newton saying: ‘No, we need a different algebra of change which I will call calculus.’

By discovering a physical predictable will Newton inspired a sophist revival in the Enlightenment Century: With moons and apples obeying their own physical will instead of that of a metaphysical patronizer, once enlightened about the difference between nature and choice, humans can do the same and do without a double patronization by the Lord at the manor house and the Lord above. Thus, two Enlightenment republics were installed, one in North America in1776 and one in France in 1789.

The US still has its first republic showing skepticism towards philosophical claims by developing American pragmatism, symbolic interactionism and grounded theory; and by allowing its citizens to uncover and develop talents through daily lesson in self-chosen half-year blocks in secondary and tertiary education.

France now has its fifth republic turned over repeatedly by their German neighbors seeing autocracy as superior to democracy and supporting Hegel’s anti-enlightenment thinking reinventing a metaphysical Spirit expressing itself through the history of different national people. To protect the republic, France established line-organized and office-directed elite schools, copied by the Prussia wanting to prevent democracy by Bildung schools meeting there criteria: The population must not be enlightened to prevent it asking for democracy as in France; instead a feeling of nationalism should be installed transforming the population into a people following the will of the Spirit by fighting other people especially the French; and finally the population elite should be extracted and receive Bildung to become a knowledge nobility for a new strong central administration to replace the inefficient blood nobility unable to stop democracy from spreading from France.

To warn against hidden patronization in institutions, France developed a post-structuralist thinking inspired by existentialist thinking (Tarp, 2016), especially as expressed in what Bauman (1992, p. ix). calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'?

> Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of 'is'. One claims existence, 'M is', one claims 'how M is' to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create 'M is N'-statements.

Thus, there are four different uses of the word 'is'. 'Is' can claim a mere existence of M, 'M is'; and 'is' can assign predicates to M, 'M is N', but this can be done in three different ways. 'Is' can point down as a 'naming-is' ('M is for example N or P or Q or …') defining M as a common name for its volume of more concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, 'is' can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its 'predicate-prison'. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a different mathematics curriculum, e.g. one based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

The philosophical and sociological background for difference research may be summed up by the Heidegger warning: In sentences, trust the subject but question the rest since it might be gossip. So, to restore its authenticity, we now return to the original subject in Greek mathematics, the physical fact

Many, and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language. The ‘pencil-paradox’ observes that placed between a ruler and a dictionary, a pencil can itself point to its length but not to its name. This shows the difference between the two languages, the word-language is for opinions, the number-language is for prediction.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many in total?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

Thus, we can ask: What happens if looking at mathematics differently as a number-language? Again, difference-research might provide an answer.

Examples of Difference-research

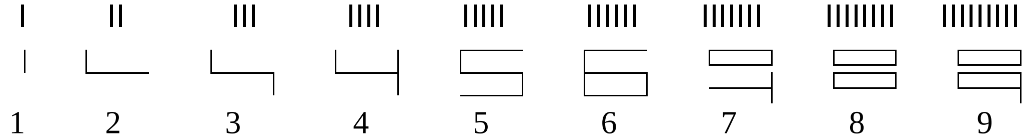
To prevent that mathematics becomes a meta-language that can be applied to describe and solve real-world problems, we must be careful with our language. Although it seems natural to talk about mathematics and its applications, this includes the logic that ‘of course mathematics must be learned before it can be applied’. Which is equivalent to saying ‘of course a grammar must be learned before it can be applied to describe a language’. This would lead to widespread illiteracy if applied to the word-language. And ‘grammar before language’ might be the cause of several problems in mathematics education. Of course, the subject must exist before the sentences can be made about it. So differences typically come from respecting that the number-language comes before its grammar and after meeting and experiencing the subject of its sentences, the total, describing the physical fact Many.

Digits as icons

A class of beginners, e.g. preschool or year 1 or migrants, is stuck in the traditional introduction of digits as symbols like letters. Some confuse the symbols, some have difficulties writing them, some can’t see why ten is written 10, some ask why eleven and twelve is not called ten-1 and ten-2.

Here a difference is to use a folding ruler to discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four-icon, and five sticks in the five-icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc.

This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning ‘1 left’ and ‘2 left’ in Danish, understood that the ten-bundle has already been counted.



Will this difference make a difference? In theory, yes, since rearranging physical entities into icons, e.g. five cars into in a five-icon, makes the icons physically before being formally written down. In his genetic epistemology, Piaget expresses a 'greifen-vor begreifen' principle, grasping physically before mentally. Thus, going from unordered cars to cars ordered into an icon to writing down the icon includes three of the four parts of his stage theory, the preoperational and the concrete operational and the formal operational stage. In practice, it works on a pilot study level thus being ready for a more formal study.

Counting sequences in different forms

A class of beginners have problems with the traditional introduction of the counting sequence and the place value system. Some count 'twenty-nine, twenty-ten, twenty-eleven'. Some mix up 23 and 32.

Here a difference is to count a total of a dozen sticks in fives using different counting sequences: '1, 2, 3, 4, bundle, 1-bundle-1, …, 2 bundles, 2-bundles-1, 2-bundles-2'. Or '01, 02, 03, 04, 10, 11, …, 22'. Or '.1, .2, .3, .4, 1., 1.1, …, 2.2'. Or '1, 2, bundle less 2, bundle less 1, bundle, bundle&1, bundle&2, 2bundle less 2, 2bundle less 1, 2bundles, 2bundles&1, 2bundles&2.'

Using a cup for the bundles, a total can be 'bundle-counted' in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and 'less 1' outside; or, if using 'bundle-writing' to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, shows that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s; and t = 3B1 tens = 3.1 tens = 31 if leaving out the unit and misplacing the decimal point

Will this difference make a difference? In theory, yes, since counting by taking away bundles and placing one stick in a cup per bundle again combines the three operational parts of Piaget's stage-theory allowing the learner to see, that a number has three parts: a unit, and some bundles inside the cup, and some unbundled outside. In practice, it works on a pilot study level thus being ready for a more formal study.

Multiplication tables made simpler

A class is stuck in multiplication tables. Some add instead of multiplying, some tries to find the answer by repeated addition, some just give random answers, and some have given up entirely to learn the tables by heart.

Here a difference is to see multiplication as a geometrical stack or block that recounted in tens increases its width and therefore decreases its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21.

Another difference is to begin by reducing the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc.

Thus, beginning with doubling visualized by LEGO bricks, T = 2 6s = 2*6 = 2*(5&1) = 10&2 = 12, or T = 2*6 = 2*(10-4) = 20-8 = 12. And T = 2 7s = 2*7 = 2*(5&2) = 10&4 = 14, or T = 2*7 = 2*(10-3) = 20 – 6 = 14. Doubling then can be followed by halving that by counting in 2s will introduce a recount-formula T = (T/B)*B saying that T/B times B may be taken away from T: So when halving 8, 8 = (8/2)*2 = 4 2s, and 9 = (9/2)*2 = (<u>8&1</u>/2)*2 = (4&1/2)*2 = 4 & ½ 2s.

As to tripling, T = 3*7 = 3*(10-3) = 30 – 9 = 21.

Proceeding with factors after 2 and 3, 2-by-2 Medieval multiplication squares can be used to see that e.g. T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Division using bundle-writing and recounting

A class is stuck in short and long division. Some subtract instead of dividing, some invent their own algorithms typically time-consuming and often without giving the correct answers, some give up because they never learned the multiplication tables.

Here a difference is to talk about 8/2 as '8 counted in 2s' instead of as '8 divided between 2'; and to rewrite the number as '10 or 5 times less something' and use the results from a multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

As to long division, here a difference is to combine renaming numbers using bundle names, e.g. sixty-five as 6ten5, with bundle-writing allowing recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

Once bundle-writing is introduced, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

Thus, by recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down all operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Proportionality as double-counting creating per-numbers

A class stuck in proportionality. Nearly all find the $-number for 12kg at a price of 2$/3kg but some cannot find the kg-number for 16$. Here a difference is to see the price as a per-number, 2$ per 3kg, bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Will this difference make a difference? In theory, yes, since proportionality is translated to a basic physical activity of counting and recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Fractions and percentages as per-numbers

A class is stuck in fractions. Rewriting fractions by shortening or enlarging, some subtract and add instead of dividing and multiplying; and some add fractions by adding numerators and denominators.

Here a difference is to see a fraction as a per-number coming from double-counting in the same unit, 3/5 = 3$ per 5$, or as percentage 3% = 3/100 = 3$ per 100%. Thus 2/3 of 12 is seen as 2$ per 3$ of 12$ that recounts in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$ of the 12$. So 2/3 of 12 is 8.

Other examples are found in economy investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$.

The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100$ in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%.

As per-numbers, also fractions are operators needing a number to give a number: a half is always a half of something as shown by the recount-formula T = (T/B)*B = T/B Bs. So also fractions must have units to be added.

If the units are different, adding fractions means finding the average fraction. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as the tradition teaches.

Taking fractions of the same quantity makes the unit the same, assumed to be already bracketed out, so that T = a/b + c/d really means T = (a/b + c/d) of (b*d). Thus adding 2/3 and 4/5 it is implied that the fractions are taken of the same total 3*5 = 15 that is bracketed out, so the real question is 'T = 2/3 of 15 + 4/5 of 15 = ? of 15, giving T = 10 + 12 = 22 = (22/15)*15 when recounted in 15s.

Thus, adding fractions is ambiguous. If taken of the same total, 2/3 + 4/5 is 22/15; if not, the answer depends on the totals: 2/3 of 3 + 4/5 of 5 is (2+4)/(3+5) of 8 or 6/8 of 8, and 2/3 of 3 + 4/5 of 10 is 10/13 of 13, thus providing three different answers, 22/15 and 6/8 and 10/13, to the question '2/3+4/5 = ?'

Hiding the ambiguity of adding fractions makes mathematics 'mathe-matism' true inside but seldom outside classrooms.

As to algebraic fractions, a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

As when adding fractions, adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph. Thus calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level. n practice, it works on a pilot study level thus being ready for a more formal study.

Will this difference make a difference? In theory, yes, if first performing double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand.

Equations as walking or recounting

A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. when equations are composite or with a reverse sign in front of the unknown.

Here a difference is to use the definitions of reverse operations to establish the basic 'OSS'-rule for solving equations, 'move to the Opposite Side with the opposite Sign'. Thus, in the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise, with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12).

As to 2+3*u = 14, a difference is to see it as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the opposite side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4 since 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14.

Another difference is to see 2+3*u =14 as a walk, first multiplying u by 3 then adding 2 to give 14. To get back to u we reverse the walk by performing the reverse operations in reverse order. Thus, first subtracting 2 and then dividing by 3 gives u = (14-2)/3 = 4, checked by repeating the walk now with a known staring number: 4*3+2 = 14. Seeing an equation as a walk motivates using the terms 'forward and backward calculation sides' for 2+3*u and 14 respectively.

With 25 – u = 14, u moves to the opposite side to have its reverse sign reversed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5 giving 40 = 5*u; 40/5 = u; 8 = u. Alternatively, recounting twice gives 40 = (40/u)*u = 5*u, and 40 = (40/5)*5, consequently u = 40/5.

Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' When building a routine, students often have fun singing:

"Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

Another difference is to introduce equations the foist year in primary school as another name for recounting from tens to icons, e.g. asking 'How many 9s are 45' or 'u*9 = 45' giving u = 45/9 since recounting 45 in 9s, the recount formula gives 45 = (45/9)*9, again showing the OppositeSide&Sign rule.

Likewise, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again obtained by moving a number to the opposite side with the opposite calculation sign.

Will this difference make a difference? In theory, yes, since equations are related to something concrete, walking or recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Geometry and algebra, always together, never apart

A class is stuck in geometry. Some mix up definitions, some find the theorems to abstract to understand, some find proofs difficult and hard to remember, some find geometry boring.

Here a difference is to use a coordinate system to coordinate geometry and algebra so they go hand in hand always and never apart, thus using algebra to predict geometrical intersection points, and vice versa, to use intersection points to solve algebraic equations. Both in accordance with the Greek meaning of mathematics as a common label for algebra and geometry.

In a coordinate-system a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates. Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side Δx = x-xo and a vertical side Δy = y-yo and a diagonal distance r from A to B, where by Pythagoras r^2 = Δx^2 + Δy^2. The angle A is found by the formula tanA = Δy/Δx = s, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: Δy/Δx = s or Δy = s*Δx, or y-yo = s*(x-xo). Vertical lines have the formula x = xo since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be Δy/Δx and -Δx/Δy respectively, so that s1*s2 = -1, called reciprocal with opposite sign.

As mentioned, geometrical intersection points are predicted algebraically by equating formulas. Thus with the lines y = 2*x and y = 6-x, equating formulas gives 2*x = 6-x, or 3*x = 6, or x = 2, which inserted in the first gives y = 2*2 = 4, thus predicting the intersection point to be (x,y) = (2,4). The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation x^2 + b*x + c = 0, solved by a solver. Or by a formula created by two x-by-(x+k) playing cards placed on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. This creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution formula is x = -b/2 ±√((b/2)^2 – c).

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

Will this difference make a difference? In theory, yes, since coordinating geometry and algebra gives equations a geometrical form and allows geometrical situations to be predicted by equations. In practice, it works on a pilot study level thus being ready for a more formal study.

Trigonometry as right triangles with sides mutually recounted

A class is stuck in trigonometry. Some find the ratios to abstract to understand, some mix up the formulas, some find the algebra difficult to use.

A difference is to introduce trigonometry as blocks halved in two by its diagonal, making a rectangle split into two right triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in the base, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C splits the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area (a*cosB)*c = a*(cosB*c) = a*a = a^2. Consequently c^2 = a^2 + b^2, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. x = √24, solving the quadratic equations x^2 = 24 = 4*6, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor's half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root √24, measured to 4.9, which checks: 4.9^2 = 24.0.

A triangle that is not right-angled transforms into a rectangle by outside right triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is 2*(4*sin(180/4)), or 2*(8*sin(180/8)) with 8 sides instead. Consequently, the circumference of a circle with radius 1 is 2*π, where π = n*sin(180/n) for n large.

Will this difference make a difference? In theory, yes, since in Greek, geometry means to measure earth, typically by dividing it into triangles, again divided into right triangles, which can be seen as rectangles halved by their diagonals; and recounting totals in new units leads directly to mutual recounting the sides in a right triangle, which leads on to a formula for calculating pi. Furthermore, the many applications of trigonometry might increase the motivation for learning more geometry where coordinate geometry uses right triangles to increase any triangle to a rectangle with horizontal and vertical sides. In practice, it works on a pilot study level thus being ready for a more formal study.

PreCalculus as constant change

A class is stuck in precalculus. Some find the function concept to abstract to understand, some sees f(2) as a variable f multiplied by 2, some cannot make sense of roots and logarithm. The tradition defines a function top-down from above as a set-relation where first-component identity implies second component identity.

A difference is to return to the original Euler-meaning of a function defining it bottom-up from below as a name for a formula containing specified and unspecified numbers. And to see a formula as the core concept of mathematics respecting that, whatever it means, in the end mathematics is but a means to an outside goal, a number-language.

As a number-language sentence, a formula contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number Δy/Δx called the gradient of the formula.

As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from the formula 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a bending line, a parabola.

The difference seeing functions as predicting number-language sentences also suggests that functions in the form of formulas should be introduced from the first class of mathematics to predict counting results by a calculator, allowing the basic operations to be introduced as icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. Also, the four basic operations uncover the original meaning of the word algebra, meaning 'to reunite' in Arabic: Addition unites unlike numbers, multiplication unites like numbers into blocks, power unites like factors, and integration unite unlike blocks.

Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s, using a cup or a decimal point to separate the 'inside' bundles from the 'outside' unbundled. This prediction holds at a manual counting:

T = 7 = I I I I I I I = III III I = 2 3s & 1.

Thus a calculator can predict a counting result by describing the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, i.e. with two formulas. The 'recount formula' T = (T/B)*B says that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B says that 'from T, T–B is left when B is taken away and placed next-to', as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. Wanting to recount a total in a new unit, the two formulas can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Recounting a total in a new unit means changing unit, also called proportionality or linearity, a core concept in mathematics at school and at university level. Thus the recount formula turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve by recounting a vertical change in a horizontal.

Will this difference make a difference? In theory, yes, since describing mathematics as the grammar of the number-language is a powerful metaphor uncovering the real outside goal of mathematics education, to develop a number-language having the same sentence structure as the word-language, which will demystify the nature of mathematics to many students. In practice, it works on a pilot study level thus being ready for a more formal study.

Calculus as adding locally constant per-numbers

A class is stuck in calculus. Some find the limit concept too abstract. Some find the applications too artificial. For some, their hate to differential calculus prevents them from learning integral calculus.

Here a difference is to postpone differential calculus till after integral calculus is presented as a means to add piecewise or locally constant per-numbers by their areas. Thus, when adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 add by their areas as 3*2 + 5*4, meaning that per-numbers add by the area under the per-number-graph. With a piecewise constant per-number this mean a small number of area strips to add. But seeing a non-constant per-number as locally constant it means adding a huge amount of area strips, only possible if we can rewrite the strips as differences since the disappearance of the middle terms makes many differences add up to one single difference between the terminal and initial number. This of course makes rewriting a formula as a difference highly interesting, thus motivating a study of differential calculus. Thus, with the area strip 2*x*dx written as d(x^2), summing up the strips gives a single difference:

$$T2-T1 = \Delta(x^2) = \Sigma\ \Delta T = \int dT = \int f(x)*dx = \int 2*x*dx\ .$$

Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip Δb*h and a horizontal strip b*Δh and a corner Δb*Δh that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T's: dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h. Therefore (x^2)'/x^2 = x'/x + x'/x = 2/x, giving (x^2)' = 2*x since x' = dx/dx = 1.

As to the limit concept, a difference is to rename it to 'local constancy': In a function y = f(x) a small change x often implies a small change in y, thus both remaining 'almost constant' or 'locally constant', a concept formalized with an 'epsilon-delta criterium', distinguishing between three forms of constancy. y is 'globally constant' c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is 'piecewise constant' c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is 'locally constant' c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

Finally, calculus allows presenting the core of the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Will this difference make a difference? In theory, yes, since presenting it as adding piecewise or locally constant per-numbers will ground integral calculus in meaningful real-world problems. Likewise, observing the enormous advantage in adding differences gives a genuine motivation for differential calculus that is lost if insisting that it comes before integral calculus. In practice, it works on a pilot study level thus being ready for a more formal study.

How Different is the Difference?

Difference research uses sociological imagination to revive the ancient sophist warning: Know nature from choice to discover choice presented as nature. Thus, true and false nature are separated by asking the tradition: Can this be different, and will the difference make a difference? Witnessed by 50 years of sterility, mathematics education research is a natural place to see if difference-research, DR, will make a difference.

The tradition says, 'To obtain its goal, to learn mathematics, mathematics education must teach mathematics!' DR objects, 'No, to obtain its goal, mastery of Many, mathematics is a means to be replaced by another means if not leading to the goal, e.g. by 'Many-matics', defining its concepts from below as abstractions from examples instead of from above as examples of abstractions as does the traditional 'meta-matics'.

The tradition says, 'The core of mathematics is to operate on numbers!' DR objects, 'No, the core of mathematics is number-language sentences describing how totals are counted and recounted before being added; and having the same sentence structure as the word-language: a subject, a verb and a predicate.'

The tradition says, 'Digits must be taught as symbols like letters!' DR objects, 'No, digits are icons containing as many strokes as they represent.'

The tradition says, 'To describe cardinality, numbers must be taught as a one-dimensional number-line!' DR objects, 'No, numbers are two-dimensional blocks counting a total in stacks of bundles and unbundled singles.'

The tradition says, 'Natural numbers must be taught as a place value system and ten-bundling is silently understood!' DR objects, 'No, numbers should be taught using bundle-writing to separate inside bundles from outside singles, making a natural number a decimal number with a unit. And ten-counting should be postponed until icon-counting and re-counting in the same and in a different unit has been experienced'.

The tradition says, 'There are four kinds of numbers, natural and integer and rational and real numbers!' DR objects, 'No, a number is a positive or negative decimal number with a unit. Rational numbers are per-numbers, i.e. operators needing a number to become a number; and real numbers are calculations to deliver as many decimals as wanted.'

The tradition says, 'Operations must be taught as functions from a set-product to the set supplying it with a structure obeying associative, commutative and distributive laws as well as neutral and inverse elements allowing equations to be solved by neutralization!' DR objects, 'Operations are icons showing the three processes of counting, bundling and stacking and removing stacks to look for unbundled singles; and adding stacks or blocks on-top or next-to. Solving equations is another word for reversing the processes by re-bundling or re-stacking'

The tradition says, 'The natural order of teaching operations is addition before subtraction before multiplication before division allowing fractions to be introduced as rational numbers to which the same operations can be applied!' DR objects, 'No, since totals must be counted before they can be added, the natural order is the opposite: first division to take away bundles many times, then multiplication to stack the bundles, then subtraction to take away the stack once to look for unbundled singles, and finally addition in its two versions, on-top and next-to. And counting also implies recounting in the same or another unit, to and from tens, and double-counting producing per-numbers as operators needing numbers to become numbers, thus being added by their areas, i.e. by integration.'

The tradition says, 'Calculators should not be allowed before all four operations are taught and learned!' DR objects, 'Calculators should be used from the start to predict counting and recounting results.'

The tradition says, 'Operations must be taught using carrying!' DR objects, 'No, operations should be taught using bundle-writing allowing totals to be recounted with overloads or underloads.'

The tradition says, 'Multiplication tables must be learned by heart!' DR objects, 'No, multiplication tables describe recounting from icon-bundles to ten-bundles; geometrically seen as changing a block by increasing the width and decreasing the height to keep the total unchanged; and algebraically sees as doubling or tripling totals written with an overload or an underload.'

The tradition says, 'Division is difficult and must be taught using constructivism to allow learners invent their own algorithms!' DR objects, 'No, division should be taught as recounting from ten-bundles to icon-bundles using bundle-writing and recounting in the same unit to benefit from the multiplication tables.'

The tradition says, 'Arithmetic comes before geometry, and they must be held apart until the introduction of the coordinate system!' DR objects, 'No, arithmetic should be seen as algebra kept together with geometry all the time and from the beginning, where numbers are a collection of blocks as well as a collection of numbers in cups; where recounting and multiplication means changing block-sizes as well as changing bundle-numbers; and where addition means adding blocks as well as bundle-numbers.'

The tradition says, 'Proportionality must be postponed until functions have been introduced!' DR objects, 'No, as another name for changing units, proportionality occurs from the beginning as recounting in another unit; and is needed when adding on-top and next-to. And reoccurring when double-counting creates per-numbers as bridges between physical units.'

The tradition says, 'Fractions must be introduced first as parts of something then as numbers by themselves!' DR objects, 'No, created by double-counting in the same unit, fractions are per-numbers and as such operators needing a number to become a number.'

The tradition says, ‘Prime-factorizing must precede adding fractions by finding a common denominator!’ DR objects, ‘No, prime-factorizing comes with recounting to another unit to find the units allowing a total to be recounted fully without any unbundled singles. And fractions should be added as operators, i.e. by integrating their areas.’

The tradition says, ‘Equations must be taught as statements about equivalent number-names, solved by the neutralizing method obeying associative, commutative and distributive laws!’ DR objects, ‘No, equations occur when recounting totals from tens to icons, and when reversing on-top and next-to addition.’

The tradition says, ‘A function must be taught as an example of a set-relation where first-component identity implies second-component identity!’ DR objects, ‘No, a function should be taught as a formula with two unspecified numbers thus respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. Later polynomials can be introduced as the number-formula containing the different formulas for constant change: T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x.’

The tradition says, ‘Linear functions must be taught before quadratic functions!’ DR objects, ‘No, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d.’

The tradition says, ‘Quadratic equations must be solved by factorizing before introducing the solution formula!’ DR objects, ‘No, when solving the quadratic equation x^2+b*x+c = 0, algebra and geometry should go hand in hand to show that inside a square with the sides x+b/2, the equation makes three rectangles disappear leaving only (b/2)^2-c, allowing possible roots to be found and used in factorization if necessary.’

The tradition says, ‘Differential calculus must be taught before integral calculus since the integral is defined as the anti-derivate.’ DR objects, ‘No, integral calculus comes before differential calculus. In primary school, next-to addition means multiplying before adding when asking e.g. T = 2 3s + 4 5s = ? 8s’, while reversing the question by asking 2 3s + ? 5s = 6 8s, or T1 + ? 5s = T, leads to differential calculus subtracting before dividing to get the answer (T-T1)/5. In middle school, fractions and per-numbers add by their areas, i.e. by integration. And in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant predictable change.’

The tradition says, ‘The epsilon-delta definition is essential in order to understand real numbers and calculus and must be learned by heart!’ DR objects, ‘No, it needs not be learned by heart. With units, it can be grounded in formalizing three ways of constancy; globally constant needing only the epsilon, piecewise constant with delta before epsilon, and locally constant with epsilon before delta.’

The tradition says, ‘Statistics and probability must be taught separately!’ DR objects, ‘No, they should be taught together aiming at pre-dicting unpredictable numbers by intervals coming from ‘post-dicting’ their previous behavior.’

In continental Europe, the tradition says, ‘Education means preparing for offices in the public or private sector. Hence the necessity of line-organized education with forced year-group classes in spite of the fact that teenage girls are two years ahead of the boys in personal development. Of course, boys and dropouts are to pity, but they all had the chance.’ North American republics object: ‘No, Education means uncovering and developing the learner’s individual talents through daily lessons in self-chosen practical or theoretical half-year blocks together with a person teaching only one subject and praising the learner for having a talent or for having courage to test it.’

In mathematics education, the tradition says, ‘Education means connecting learners to the canonical correctness through scaffolding from the learner’s zone of proximal development as described in social constructivism by Bruner and Vygotsky.’ DR objects, ‘No, education means bringing outside phenomena inside a classroom to be assimilated or accommodated by the learners thus respecting that in a sentence, the subject is objective but the rest might be subjective as described in radical constructivism by Piaget and Grounded Theory and Heidegger existentialism.

In mathematics education, the tradition says, ‘Research means applying or extending existing theory.’ DR objects, ‘No, where master level work means applying existing theory, research level means questioning existing theory, e.g. by asking if it could be different.’

How to Improve PISA Performance

PISA performance (Tarp, 2015a) can be improved in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based core-curriculum for outsiders.

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as ‘Many-matics’ in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as ‘meta-matism’ mixing ‘meta-matics’, defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with ‘mathe-matism’, true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle ‘through the hand to the head’.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask ‘How many in total?’ To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter ‘Examples of difference-research’.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition's 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying 'T = 2*3' instead of just '2*3'.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with bundle-counting and bundle-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as 'mathematism' true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2**u* = 8	an open statement about two equivalent number-names
(2**u*)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(*u**2)*(1/2) = 4	since multiplication is commutative
*u**(2*(1/2)) = 4	since multiplication is associative

*u**1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation ‘2*u = 8’ means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the ‘opposite side&sign’ method. A method that allows the equation ‘20/u = 5’ to be solved quickly by moving across twice; 20 = 5*u’ and 20/5 = u’, or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers.

And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n* *n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Thus, as described above in the chapter 'examples of difference-research', in primary school, problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.
- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.
- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336, or T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336
- when dividing, using bundle-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48
- when subtracting, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38
- when adding, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

In middle school, problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

In high school, problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus, calculus and statistics as pre- or post-dicting constant, non-constant and non-predictable change

A Macro STEM-based Core Curriculum

A macro-curriculum (Tarp, 2017) was designed as an answer to a fictitious curriculum architect contest set up by a Swedish university wanting to help the increasing number of young male migrants coming to Europe each year: 'The contenders will design a STEM-based core mathematics curriculum for a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.'

The design was inspired by an article on STEM (Han et al, 2014). Thus te curriculum goal is mastery of Many in a STEM context for learners with no background. As to STEM, OECD writes:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules, called heat; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial forms, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, mastery of Many means counting and recounting and adding and reversing addition and describing change and spatial shapes.

A STEM-based core curriculum can be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Teaching Differences to Teachers

A group of teachers wanting to bring difference-research findings to the classroom might want first to watch some YouTube videos at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many.

Then to try out the ‘Free 1day SKYPE Teacher Seminar: Cure Math Dislike by 1 cup and 5 sticks’ where, in the morning, a power point presentation ‘Curing Math Dislike’ is watched and discussed locally, and at a Skype conference with a coach. After lunch the group tries out a ‘BundleCount before you Add booklet’ to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in the three genres of quantitative literature, fact and fiction and fiddle. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Being a Difference-Researcher

In mathematics education, difference-research can be used by teachers observing problems in the classroom, or by teacher-researchers splitting their time between academic work at a university and intervention research in a classroom. Or by full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, working out a different micro-curriculum together to be tested by the teacher and reported by the researcher conducting a pretest-posttest study.

Thus, a typical difference-researcher begins as an ordinary teacher observing learning problems in his classroom and wondering if he could teach differently. Personally, in a precalculus class I taught linear and exponential functions by following the textbook order presenting them as examples of functions, again presented as examples of relations between two sets assigning one and only one element in one set to each element in the other set. I realized that by defining concepts as examples of abstractions instead of as abstractions from examples, I basically taught that 'bublibub is an example of bablibab' which some learners just memorized while others refused to learn before I gave them some applications. Talking about the difference between saving at home and in a bank, some asked me: Instead of calling it linear and exponential functions, why don't you just call it change by adding and by multiplying since that is what it is?'

So here the students themselves invented a difference that makes sense since historically, functions came after calculus. And the difference made two differences. Nobody had problems with learning about change by adding and by multiplying. And the Ministry of Education followed my suggestion to replace functions with variables instead of making pre-calculus non-compulsory, which was the plan because of the high number of low marks.

So one way to become a difference-teacher is to combine elements from action learning and action research and intervention research and design research. First you identify a difference, then you design a micro-curriculum, then you teach it to learn what difference the difference makes, then you learn from reporting and discussing it internally with colleagues. After having repeated this cycle of teaching and reporting the difference, the difference and the difference it makes in a posttest or a pretest-posttest setting is reported externally to teacher magazines or to conferences or to research journals.

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. This raises two questions: Can a teacher produce research, and can research produce teaching? (Hammersley, 1993, p. 215). Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

Often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015b, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a

> contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report 'Improving Schools in Sweden' made this paper ask: Apparently half a century's research in mathematics education has not prevented low and declining PISA performance. Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? Seeking guidance by difference-research searching traditions for hidden differences that make a difference, the answer is: Yes, mathematics can be different, education can be different, and research can be different.

Looking back, mathematics has meant different things through its long history, from a common label for knowledge in ancient Greece to today's 'meta-matism' combining 'meta-matics' defining concepts by meaningless self-reference, and 'mathe-matism' adding numbers without units thus lacking outside validity. So, looking for a difference to traditional set-based meta-matism, one alternative is the original Greek meaning of mathematics: Knowledge about Many in time and space.

Observing Many, allows rebuilding mathematics as a 'many-matics', i.e. as a natural science about the physical fact Many, where counting by bundling and stacking leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus. And where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition.

Observing classes in continental Europe and in North America shows that education can be line-organized with forced year-group classes aiming at fulfilling the nation's need for officials for the public or private sector; or education can be block-organized with self-chosen half-year classes aiming at uncovering and developing the learner's individual talent. In mathematics education, the tradition sees learning mathematics as the goal of teaching mathematics and defines its concepts from above as examples of abstractions, part of the ruling canonical correctness, to be reached by learners through scaffolding. Here a difference is to accept that concepts historically arose from below as abstractions from examples, thus allowing new concepts to connect to existing.

Observing conference proceedings, shows that research papers may instead be master level papers applying instead of questioning existing theory and aiming at explaining instead of solving educational problems. Here a difference is difference-research searching traditions for hidden differences that make a difference.

So yes, as to mathematics education research, all three components can be different. Bottom-up many-matics can replace top-down meta-matism. In teenage education, daily lessons in self-chosen half-year blocks can replace periodic lessons in forced year-group lines. And, searching for useable differences can replace attempts at understanding the lack of understanding non-understandable self-reference.

Consequently, PISA performance may increase instead of decrease, and Swedish schools might improve dramatically by respecting that education means preparing learners for the outside world, brought inside to change the classroom from a library with self-referring textbooks to be learned by hart into a laboratory allowing the learner to meet the educational subject directly instead of indirectly through textbook 'gossip'. And by avoiding a goal displacement seeing mathematics as the goal for mathematics education, thus hiding the real goal, a number-language about Many in time and space.

To teach many-matics instead of meta-matism, big-scale in-service teacher training is needed, e.g. through the MATHeCADEMY.net, designed to teach teachers to teach mathematics as a natural science about Many by the CATS-approach, Count & Add in Time & Space, using PYRAMIDeDUCATION, where learners learn by being taught by the subject directly instead of indirectly by a sentence.

So, if a society as Sweden really wants to improve mathematics education, extra funding should force its universities to arrange curriculum architect contests to allow differences to compete as to imagination, creativity and effectiveness, thus allowing universities to rediscover their original external goal and to change their internal routines accordingly. A situation described in several fairy tales: The Beauty Sleeping behind the thorns of routines becoming rituals; and Cinderella making the prince dance, but only found when searching outside the canonical correctness.

With 2017 as the 500th anniversary of Luther's 95 theses, the recommendation of difference-research to mathematics education research could be the following theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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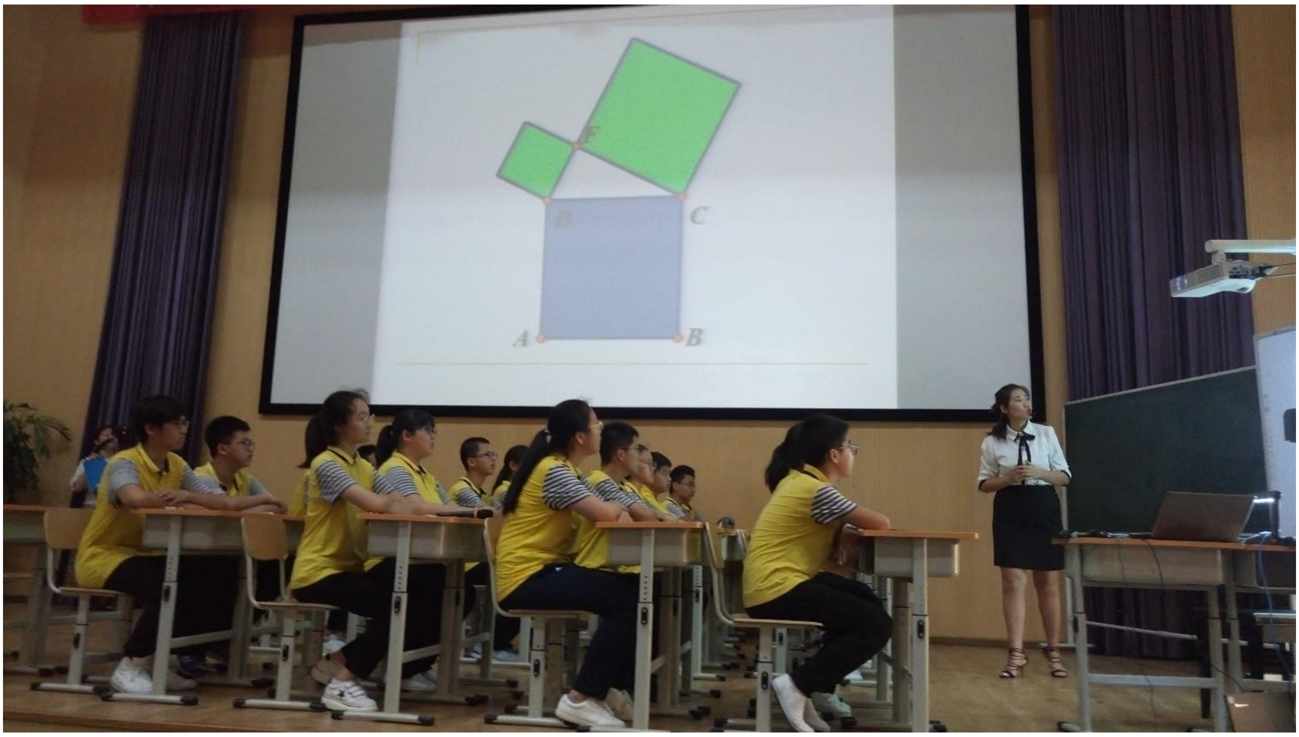
Reflections from the CTRAS 2017 Conference in China

Examples of Goal Displacements in Mathematics Education

At the annual conference of the Classroom Teaching for All Students Research Working Group (CTRAS), the 2017 conference theme was to promote classroom teaching research on exploring effective teaching strategies to support all students' mathematics learning. The two conference days contained half a day of plenary lectures. The first day also contained four examples of classroom teaching where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom lessons in China and the US. This paper reflects upon the lessons and some of the plenary lectures from a difference-research perspective looking for differences making a difference.



Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4th year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other countries also experience a low and declining PISA performance. And apparently research can do nothing about it. At a plenary discussion, it was mentioned that according to an American Educational Research Association report, many research studies on teacher education does not have value to classroom teachers and classroom teaching. So, to improve student performance, maybe a different kind of research is needed to rise questions as: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, maybe it is time to seek guidance by difference-research, searching for differences making a difference.

Searching for Hidden Differences, Difference-Research looks at Mathematics Education

Difference-research (Tarp, 2017) asks two questions: ‘Can this be different – and will the difference make a difference?’ Difference-research is inspired by the ancient Greek sophists looking for hidden differences to unmask choice masked as nature. If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

As to mathematics education, education is a social institution, and perhaps the most intervening one considering the numbers of hours spent there per week and during childhood and adolescence. As to institutions, Bauman talks about rationality and goal displacements in social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) It may happen in effect that the task originally seen as the reason to establish it is relegated to a secondary position by the all-powerful interest of the organization in self-perpetuation and self-aggrandizement. The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

So, in a social institution, its goal cannot be different unless a means is masked as a fake goal, to be unmasked and replaced by the original goal by difference-research finding hidden differences.

As an institution, mathematics education is a social organization with a ‘rational action in which the end to be achieved is clearly spelled out’, apparently aiming at educating students in mathematics, ‘we teach you mathematics so you can learn mathematics’. But this is a goal displacement created by meaningless self-reference (we teach you bublibub so you can learn bublibub). So, if mathematics isn’t the goal in mathematics education, what is? And, how well-defined is mathematics after all?

How Well-Defined is Mathematics?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, algebra, replacing arithmetic because of smarter numbers, and geometry, both rooted in the physical fact Many through their original meanings, ‘to reunite’ in Arabic and ‘to measure earth’ in Greek. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the ‘New Mathematics’.

Here the invention of the concept Set created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant definition by self-reference, i.e. defining a concept top-down as examples of abstractions instead of bottom-up as abstractions from examples. Thus the concept ‘function’, originally labeling a calculation containing both specified and unspecified numbers, was turned into a subset of a set-product where first-component identity implies second-component identity.

However, looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts: You can eat an example of an apple, but not the word 'apple'.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics, defining concepts as examples of abstractions instead of as abstractions from examples; and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units, as '1 + 2 IS 3', meets counter-examples as e.g. 1 week + 2 days is 9 days.

So, looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach that 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the number-language sentence 'The total is 3 8s' or 'T = 3*8' we treat the predicate, 3 8s or 3*8, as if it was the subject, which is a clear indication of a goal displacement. Thus, the total of fingers on a hand cannot be different, but the way they are counted can: T = 5 1s = 2 2s &1 = 1 2s & 3 = 3 2s less 1 = 1 3s & 2 etc.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers and operations and calculations.

However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. Furthermore, we are sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens. This might speed up writing, but might also slow down learning; together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, Lincolns Gettysburg address, 'Four scores and ten years ago' shows that not all count in tens. Thus in Denmark, seventy is called 'half four' with scores understood.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to unmask choice presented as nature is precisely the aim of difference-research.

How to find Hidden Differences?

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality, which then is used to collect reliable data and to test the validity of its findings by falsification attempts.

Hidden differences might be found by sociological imagination, seen as the core of sociology by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.'

However, often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research, seen by many teachers as useless because of its many references. In a Swedish context, this has been called the ‘irrelevance of the research industry’ (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don’t dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology’s great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are three examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead, thus marginalizing or forgetting its original outside goal.

Here Heidegger gives a tool to tell goals from means by pointing out, that in defining is-statements we should trust the subject but question the predicate since the subject, by its existence, cannot be different whereas the predicate is a judgement that might be a prejudice, i.e. one among several means that can be different, as illustrated above when reporting on the number of fingers on a hand.

Heidegger sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ (Dasein) must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (..) Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer you curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to your world and yourself (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

Thus Foucault (1995) sees a school as a ‘pris-pital’, i.e. a mixture of a prison and a hospital. A school is prison-like by forcing students to stay together in classes for a long period of time, where continental Europe uses multi-year lines based upon age, in contrast to North America that from secondary school uses self-chosen half-year blocks.

And a school is hospital-like by wanting to cure the students by treating them for a diagnose that is not always that well-defined, and in many cases self-referring as when saying: we teach you mathematics so you can learn mathematics.

So, to make education a meaningful and civilized ‘pris-pital’, a diagnose must refer to a lack of or in knowledge about outside things or phenomena that students will meet when leaving school.

Thus, the original educational goal, to prepare children and adolescents for mastering the outside world, leads to two questions: What should the students meet in the classroom, the outside world brought inside, or descriptions of it in textbooks? And should all students meet the same in forced multi-year classes or be allowed to choose individually between half-year blocks?

And, to make its education a meaningful and civilized cure we must confront mathematics, seen as a collection of definitions and truth-claims, with two questions: Are the definitions self-referring or rooted in the outside goal, Many? Has the inside truth outside validity also?

Heidegger’s warning ‘In sentences, trust the subject but question the rest’ implies that to discover the true nature of the subject hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its ‘predicate-prison’. By opening us, Many will appear with its nature undisguised, thus allowing us to construct different mathematics micro- and macro-curricula.

So we now return to the original subject in Greek mathematics, the physical fact Many, and use sociological imagination and Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties (Tarp, 2017). We do so to answer the question: How to find differences debunking mathematics from a goal to an inside means to the real outside goal, mastery of Many.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we count and add to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence. We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to direct or reversed addition. Thus, to count we take away bundles (thus rooting division) to be stacked (thus rooting multiplication) to be moved away to look for unbundled singles (thus rooting subtraction); finally we answer using bundle-writing for the bundles inside the bundle-cup and the singles outside, possibly with an overload or an underload to be removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s = 2 1/3 3s = 2.1 3s (thus rooting fractions and decimals to describe the singles). The result is predicted by a re-count formula T = (T/B)*B saying that ‘from T, T/B times B can be taken away’. Re-counting in another unit roots proportionality. A total counted in icons can be re-counted in tens (thus rooting multiplication tables), or a total counted in tens can be re-counted in icons (thus rooting equations).

Double-counting in physical units creates per-numbers (again rooting proportionality) becoming fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas (thus rooting calculus).

Once counted or re-counted, totals can be added on-top after being re-counted in the same unit (again rooting proportionality); or next-to as areas (again rooting integral calculus). Then both on-top and next-to addition can be reversed (thus rooting equations and differential calculus).

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Thus, a Count&Multiply&Add curriculum differs from the tradition by presenting counting and multiplication before addition, and by using calculus to add fractions as per-numbers (Tarp, 2017).

Classroom Lessons

At the CTRAS 2017 conference, the first day contained four example of classroom lessons where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom teaching in China and the US.

B. China Teacher Lesson Display, Grade 5

The first lesson was a China teacher lesson display with a grade 5 class. The task was to fill a 3x3 square with the numbers 1-9 are so that they add up to 15 horizontally, vertically and on the diagonals, motivated by a video sequence from a fairy tale showing that this would lift a spell.

Personally, I found this an interesting task allowing the children to use their imagination and creativity. Likewise, a motivating video was a good idea. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

Based upon the principle ‘algebra & geometry, always together, never apart’, symmetry is present on the geometry part, so it ought also to be present on the algebra part, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as ‘Bundle less or plus’ using five as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4, inspired by the Roman numbers and a Chinese or Japanese abacus.

By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to ‘adding up to zero’. Because of the symmetry in geometry and algebra, 0 must be in the middle. Seeing zero as an even number, the three terms must be odd+odd+even, so the corners must be odd numbers.

Thus, the task could split up in several subtasks:

1. Starting by 5, find a symmetrical way to count from 1 to 9. Describe the symmetry.

2. Reformulate the task using these new numbers. Which number must be placed in the middle?

3. Adding two numbers to an odd number, how can the result be an even number?

4. Which numbers must be placed in the corners?

5. Show and test the answer using the numbers 1-9.

In a Count&Multiply&Add curriculum, re-counting the numbers from 1 to 9 in 5s is a routine task since the fingers on a hand is counted as ‘1 or bundle less 4; 2 or B-3 etc.’. Bundle-counting implies that you chose a bundle-size for the cup. In this case the sum 15 is obtained by three numbers, so 5 would be a natural choice as bundle-size allowing re-counting as T = 3 = 1B-2 5s, and T = 8 = 1B3 5s, etc. In such a class, the first subtask would be: ‘1. With the sum 15 obtained by three numbers, chose a bundle-size and reformulate the task.’

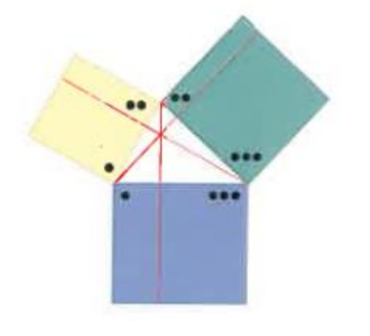
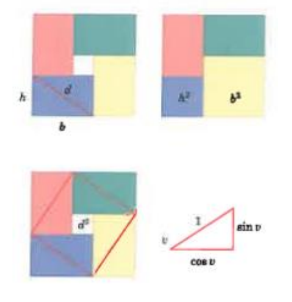
B. China Teacher Lesson Display, Grade 8

The second lesson was a China teacher lesson display with a grade 8 class. The task was to give a geometrical proof of the Pythagoras Theorem

Personally, I found this an interesting task allowing the adolescents to use their imagination and creativity. A proof is a core task in classical geometry; and choosing the Pythagoras Theorem as a core theorem is a good idea. I observed that some students seemed to find the task difficult. This raises the question: 'Will a different approach make a difference as to how many students succeed?' So, from the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if the Pythagoras Theorem is a goal or a means. Thus the Pythagoras Theorem may be seen as an inside means to the outside goal of adding travel-distances. If parallel, two distances add or subtract directly. But if perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps since $3^2 + 4^2 = 5^2$.

Based upon the principle 'algebra & geometry, always together, never apart', the task could contain both a geometrical and an algebraical proof. If it is correct that the theorem can be proved in more than 100 ways, two easy proofs could be used first to include all students, and two more difficult proofs could be added later, as could a proof using trigonometry.

As to the background, three cases can be mentioned: an isosceles right-angled triangle, a right-angled triangle, and an arbitrary triangle leading to $a^2 + a^2 = c^2$ (or $c = a*\sqrt{2}$), $a^2 + b^2 = c^2$; and $a^2 + b^2 – 2*a*b*\cos C = c^2$, in its algebraic versions; or in its geometrical version: the three heights split the opposite squares in parts that are like to its outside neighbors. An easy algebraic proof is the one showing that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled.	
An easy geometrical proof could be the one presented the next day in the plenary lecture 'The wisdom of Traditional Mathematical Teaching in China', shown with playing cards: Place four h-by-b playing cards after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. The full area can be expressed in two ways, as c^2 + 4 half cards, or as $h^2 + b^2$ plus two full cards. Consequently $h^2+b^2 = c^2$.	

Thus, the task splits up in several subtasks:

1. A 1.4-by-1.4 square is split into four triangles by the two diagonals. Prove that the triangles are isosceles and right-angled. Prove geometrically and algebraically that the Pythagoras Theorem $a^2+b^2 = c^2$ applies here. Measure the length of the diameter - are you surprised?

2. Draw a triangle with three angles less than 90 degrees. The three heights split the opposite squares in two parts. What can be said about the areas of two outside neighbors? Does this also apply to a right-angled triangle?

3. A geometrical proof of the Pythagoras Theorem uses four h-by-b playing cards placed after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. How can the total area be expressed?

4. Give an algebraic proof of the Pythagoras Theorem by using the result from question 2 and by splitting c in c1 and c2.

5. Tossing two dices gives the number of steps horizontally and vertically on a squared paper. Predict the length of the shortcut and test by measuring.

6. A 2meter bar is carried around a right-angled corner. How wide must the corridor be?

In a Count&Multiply&Add curriculum, counting includes a mutual re-counting of the sides in a right-angled triangle, seen as a rectangle halved by a diagonal. This allows trigonometry to be taught before geometry in accordance with the Greek meaning, earth-measuring. Thus, in a triangle ABC with C as the right angle and the side c split in c1 and c2 by the height, cosA = c1/b = b/c, or b^2 = c*c1 and likewise with cosB. This shows that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled. So in this case c^2 = a^2 + b^2 – 2*a*b*cosC. The subtasks would be the same.

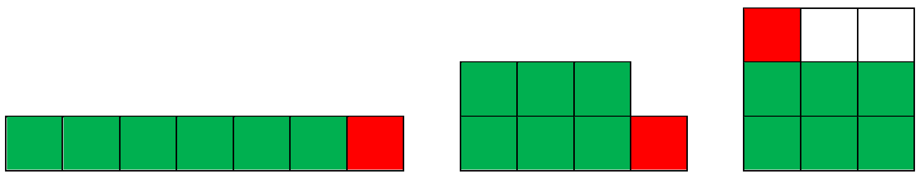
C. American Teacher Lesson Display, Grade 3

The third lesson was an American teacher lesson display with a grade 3 class. The task was to learn about and apply fractions, a core concept in algebra. I observed that some students seemed to find the task difficult. This raises the question: 'Will a different approach make a difference as to how many students succeed?' So, from the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if fractions is a goal or a means. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting and switching units and parting.

Thus 'icon-counting fractions' occur when counting a total by bundling and stacking, which creates a double-counting of bundles and unbundled leftovers that can be placed in a separate stack for unbundled ones, separated by a decimal point, or on-top of the stack counted in 3s as a fraction creating a mixed number, since counting in 3s means taking away 3s, i.e. divide by 3.

Thus, a total of 7 can be counted as:



T = 7 = 2.1 **3s** = 2 1/3 **3s**

Re-counting 6 7s in tens gives T = 6 7s = 6*7 = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and misplacing the decimal point.

'Per-number fractions' or 'unit switching fractions' occur when double-counting something in the same or in different units. Counting in different units, per-numbers as 4$/5kg or 4/5 $/kg allows bridging the units by re-counting in the per-number:

10$ = (10/4)*4$ = (10/4)*5kg = 12.5kg; and 20kg = (20/5)*5kg = (20/5)*4$ = 16$.

'Parting fractions' are per-numbers coming from double-counting a part and the total in the same unit: If 5 of 7 apples are green, the fraction 5/7 of the 7 apples is the green part. Splitting a total in the ratio 2:3 means getting the fractions 2/5 and 3/5 of the total.

An outside sharing-situation can be a root or an application.

Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. In this case, the outside goal sharing roots the inside means icon-counting and per-numbers.

Or, sharing 8 apples between 4 persons may be presented as an application of getting the fraction ¼ of 8, found by dividing 8 by 4. Postponed to after division and fractions have been taught and learned, this is an example of a goal displacement, where the inside means, divisions and fractions, are treated as goals in need of outside applications as means for student motivation.

In education, a choice should be made as to which fraction should be taught first. In the actual lesson, the choice seemed to be teaching parting fractions as 2/7 by double-counting the part and the total, and to apply them to describe a self-designed packman, although the lesson also contained examples of sharing fractions when dividing a geometrical figure.

Observing the Piaget principle ‘through the hand to the head (greifen vor begreifen)’, one way to introduce parting fractions could be using the biological counters, the fingers: On my left hand, the fingers can be straight or bent. If 2 of the 5 fingers are bent I will say that the fraction 2 of 5 or 2/5 of my fingers are bent. If no fingers are bent, the fraction is 0/5. If all five fingers are bent, the fraction is 5/5. Thus, a fraction is used to describe a double-counting of a part in the total. In the fraction 2/5, 2 is called a numerator since it numbers the specials; and 5 is called a denominator since it names the total, and ‘nomen’ is ‘name’ in latin.

Later both hands can be used to illustrate fractions as 7/10, or 5/8 if excluding the thumbs.

Next step could be to discuss what is meant by saying that 3/5 of my ten fingers are bent. Here a choice must be made between parting and per-number fractions.

As to parting fractions, looking at the ten fingers I must apply my mathematical knowledge to say: I find the fraction 1/5 by splitting the total in 5 equal parts, which is done by dividing 10 by 5 giving 2. Now I can multiply with 3 to get 6. So I bend 6 fingers’.

As to per-number fractions, looking at the ten fingers I reformulate the task: the fraction 3/5 means taking 3 per 5, and with ten as two 5s, I just bend 3 fingers on both hands, i.e. 6 fingers’

As alternative means to the same goal, both should be presented in the class to observe differences as to effect.

Another option is to introduce parting fractions in a symmetry context using a dice and writing a cross if the dice shows an even number. Such a task splits up in several subtasks:

1. Put your left hand flat on the table with all finger straight.

2. On a dice, which numbers are even and why? The rest are called odd.

3. On this paper you find 10 rows with 5 squares in each row. Throw a dice five times. If even, bend a finger and write a cross; else leave the finger straight and write nothing. And report the number of crosses as a fraction 3 of 5 and as 3/5. Each time, mention which is the numerator and which is the denominator.

4. Please do the same with the next 9 rows.

5. At the bottom line, please fill in the report saying: The result 0/5 I got ?/10 times, etc.

6. Among the 10 rows, how many are identical? How many are symmetrical?

A third option is to introduce parting fractions in a probability context and continue with the following subtasks:

6. Use centi-cubes or double centi-cubes to show the answer to question 5.

7. In groups of fours, build your centi-cubes together vertically and write a report: The result 0/5 we got ?/40 times, etc.

8. In the class, arrange all the structures behind each other horizontally. Are they like?

In a Count&Multiply&Add curriculum, counting by bundling and stacking implies fractions and decimals to account for the unbundled singles placed on-top of or next-to the stack thus creating mixed numbers as e.g. T = 7 = 2 1/3 3s. Later fractions occur as per-numbers coming from double-counting in the same unit, as e.g. 2$/5$ = 2/5 = 2 per 5. Taking the fraction 2/5 of 20 means taking 2$ per 5$ of 20$, so we just re-count 20 in 5s as T = 20$ = (20/5)*5$ giving (20/5)*2$ = 8$.

As to addition, fractions add as per-numbers, both being operators needing a number to become a number. Multiplying before adding creates areas to be added, thus rooting integral calculus.

D. American Teacher Lesson Display, Grade 8

The fourth lesson was an American teacher lesson display with a grade 8 class. The task was to find a formula connecting the number of angles with the angle sum in a polygon.

Personally, I found this a core task in geometry, allowing the adolescents to use their imagination and creativity. I observed that some students seemed to find the task difficult. This raises the question: 'Will a different approach make a difference as to how many students succeed?' So, from the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if finding the angle sum in a polygon is a goal or a means. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled A and B and C and D and E.

The triangle allows showing that the angle sum is 180 degrees from a new perspective: Inserting an extra point P between A and B transforms the line segment AB into a tri-angle APB where P adds 180 degrees to the angle sum zero. Pulling P out makes P decrease with what A and B increase, so the angle sum remains 180 degrees.

Likewise, on a triangle ABC, inserting an extra point P between A and B transforms the triangle into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

And again the angle sum is increased by 180 degrees by inserting an extra point Q between A and P in the four-angle APBC. So each time an angle is added to the original 3, the angle sum gets 180 added to the original 180 degrees. Consequently, the total angel sum is 180 + 180*(angle number – 3).

Thus, as to teaching, the task could split up in several subtasks:

On this sheet, you see three different polygons. We would like to find a formula connecting the angle number with the angle sum in a polygon.

1. The word 'polygon' is Greek. What does it mean in English? In German a triangle is called a 'Dreieck'. Are the words describing the same?

2. On a line segment AB, insert an extra point P between A and B to transform the line segment into a 3-angle APB. What is the angle sum in APB?

3. Pulling P away from the line segment makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APB?

4. On a triangle ABC, insert an extra point P between A and B to transform the 3-angle into a 4-angle APBC. What is the angle sum in APBC?

5. Pulling P away from the triangle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBC?

6. On a 4-angle polygon ABCD, insert an extra point P between A and B to transform the 4-angle into a 5-angle APBCD. What is the angle sum in APBCD?

7. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCD?

8. On the 5-angle polygon ABCDE, insert an extra point P between A and B to transform the 5-angle into a 6-angle APBCDE. What is the angle sum in APBCDE?

9. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCDE?

10. Try formulating a formula connecting the angle number n with the angle sum S.

11. Are any of these formulas correct?

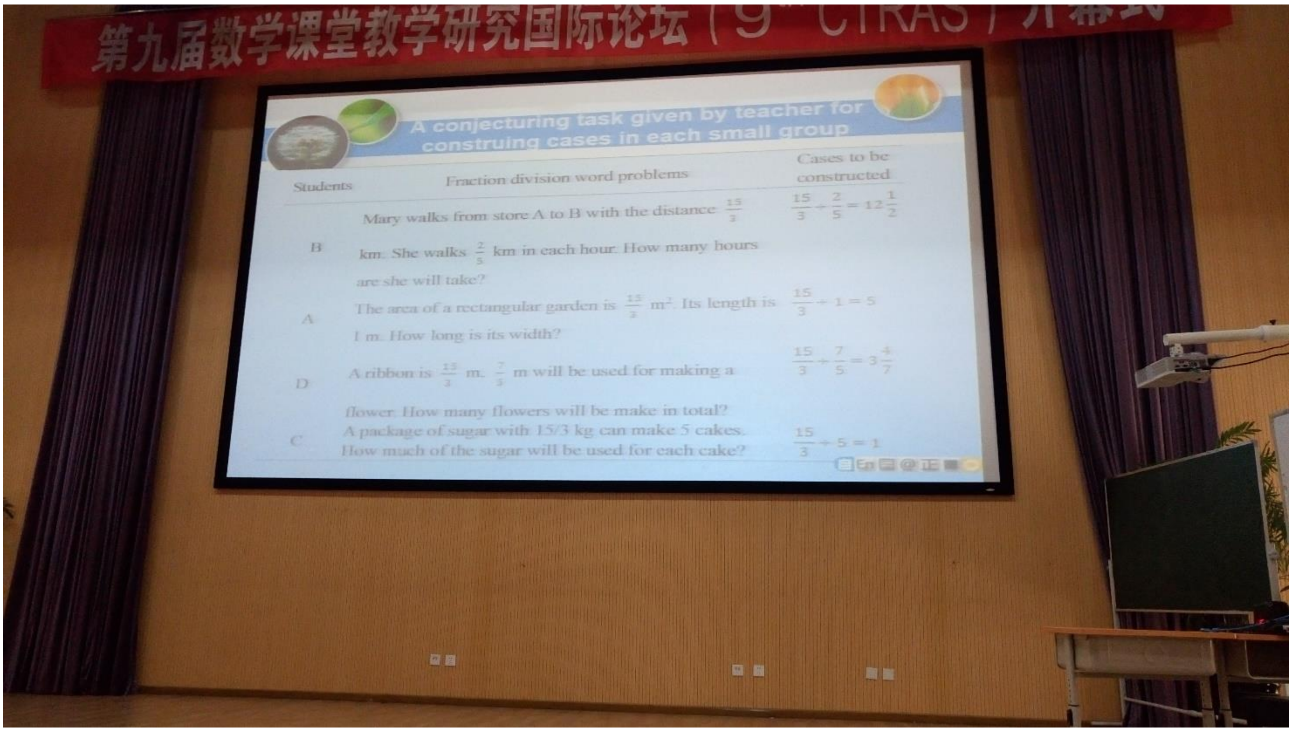
S = n*180 -3	S = n*180 - 360	S = 180 + 180*(n-2)

In a Count&Multiply&Add curriculum, the 'Geometry & algebra, always together, never apart' principle is observed. Thus a polygon will be lines connecting angles with given coordinates. So an angle is found by solving the equation tanA = slope. If all angles are to be found, in the end the rule for the angle sum can be used for checking.

Fractions and Mixed Numbers

Two plenary presentations contained mixed numbers. The 'Using sharing brownies task for mixed number concept development' presentation discussed the task: How to split 13 cookies between 4 children? The 'The conjecturing contributing to the group argumentation in primary classrooms' presentation contained a slide with three parts.

Students	Fraction division word problems	Cases to be constructed
B	Mary walks from store A to B with the distance 15/3 km. She walks 2/5 km in each hour. How many hours are she will take.	15/3 ÷ 2/5 = 12½
A	The area of a rectangular garden is 15/3 m^2. Its length is 1 m. How long is its width?	15/3 ÷ 1 = 15/3
C	A ribbon is 15/3 m. 7/5 m will be used for making a flower. How many flowers will be make in total?	15/3 ÷ 7/5 = 3 4/7
D	A package with 15/3 kg can make 5 cakes. How much of the sugar will be used for each cake?	15/3 ÷ 5 = 1

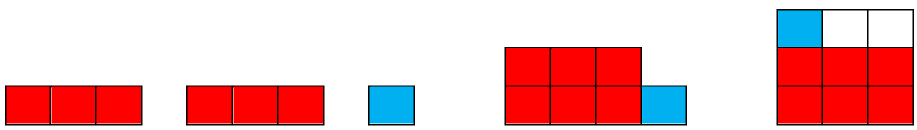


From the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if mixed numbers is a goal or a means. Looking for the outside root of mixed numbers we look for the roots of fractions, again rooted in division.

Seeing ‘Mastering Many’ as the outside root of mathematics, meeting Many leads to the question ‘How many in total?’. To answer we total by counting and adding. We count by bundling and stacking, predicting the resulting number-block by a re-count formula, T = (T/B)*B, saying ‘from T, T/B times B can be taken away’, thus rooting division and multiplication. Thus, a total of 8 can be re-counted in 4s as T = 8 = (8/4)*4 = 2*4 = 2 4s. So the root of division is counting by bundling.

Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s; or T = 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

So the root of mixed numbers is double-counting a total in bundles and parts, expressing the part as a fraction or by a decimal point. Counting in icon-bundles different from ten, the fraction remains unchanged, T = 7 = 2 1/3 3s. But counting in tens, the fraction is changed into decimals: T = 6 7s = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and the decimal point.

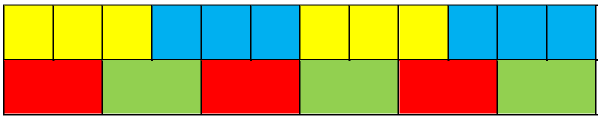
Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. Thus the root of fractions is per-numbers.

So, the sharing question can be reformulated to ‘How manty times can 4 be served by 8 items?’ or ‘8 = ?*4’ or ‘8 = u*4’ which is an equation solved by re-counting 8 in 4s: u*4 = 8 = (8/4)*4 giving u = 8/4 = 2, showing that an equation is solved by moving a number to the opposite side with the opposite sign.

Seeing 8/4 as ‘8 counted in 4s’ thus reflects what takes place psychically when sharing. However, the tradition says that ‘8/4 = 2’ means ‘8 shared between 4’ giving 4 2s and not 2 4s.

Thus, seeing division as THE sharing tool will exclude students unable to learn division, normally considered the difficult of the four operations; and introduced as the last operation, despite introducing it as the first is the natural approach if respecting that the natural way to share is to count the total in shares.

Furthermore, the sharing-understanding of division does not allow problems as ‘4 shared between 1/3’, to which the counting-understanding has the natural answer T = 4/(1/3) = 4*3 = 12. This resonates with the re-count formula saying T = (4/(1/3))*1/3, so 4/(1/3) must mean 4*3. Likewise, counting 4 in 2/3s halves the result, so T = 4/(2/3) = 4*3/2 = 6, or k/(2/3) = k*(3/2).



4 counted in 1/3s gives 12, so 4/(1/3) = 4*3

4 counted in 2/3s gives 6 , so 4/(2/3) = 4*3/2

Here the difference-research question is ‘Will presenting division as a counting means instead of as a sharing means make a difference?’

Returning to the discussion about outside goals and inside means we can ask: With ten as the international standard for bundles, does mixed numbers occur outside or only inside classrooms?

Sharing 13 brownies between 4, each get 3 ¼ brownie, which makes sense since a brownie can split in 4 equal parts. However, the answer could also be ‘3 each and 1 leftover’ as would be the case if sharing 13 cats instead. So whether 13 shared by 4 is 3 ¼ or ‘3 & 1 left’ depends on the unit.

To study the difference in concept development, difference-research would arrange two additional introductions ’13 brownies are arranged in boxes of fours; how many boxes are needed?’; and ’13 brownies are served in quarters, how many can be served? How many boxes are needed?’

Measuring lengths in inches, it makes sense to talk about 3 ¼ inch since 1inch splits into parts by repeated halving. Whereas 3 1/5 inch makes no sense.

However, internationally, length is measured in meters that splits into ten-parts, that split into ten-parts etc., making fractions of tens transform into decimals.

In the first slide task, a distance of 15/3 km only makes sense if sharing 15 km between 3 persons or parts. And presenting a velocity as 2/5 km per hour only makes sense if presented as a per-number 2 km per 5 hours. But both are rare cases that should be presented as footnotes to the typical outside problems using decimal numbers.

The next slide tasks also contain mixed numbers: ‘The area of a rectangular garden is 15/3 m^2’; and ‘A ribbon is 15/3 m’; and ‘7/5 m will be used for making a flower’; and ‘A package with 15/3 kg can make 5 cakes’. By geometrical constructions it is possible to construct 15/5 in the case of a garden and a ribbon. It is however not possible to find precisely 1/3 of 15 kg without first calculating 15/3. So, again we can ask: Are these typical situations in need of mixed numbers, or will decimal numbers be more frequently used in such situations?

Likewise, we can ask if problems describing outside phenomena with mixed numbers are examples of a goal displacement where the outside goal has become a means to motivate learning an inside means presented as a goal?

Basically, a mixed number as 23 ¼ is a mixture of two different bundle-sizes, 23 is counted in tens and ¼ is counted in 4s. This only has meaning when measuring length in inches but since meters has become the international unit, maybe mixed numbers should play only a minor role as footnotes to decimal numbers especially if mathematics education should include all.

Adding mixed numbers directly have meaning when adding inches. Elsewhere, by containing fractions, which are not numbers but operators needing a number to become a number, they should be added by areas, i.e. by integration.

In a Count&Multiply&Add curriculum, mixed numbers thus occur from day one when counting a total by bundling leaves some unbundles singles described by a fraction or a decimal point. Later when ten bundling takes over, mixed numbers become decimal numbers. And from the beginning, all numbers are seen as mixed decimal numbers in disguise, T = 43 = 4.3 tens just as it is said, 4ten3, and as it is written in China.

Fractions: Numbers or Operators

At the end of the first day plenary presentation a discussion took place about the nature of fractions. Arguing that fractions are not numbers but operators needing a number to become a number, I used 5 water bottles for illustration: 'To my right I have 2 bottles, 1 is horizontal since it is empty; to my left I have 3 bottles, 2 are empty. So to the right ½ of my 2 bottles are empty, and to the left 2/3 of my 3 bottles are empty. In total 1+2 = 3 of my 2+3 = 5 bottles are empty, so in this case, adding ½ and 2/3 gives 3/5 of my bottles, the same answer as many students give when adding fractions by adding the numerators and adding the denominators. But the school teaches that ½ + 2/3 = 7/6, meaning that when added, I have 6 bottles and 7 of them are empty. This is perhaps why students dislike fractions. We teach fractions as if they are numbers. But fractions are not numbers, fractions are operators needing a number to become a number. So maybe we should teach fractions that way.'

A gentleman gave as a counter argument that I was mixing fractions with ratios: The example should be described, not by fractions but by ratios, to the right the ratio of empty bottles is 1:2, and to the left it is 2:3, and since ratios do not add it was meaningless to ask for the total. I replied that ratios describe sharing situations which was not the case here. But I thanked him for disagreeing and asked the conference organizers to include in the next conference a debate between persons with different views on mathematics and its education, e.g. on the nature of fractions, or on other issues. In the break, we continued the discussion and agreed on considering writing a common paper on fractions and ratios.

Decimal Multiplication in Grade 5

The second day, a plenary presentation presented a study om fifth graders' learning of decimal multiplication, a core task in algebra, but causing problems to some students when asked to do the multiplication '110*2.54'. From the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if multiplying and decimal numbers is a goal or a means. Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding. To count, we take away bundles to be stacked, thus rooting division and multiplication, allowing the result to be predicted by a re-count formula T = (T/B)*B saying 'from T, T/B times, B can be taken away'. A total of e.g. 8 can be re-counted in 4s as a block-number T = (8/4)*4 = 2*4 = 2 4s.

Multiplication thus is a means to stack six 7s as T = 6 7s = 6*7; and a means to re-count 6 7s in tens: T = 6 7s = 6*7 = 42 = 4.2 tens if including the unit and the decimal point. So, looking for the outside root of multiplication we find stacking and shifting units. Thus, the present task is to re-count 110 2.54s in tens, or to re-count 2,54 110s in tens.

Based upon the principle 'algebra & geometry, always together, never apart', this task can be reformulated to changing the size of a number block: Re-counted in tens, a block of 110 2.54s will increase the base 2.54 with a factor close to 4 and decrease the height with the same factor, so the result will be close to 110/4 tens or 27.5 tens or 275. Or Re-counted in tens, a block of 2.54 110s will decrease the base 110 with a factor 11 and increase the height with the same factor, so the result will be close to 2.54*11 tens close to 27.5 tens or 275.

Using pure algebra, the ten-units can be shown as factors: 110*2.54 = 11 tens*2.54 = 11*25.4 = 1.1 tens 25.4 = 1.1*254 = 279.4'

Thus, the task could split up in several subtasks:

1. Geometrically, show the product 110*2.54 as two number-block with a base and a height.

2. In each case, what is the factor needed to change the base to tens.

3. How will this factor change the height?

4. Factorize the product to show the ten-units.

6. Include the ten-factors in the other factor.

7. Write the product with and without the unit tens.

Recommending counting and multiplying before adding, multiplication and decimal numbers are part of counting in a Count&Multiply&Add curriculum seeing mastering Many as the outside goal of mathematics. Meeting Many, we ask 'How many in Total?' Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s, or 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.

To answer the question 'How many in Total?' we use a number-language sentence with a subject and a verb and a predicate as has word-language sentences. Thus T = 6*7 means that the total is counted by bundling and stacking as a block of 6 7s, that may or may not be re-counted in tens as T = 6 7s = 6*7 = 4ten2 = 4Bundle2 = 4B2 = 4.2 tens = 4 2/ten tens, or 42 if we ask a calculator, leaving out the unit and the decimal point.

We see that a decimal point is an inside means to the outside goal of separating parts from bundles. Thus, counting in 3s, 1 single is described by a decimal number or a fraction as 0B1 or 0.1 or 1/3. And, when counted in tens, 1 single becomes 0B1 or 0.1 or 1/10.

Counting in tens, a bundle-of-bundles, a BB, is called a ten-tens or a hundred; and a bundle-of-bundles-of-bundles, a BBB, is called a ten-ten-tens or a thousand. A bundle-of-bundles-of-bundles-of-bundles, a BBBB, is called a wan in Chinese probably describing a standard army unit of hundred hundreds.

Thus, a total of 2 thousands and 3 hundreds and 4 tens and 5 ones, written shortly as T = 2345 with 1s as the unit, can also be written as T = 2BBB3BB4B5 = 234.5 tens = 234.5*10. We see that multiplying with the bundle-number 10 moves the decimal point one place to the right. And reversely, dividing with (or counting in) the bundle-number 10 moves the decimal point one place to the left.

Using cups for the bundles and the bundles-of-bundles etc. allows a total to be reported by bundle-writing, where T = 2345 = 2BBB3BB4B5 tens = 234.5 tens.

Changing the unit to hundreds where H = BB, we get T = 2345 = 2BH3H4B5 = 2BB3B45 hundreds = 23.45 hundreds = 23.45*100. Changing the unit to thousands where M = BBB, we get T = 2345 = 2M3BB4B5 = 2B345 thousands = 2.345 thousands = 2.345*1000. Again, we see that the decimal point moves one place to the right each time we multiply with the bundle-number 10.

With a ten-bundle as a ten-part of a hundred-bundle we can write T = 10 = 0H1P = 0.1 hundreds, again using the decimal point to separate the parts. And with 1 as a ten-part of a ten-part, we can write T = 1 = 0H0P1PP = 0.01 hundreds. So counting in hundred-bundles, T = 345 = 3B4P5PP = 3.45 hundreds = 3.45*100.

Some physical units can be divided in parts. The length 1meter divides into ten ten-parts called a decimeter, dm, that divides into ten ten-parts called a centimeter, cm, that divides into ten ten-parts called a millimeter, mm. Thus T = 2345 mm = 234.5 cm = 23.45 dm = 2.345 m. Or counted in decimeters, T = 23.45 dm = 2B3.4P5PP, again using a decimal point to separate the parts.

So a number can change to a number between 1 and 10 by factoring ten-units in or out:

T = 2.3*75.6 = 2.3*7.56*10 = 17.388*10 = 173.88

T = 0.023*7560 = 2.3/10/10*7.65*10*10*10 = 17.388*10 = 173.88

The multiplication table is an inside means to the outside goal to change the unit from icons to tens by asking e.g. T = 6 7s = ? tens, or T = 6*7 = ?*10.

One way is to memorize the full ten-by-ten table, another way is to reduce it to a small 2-by-8 table containing doubling (and halving) and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus

T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14, or 2*(½B2) = 1B4 = 14.

T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21, or 3*(B-3) = 3B-9 = 21.

T = 6*7 = 6*(½B2) = 3B12 = 4B2 = 42, or 6*7 = 6*(B-3) = 6B-18 = 4B2 = 42.

T = 6*7 = (5+1) * (10-3) = 50 – 15 +10 – 3 = 42, or

T = 6*7 = (10-4)*(10-3) = 100 – 30 – 40 + 12 = 42.

These results generalize to a*(b – c) = a*b – a*c and vice versa; and to (a – d)*(b – c) = a*b – a*c – b*d + d*c.

5	1	
50	10	B
-15	-3	-3

Multiplying often creates an overload to be removed by stepwise bundling

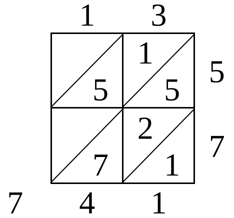
T = 3 57s = 3*57 = 3* 5ten7 = 15ten21 =15ten2ten1 = 17ten1 = 171, or

T = 3 57s = 3*57 = 3* 5B7 = 15B21 = 15B2B1 = 17B1 = 171

T = 13 57s = 13*57 = 13* 5B7 = 65B91 = 74B1 = 741, or

T = 13 57s = 13*57 = 1B3 * 5B7 = 5BB+7B+15B+21 = 5BB22B21 = 5BB24B1 = 7BB4B1 = 741.

The same result comes from using Renaissance-multiplication, also useful with multi-digit multiplication and when multiplying polynomials.



	2a	-3b	
	8a^2	-12ab	4a
	10ab	-15b^2	+5b
8a^2	-2ab	-15b^2	

*Renaissance-multiplication showing that 13*57 = 741 and that (2a-3b)*(4a+5b) = 8a^2-2ab-15b^2*

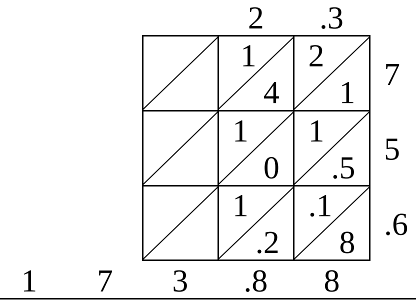
Creating and removing overloads also applies for decimal numbers as 523.47 = 5BB2B3.4P7PP.

T = 6* 523.47 = 6* 5BB2B3.4P7PP = 30BB12B18.24P42PP = 30BB12B18.28P2PP = 30BB12B20.8P2PP = 30BB14B0.8P2PP = 3140.82

Or, with cup-writing: T = 6* 523.47 = 6* 5]2]3.4]7] = 30]12]18.24]42] = 30]12]18.28]2] = 30]12]20.8]2] = 30]14]0.8]2] = 3140.82

The same when multiplying multi-digit numbers:

T = 2.3*75.6 = 2.3P * 7B5.6P = 14B10.12P + 21BP+15P+18PP = 14B + (10+21) + (12+15)P + 18PP = 14B31.27P18PP = 7B3.8P8PP= 173.88 since BP = 1

<table>
<tr><td>The same result comes from using Renaissance multiplication also useful with many-digit multiplication and multiplying polynomials.</td><td>
</td></tr>
<tr><td>The same using B for bundles and P for parts, and where 1BP = 1:

T = 2.3*75.6 = 75.6*2.3 = 7B5.6P * 2.3P</td><td><table>
<tr><td></td><td></td><td>7B</td><td>5</td><td>6P</td><td></td></tr>
<tr><td></td><td></td><td>14B</td><td>10</td><td>12P</td><td>2</td></tr>
<tr><td></td><td></td><td>21</td><td>15P</td><td>18PP</td><td>3P</td></tr>
<tr><td></td><td>14B</td><td>31</td><td>27P</td><td>18PP</td><td></td></tr>
<tr><td></td><td>14B</td><td>31</td><td>28P</td><td>8PP</td><td></td></tr>
<tr><td></td><td>14B</td><td>33</td><td>8P</td><td>8PP</td><td></td></tr>
<tr><td></td><td>17B</td><td>3</td><td>8P</td><td>8PP</td><td></td></tr>
<tr><td>1BB</td><td>7B</td><td>3</td><td>8P</td><td>8PP</td><td></td></tr>
<tr><td>1</td><td>7</td><td>3.</td><td>8</td><td>8</td><td></td></tr>
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Difference-Research Presentation

My own plenary presentation (Tarp, 2017) was called ‘Difference-Research Powering PISA Performance: Count and Multiply before you Add’. Seeing poor PISA performance as the result of 50 years of low-performing Mathematics Education Research, I asked if this could be different.

First I talked about different education, comparing two types of classrooms: Half-year self-chosen blocks in North America versus multi-year forced lines in Continental Europe.

Then I talked about different kinds of mathematics, comparing bottom-up Many-based ‘Many-matics’ from below with top-down Set-based ‘meta-matics’ from above.

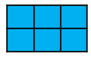
Next, I pointed to ancient Sophism, Renaissance natural science, and (post)modern existentialism as the inspiration for difference-research searching for differences making a difference.

Finally, I talked about a different mathematics education, showing the beauty of the simplicity of mathematics: To master Many, count and re-count and multiply before you add; and when you add forwards & reverse, add block-numbers next-to & on-top, and add per-numbers and fractions by their areas, i.e. by calculus present in both primary and middle and high school.

Inspired by The Greek Sophist saying ‘Beware of choice masked as nature’, I warned against a Goal Displacement in mathematics education, occurring when a means becomes the goal; and unmasking means masked as goals is what difference-research is aiming at.

As to the main finding of difference-research, I showed the following slide unveiling the simplicity of mathematics when presented as tales of Many:

Difference-Research, Main Finding: The Simplicity of Math – Math as Tales of Many



Meeting Many we ask: 'How Many in Total'

- To answer, we math. *Oops, sorry, math is not an action word but a predicate.*
- Take II. To answer, we **Count & Add**. And report with Tales of Many (Number-Language sentences): T = 2 **3s** = 2*3

Three ways to Count: CupCount & ReCount & DoubleCount

- CupCount gives units. ReCount changes units. Double-count bridges units by per-numbers as 2$/3kg

Recount to & from tens gives Multiplication & Equations, coming before Addition

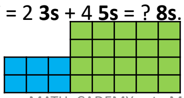
- To tens: T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens** . From tens: T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** (ReCount-Formula)

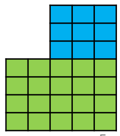
Counting gives variable or constant unit- or per-numbers, to be Added in 4 ways

- Addition & multiplication unites variable & constant unit-numbers.
- Integration & power unites variable & constant per-numbers.

Adding NextTo & OnTop roots Early Childhood Calculus & Proportionality

- EarlyChildhood-Calculus: T = 2 **3s** + 4 **5s** = ? **8s**. EarlyChildhood-Proportionality: T = 2 **3s** + 4 **5s** = ? **5s**





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As to the main warning of difference-research, the following slide shows the 3x3 goal displacements in mathematics education in primary, middle and high school:

Difference-Research, Main Warning: The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

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As to a different mathematics, the following slide shows the beauty of the simplicity of mathematics in 8 areas:

20. Different Mathematics

The Beauty of the Simplicity of Mathematics

21. The Goal & Means of Mathematics Education
22. Totals as Blocks. Digits as Icons. Operations as CupCounting Icons
23. ReCounting gives Proportionality & Multiplication & Equations
24. Multiplication tables simplified by ReCounting
25. DoubleCounting in different & same units creates PerNumbers & Fractions
26. Geometry: Counting Earth in HalfBlocks
27. Once Counted, Totals can be Added. But counting and double-counting gives 4 number-types (constant & variable unit-numbers & per-numbers) to add in 4 ways
28. How Different is the Difference? Set-based versus Many-based Mathematics

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As to the goals and means of mathematics education, the following slide shows the difference between the Set-based top-down tradition and the Many-based bottom-up difference:

21. Different Mathematics

The Goal and Means of Mathematics Education

The Set-based Top-Down Tradition:

- Mathematics exists as a collection of well-proven statements about well-defined concepts, all derived from the mother concept SET
- Mathematics is surprisingly useful to modern society
- Consequently, mathematics must be taught and learned

The Many-based Bottom-Up Difference:

- Many exists; to master Many we develop a number-language with Tales of Many, a 'ManyMatics'.
- **Many-matics**, defining concepts from below as **abstractions from examples**, is a more successful means to the goal of mastering Many than
- '**Meta-matics**' defining concepts from above as **examples from abstractions**

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The following slide compares the Set-based top-down tradition and the Many-based bottom-up difference:

28a. Different Mathematics

How Different is the Difference?

Set-based Math versus Many-based Math

	SET-based Tradition	**Many-based Difference**
Goal/Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Math as Tales of Many
Digits	Symbols as letters	Icons with as many sticks as they represent
Numbers	Place-value number line names. Never with units	A union of blocks of stacked singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive and negative decimal numbers with units
Operations	Mapping from a set-product to the set	Counting-icons: /,*,-,+ (bundle, stack, remove, unite)
Order	Addition, subtraction, multiplication, division	The opposite
Fractions	Rational numbers, add without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equations	Statement about equvalent number-names	Recounting from tens to icons, reversing operations
Functions	Mappings between sets	Number-language sentences with a subject, a verb and a predicate
Proportionality	A linear function	A name for double-counting to different units
Calculus	Differential before integral (anti-differentiation)	Integration adds locally constant per-numbers.

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Finally, a slide showed the main parts of a curriculum in 'ManyMath' seeing mathematics as a natural science about the physical fact Many

28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- CupCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: / x - +

Middle school – integrating algebra and geometry, the content of the label math

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

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The PowerPoint presentation was supplemented with a paper (Tarp, 2017) carrying the same title, describing in detail how PISA performance can improve in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based (Han et al, 2014) core-curriculum for outsiders.

Conclusion

Mathematics education is a social institution with one goal and many means; and as such running the risk of a goal displacement where the original goal becomes a means to a means becoming the goal instead, seduced by a persuasive logic: Mathematics is highly applicable to the outside world, but of course, mathematics must be learned before it can be applied. So of course, mathematics, as defined by the mathematicians, is the goal, and outside applications may be included as a means to motivate the students for learning mathematics even if it is a hard subject demanding a serious commitment, as witnessed by poor PISA results even after 50 years of mathematics education research.

To this compelling argument, difference-research, searching for differences making a difference, will ask: maybe it is the other way around. Maybe there are several forms of mathematics and has been so during its long history, all leading to the same outside goal described in ancient Greece as four knowledge areas about Many in time and space, together labeled 'mathematics'.

So maybe mathematics becomes simple and easy to learn for all, if once again it accepts itself as a means to the outside goal, mastery of Many, accessible through a Many-matics answering the basic question 'How many in total?' by number-language sentences with a subject and a verb and a predicate in the form of a calculation uniting constant or variable unit- or per-numbers.

Therefore, if mathematics for all is a social goal, society must remind mathematics about its role as a means serving the outside goal, mastery of Many, by constantly asking the basic question from the fairy tale Cinderella: Are there other alternatives outside the saloons of present correctness? This precisely is the aim of difference-research searching for differences making a difference. This entails two tasks, to find differences and to test them in a classroom. In this paper only the first task was conducted. In doing so, hidden differences were located within:

- Number sequences. The tradition counts the fingers on a hand as 1, 2, 3, 4, 5. A difference is to count 1, 2, 3, 4, B (bundle); or 1, 2, 3, B less 1, B; or B less 4, B-3, B-2, B-1, B. Emphasizing the word 'bundle' allows showing the nature of counting as bundling, might make a difference in micro-studies.
- Multiplication. The tradition says that 6*7 is 42. A difference is to say that 6*7 is 6 7s that may stay as it is or be recounted in another unit. If recounted in tens, 6 7s is 4.2 tens, shown geometrically as a block where an increase of the base from 7 to ten means a decrease of the height from 6 to 4.2 to keep the total unchanged. Multiplication thus becomes an inside means for two outside goals, to stack bundles and to change the unit to tens. Presenting multiplication before addition as a means to stack and change unit might make a difference in micro-studies.
- Multiplication tables. The tradition says that 6*7 is 42, which is a part of a ten-by ten multiplication table. A difference is to include the total behind and to recount 6 and 7 by saying T = 6*7 = 6 7s to be recounted in tens = (ten less 4)*(ten less 3) = tenten, less 4ten, less 3ten, and 4 3s = 100 – 40 – 30 + 12 = 42; or T = 6*7 = (½ten & 1)*(ten less 3) = ½tenten, ten, less ½ 3ten, less 3 = 50 + 10 – 15 – 3 = 42; or with bundle-writing, T = 6*(1B-3) = 6B-18 = 4B2 = 42; or counting in 5s, T = 6*(½B2) = 3B12 = 4B2 = 42. Allowing numbers to be recounted before multiplied might make a difference in micro-studies.
- Multiplying decimal numbers. The tradition says that multiplying decimal numbers is like multiplying numbers, only keeping track of the place of the decimal point. A difference is to see both factors as numbers between 1 and 10 with ten-units factored in or out. Another difference is to use bundle-writing and allow overloads in the different cups by gradual re-counting. Presented in this way it might make a difference in micro-studies.
- Division. The tradition says that 9/4 is 9 shared by 4 giving each the mixed number 2 ¼ . A difference is to say that 9/4 is 9 counted in 4s giving a total of T = (9/4)*4 = 2*4 +1 = 2 ¼ 4s = 2.1 4s. And to realize that sharing 9 between 4 involves two take-steps. First 4-bundles are taken

away from 9 to re-count 9 in 4s; then, in a 4-bundle, each takes 1 part of 4, i.e. ¼. Sharing thus does not root the traditional division-understanding, instead sharing roots both counting in icons and taking fractions. Presented in this way it might make a difference in micro-studies.

- Fractions. The tradition says that the fraction 3/5 is a rational number describing 3 as a part of 5. A difference is to say that the fraction 3/5 is a per-number coming from double-counting in the same unit; and as per-numbers, fractions are not numbers but operators needing a number to become a number thus adding by their areas as in calculus. Using the fingers on both hand, you quickly learn about 2/5 of 5 and 2/5 of ten. Presenting fractions as per-numbers occurring in sharing situations might make a difference in micro-studies
- Pythagoras. The tradition says that the Pythagoras Theorem allows calculating the hypotenuse from the two other sides in a right-angled triangle. A difference says that parallel distances add directly but perpendicular distances add by their areas. Presented in this way it might make a difference in micro-studies.

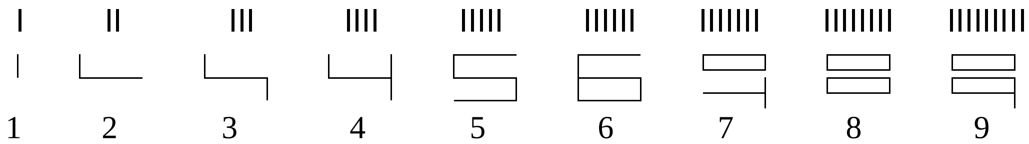
When teaching children to obtain mastery of Many, two options are available.

One option is to see mathematics as an unavoidable means that therefore might be a goal as well, leading to traditional teaching of line-numbers to be added, subtracted, multiplied and divided; and to fractions as rational numbers to be added directly without units, etc.

Another option is to build on what the children already know about mastering Many from being exposed to Many for several years before beginning school. Asking ‘How old will you be next time?’ a 3year old child answers ‘four’ with four fingers shown; but reacts to four fingers held together 2 by 2 with a ‘That is not four, that is two twos.’

So children come to school with 2dimensional number-blocks where all numbers have a unit as with the Arabic numbers they are supposed to learn, T = 345 = 3*10^2 + 4*10 + 5*1.

This allows school to practice guided discovery so the child can see that the digits are, not symbols as letters, but icons with as many strokes as they represent if written less sloppy, thus allowing the child to discover the transition from 4 1s to 1 4s, that can serve as a bundle when counting and re-counting.



Then school can practice double representation of totals using Lego blocks and bundle-writing with full number-language sentences as T = 2*3; and practice re-counting in the same unit to create or remove overloads, allowing the child to see that a total can be counted in different ways, as e.g. T = 7 = 5&2 = ten less 3 = ½bubndle & 2 = bundle less 3; or T = 12 = bundle & 2 = 2 ½bundles & 2 = 1 ½bundle & 7 = 3 ½bundles less 3 = 2 left (twelve = two left, ‘twe levnet’ in Viking language)

Then school can practice re-counting in a different unit so the child can experience the operations as means for a calculator-prediction using the re-count formula T = (T/B)xB, saying that ‘from T, T/B times B can be taken away’, presenting division as a broom wiping away the bundles, and multiplication as a lift stacking the bundles in a block to be removed by subtraction to count the unbundled singles: Asking ‘7 is how many 3s’, first we take 3s a number of times, predicted by 7/3 as 2. Then we take away the stack of 2 3s to count the leftovers, predicted by 7 – 2x3 as 1:

T = 7 = ? 3s. First 7/3 gives 2.some; next 7 – 2x3 gives 1; so T = 7 = 2.1 3s = 2 1/3 3s

Then school can practice recounting between icon-bundles and ten-bundles. Recounting in ten-bundles allows the multiplication table to be built slowly by beginning with doubling and halving and tripling. And recounting from ten-bundles to icon bundles allows the child to solve equation by recounting. So to answer the question ‘how many 8s is 24’ we juts re-count 24 in 8s to get the answer 3, thus moving a number to the opposite side with opposite sign:

?*8 = 24 = (24/8)*8 = 3*8 = 3 8s; so ? = 24/8 = 3

Then school can practice double-counting to create per-numbers bridging countings in different units, and becoming fractions if the units are the same.

Finally, once counted and re-counted, totals can add; either on-top after being re-counted to the same unit, later called proportionality, or next-to as areas also used when adding per-numbers and fractions, later called integral calculus. And then addition can be reversed, later called equations and differential calculus.

Thus, if the school allows children to develop their own number-language they will learn core subjects as proportionality and calculus and solving equations in the first year or two.

So why not celebrate the beauty of the simplicity of the child’s own mathematics? Why replace the child’s own ‘Many-matics’ with the school’s traditional ‘meta-matism,’ mixing ‘meta-matics’, defining concepts as example of abstractions instead of as abstraction form examples, with ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units meet countless counterexamples: T = 2weeks + 3days is not 5 but 17 days; in contrast to this, T = 2*3 = 6 says that 2 3s can be re-counted as 6 1s which is universally true by including the unit 3.

Of course, an ethical issue occurs when depriving the child of its natural number-language, and forcing upon the child an alien language consisting of self-referring definitions and statements with uncertain validity.

In the second enlightenment republic France, Bourdieu calls this ‘symbolic violence’; and Foucault, seeing the school as a ‘pris-pital’ mixing power techniques from a prison and a hospital, would warn against curing children not properly diagnosed, and against accepting self-reference when diagnosing (Bourdieu, 1970. Foucault, 1995).

So, wanting mathematics education to be for all, Many-based Many-matics from below should be preferred to set-based meta-matism from above.

This is how the Count&Multiply&Add curriculum was designed to allow children to develop their own number-language by the natural tasks of counting and re-counting and double-counting and multiplying before performing on-top and next-to addition and reversed addition.

So, a conjecture to be tested and researched is: PISA-like testing will improve if letting a Many-based Bottom-up Count&Multiply&Add curriculum replace the traditional Set-based top-down curriculum presenting 1dimensional numbers to be treated by addition firsts, then subtraction, then multiplication, then division leading to fractions added without units.

Of course, it will take many years to see the effect of a full curriculum, so in the meantime micro-curricula can be designed and tested via intervention research. Or the full curriculum can be tested as a 1year ‘migrant-mathematics’ course allowing young male migrants coming to Europe in high numbers to acquire competence as a pre-teacher or a pre-engineer to return help develop or rebuild their homeland after two years (Tarp, 2017).

As to online in-service teacher education, the MATHeCADEMY.net has been designed to teach teachers to teach mathematics as Many-matics, a natural science about Many, using the CATS-approach, Count&Add in Time&Space, partly described in DrAlTarp YouKu and YouTube videos; and organizing learners in groups of 8 using PYRAMIDeDUCATION.

Recommendation

With only a small percentage of mathematics education research having value to the classroom we must ask if research can be conducted differently. Here difference-research is a difference that might make a difference. Difference-research goes to the classroom to observe problems, allowing it to ask its basic question: Find a difference that makes a difference. Seeing education as preparing students for the outside world leads to accepting mathematics as it arose historically, an inside means to an outside goal, mastery of Many. This allows using intervention research to construct a different micro-curriculum to be tested and adapted in the classroom to see if it makes a difference.

Becoming a difference-researcher is straight forward. You begin as a teacher wanting to teach mathematics for all. At the master level, you read conflicting theory within sociology, philosophy and psychology. In sociology, you focus on the difference between patronizing and enlightening societies as described e.g. by Bauman and Giddens. In philosophy, you focus on the difference between a Platonic top-down view and a sophist bottom-up view as described e.g. by existentialism and post-structuralism. In Psychology, you focus on the difference between mediation and discovery as described e.g. by Vygotsky and Piaget. At the research level, you focus on the difference between testing existing theory and generating new theory as described e.g. by top-down deductive operationalization and a bottom-up inductive grounded theory. And you conduct intervention research by deigning different micro-curricula inspired by thinking differently within sociology, philosophy and psychology.

So, to improve mathematics education worldwide, China could educate ten-ten difference-researchers to spread along the coming new silk road where they each educate ten difference-researchers to help the local population implement a mathematics education for all, rooted in everyday experiences, thus allowing all to enjoy the beauty of the simplicity of mastering many.

With 2017 as the 500year anniversary for Luther's 95 theses, the recommendation can be given as 12 or 20 theses (Tarp, 2017), here reduced to 7 theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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CTRAS 2018 in China

Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'. Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

Mathematics, before and now

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2x3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Difference Research Looking at Mathematics Education

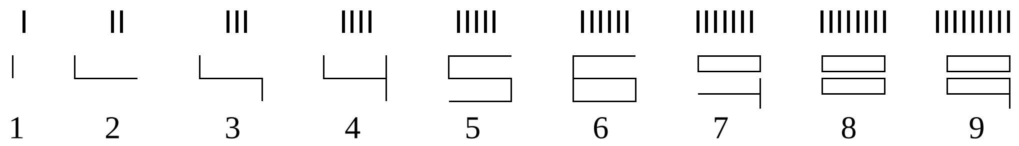
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a list of remedial micro-curricula for classes stuck in a traditional mathematics curriculum.

So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a ‘Count-before-Adding’ Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create number-language sentences, T = 2 3s, with a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b). Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:



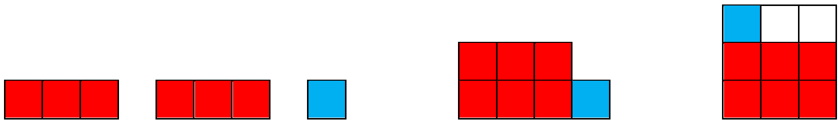
We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T, we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away).

A calculator predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

7 / 3	2.some
7 – 2 x 3	1

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A typical mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precede bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, let us use difference research and imagination to uncover or invent remedial micro-curricula for classes stuck in the tradition.

Remedial micro-curricula for classes stuck in the tradition

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to use two-dimensional block-numbers and bundle-counting, recounting in the same and in a different unit, and calculator prediction before next-to and on-top addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before

adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a

piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ A hymn can be created: “Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to ‘pre-dict’ a number, instead statistics can ‘post-dict’ its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5th root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Inspirational purposes have led to the creation of several DrAlTarp YouKu.com, SoKu.com videos, and MrAlTarp YouTube videso: Deconstructing Fractions, Deconstructing Calculus, Deconstructing PreCalculus Mathematics, Missing Links in Primary Mathematics, Missing Links in Secondary Mathematics, Postmodern Mathematics, PreSchool Math.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, one alternative presents itself directly for future studies creating a paradigm shift (Kuhn, 1962): to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining the misery coming from teaching meta-matism. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research should search for differences and test if they make a difference. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a is-sentence, trust the subject since it exists, but doubt the predicate, it is a verdict that might be gossip. So, maybe we should stop teaching essence and instead start letting learners meet and experience existence.

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Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (p. 3).

This may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference. So, inspired by sociology we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?'

In short, could there be different kinds of mathematics? And could it be that among them, one is good, and one is bad, and one is evil? In other words, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

Good and Bad and Evil Mathematics

The existence of three different versions of mathematics, many-matics and meta-matics and mathe-matism, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world. An example is *T* = 2*3 = 6 stating that a total of 2 3s can be re-counted as 6 1s. So good mathematics is tales about totals, and how to count and unite them.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming that 2+3 = 5, only valid if the units are the same, else meeting contradictions as 2weeks + 3days = 17days. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as claiming that 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3apples total 3reds of 5 apples and not 7reds of 6apples. So bad mathematics is tales about fractions as numbers.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question:

How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, *T* = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to use as units when counting:

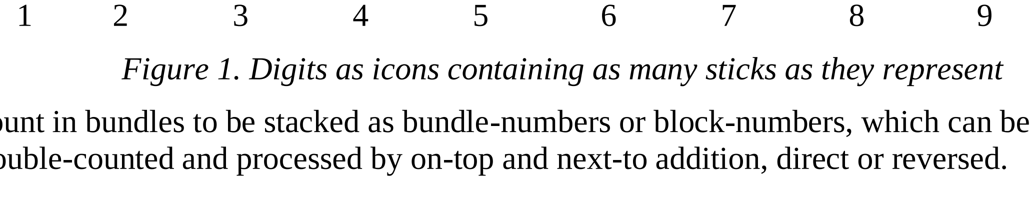


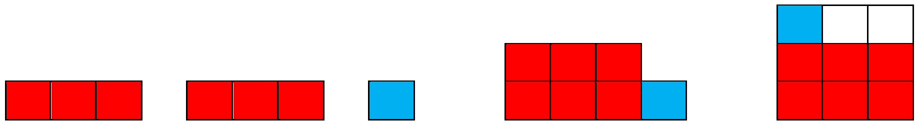
Figure 1. Digits as icons containing as many sticks as they represent

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total *T* we take away bundles *B* thus rooting and iconizing division as a broom wiping away the bundles. Stacking the bundles roots and iconizes multiplication as a lift stacking the bundles into a block. Moving the stack away to look for unbundled singles roots and iconizes subtraction as a trace left when dragging the block away. A calculator predicts the counting result by a 're-count formula' *T = (T/B)*B* saying that 'from *T*, *T/B* times, *B* can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so *T* = 7 = 2*B*1 3s.

Placing the unbundled singles next-to or on-top of the stack of 3s roots decimals and fractions:



T = 7 = 2 3s & 1 = 2*B*1 3s = 2.1 3s = 2 1/3 3s

Figure 2. Re-counting a total of 7 in 3s, the unbundled single can be placed in three different ways

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, *T* = 42 = ? 7s = *u**7, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations, and differential calculus:

T = 2 3s + ? 4s = 5 7s gives differentiation: ? = (5*7 – 2*3)/4 = $\Delta T/4$

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers *sine*, *cosine* and *tangent*. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the *y*-change in the *x*-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A Short Version of a Curriculum in Good Mathematics, Grounded Many-matics

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard, and with over- and underload): *T* = 2*B*1 5s = 1*B*3 5s = 3*B*-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use overload counting after 5: *T* = 4 7s = 4*7 = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: *T* = 336 /7 = 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48

06. Solving proportional equations as 3**x* = 12 should be formulated as re-counting from tens to 3s: 3**x* = 12 = (12/3)*3 giving *x* = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, *T* = 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and *T* = 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, so T = 60$ = (60/3)*3$ gives (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. $a = (a/c)*c = \sin A*c$.

Good and Bad Mathematics

Today's tradition begins with arithmetic telling about line-numbers, processed by four basic operations, later extended with negative numbers and rational numbers and reel numbers. Algebra then repeats it all with letters instead. Geometry begins with plane geometry followed by coordinate geometry and trigonometry later. Functions are special set-products, and differential calculus precedes integral calculus.

In general, we see mathematics as truths about well-defined concepts. So we begin by discussing what can be meant by good and bad concepts.

Good and Bad Concepts

As an example, let us look at a core concept in mathematics, a calculation. To differentiate between $y = 2*3$ and $y = 2*x$, around 1750 Euler defined the concept 'function' as a calculation containing unspecified numbers. Later, around 1900, set-based mathematics defined a function as an example of a set-product where first component identity implies second component identity.

So where the former is a bottom-up definition of a concept as an abstraction from examples, the latter is a top-down definition of a concept as an example of an abstraction.

Since examples are in the world and since Russell warned that by its self-reference the set-concept is meaningless, we can label bottom-up and top-down definitions good and bad concepts respectively.

Good and Bad Numbers

Good numbers should reflect that our number-language describes a total as counted in bundles and expressing the result in a full sentence with subject and verb and predicate as in the word-language, as e.g. *T* = 2 3s. These are the numbers that children bring to school, two-dimensional block-numbers that contain three different number-types: a 'unit-number' for the size, a 'bundle-number' and a 'single-number' for the number of bundles and unbundled singles. Totals then are written in bundle- form or in decimal-form with a unit where a bundle-B or a decimal point separates the inside bundles from the outside singles, as e.g. T = 3*B*2 tens = 3.2 tens.

Good numbers are flexible to allow a total to be re-counted in a different unit; or in the same unit to create an overload or underload to make calculations easier, as e.g. T = 3*B*2 tens = 2*B*12 tens = 4*B*-8 tens. Good numbers are shown in two ways: an algebraic with bundles, and a geometrical with blocks. Good numbers also tell that eleven and twelve come from the Vikings saying 'one left' and 'two left'.

Bad numbers do not respect the children's own two-dimensional block-numbers by insisting on one-dimensional line-numbers be introduced as names along a line without practicing bundling. Numbers follow a place value system with different places for the ones, tens, hundreds, and thousands; but seldom renaming them as bundles, bundle of bundles, and bundles of bundles of bundles.

Good and Bad Counting

A good counting sequence includes bundles in the names, as e.g. 01, 02, …, Bundle, 1*B*1, etc.; or 0Bundle1, 0*B*2, etc. Another sequence respects the nearness of a bundle by saying 0*B*6, 1*B*less3, 1*B*-2, etc. Good counting lets counting and re-counting and double-counting precede addition; and allows the re-count formula to predict the counting-result; and it presents the symbols for division, multiplication and subtraction as icons coming from the counting process, thus introducing the operations in the opposite order. Bad counting neglects the different forms of counting by going directly to adding, thus not respecting that totals must be counted before they can be added. Bad counting treats numbers as names thus hiding their bundle nature by a place value system. This leads some to count 'twenty-ten' instead of 'thirty', and to confuse 23 and 32.

Good and Bad Addition

Good addition waits until after totals have been counted and re-counted in the same and in a different unit, to and from tens, and double-counted in two units to create per-numbers bridging the units. Likewise, good addition respects its two forms: on-top rooting proportionality since changing the units might be need; and next-to rooting integral calculus by being added by the areas.

Bad addition claims it priority as the fundamental operation defining the others: multiplication as repeated addition, and subtraction and division as reversed addition and multiplication. It insists on being the first operation being taught. Numbers must be counted in tens. Therefore there is no need to change or mention the unit; nor is there a need to add next-to as twenties.

Bad addition does not respect that in block-numbers as T = 2*B*3 4s, the three digits add differently. Unit-numbers, as 4, only add if adding next-to. Bundle-numbers, as 2, only add if the units are the same; else re-counting must make them so. Single-numbers, as 3, always add, but might be re-counted because of an overload.

Good and Bad Subtraction

Good subtraction sees its sign as iconizing the trace left when dragging away a stack to look for unbundled singles, thus leading on to division as repeated subtraction moving bundles away. It does not mind taking too much away and leaving an underload, as in 3*B*2 - 1*B*5 = 2*B*-3.

Bad subtraction sees its sign as a mere symbol; and sees itself as reversed addition; and doesn't mind subtracting numbers without units.

Good and Bad Multiplication

Good multiplication sees its sign as iconizing a lift stacking bundles. It sees 5*7 as a block of 5 7s that may or may not be re-counted in tens as 3.5 tens or 35; and that has the width 7 and the height 5 that, if recounted in tens, must widen it width and consequently shorten its height. Thus, it always sees the last factor as the unit.

Good multiplication uses flexible numbers when re-counting in tens by multiplying, as e.g. *T* = 6*8 = 6*(ten-2) = (ten-4)*8 = (ten-4)*(ten-2). This allows reducing the ten by ten multiplication table to a five by five table.

Bad multiplication sees its sign as a mere symbol; and insists that all blocks must be re-counted in tens by saying that 5*7 IS 35. It insists that multiplication tables must be learned by heart.

Good and Bad Division

Good division sees its sign as iconizing a broom wiping away the 2s in *T* = 8/2. It sees 8/2 as 8 counted in 2s; and it finds it natural to be the first operation since when counting, bundling by division comes before stacking by multiplication and removing stacks by subtraction to look for unbundled singles.

Bad division sees its sign as a mere symbol; and teaches that 8/2 means 8 split between 2 instead of 8 counted in 2s. Bad division accepts to be last by saying that division is reversed multiplication; and insists that fractions cannot be introduced until after division.

Good and Bad Calculations

Good calculations use the re-count formula to allow a calculator to predict counting-results.

Bad calculations insist on using carrying so that the result comes out without overloads or underloads.

Good and Bad Proportionality

Good proportionality is introduced in grade 1 as re-counting in another unit predicted by the re-count formula. It is re-introduced when adding blocks on-top; and when double-counting in two units to create a per-number bridging the units by becoming a proportionality factor.

Bad proportionality is introduced in secondary school as an example of multiplicative thinking or of a linear function.

Good and Bad Equations

Good equations see equations as reversed calculations applying the opposite operations on the opposite side thus using the ‘opposite side and sign’ method in accordance with the definitions of opposite operations: 8-3 is the number x that added to 3 gives 8; thus if $x+3 = 8$ then $x = 8-3$. Likewise with the other operations.

Good equations sees equations as rooted in re-counting from tens to icons, as e.g. 40 = ? 8s, leading to an equation solved by re-counting 40 in 8s: $x*8 = 40 = (40/8)*8$, thus $x = 40/8 = 5$.

Bad equations insist that the group definition of abstract algebra be used fully or partwise when solving an equation. It thus sees an equation as an open statement expressing identity between two number-names. The statements are transformed by identical operations aiming at neutralizing the numbers next to the unknown by applying commutative and associative laws.

$2*x = 8$	an open statement about the identity of two number-names
$(2*x)*(1/2) = 8*(1/2)$	½, the inverse element of 2, is multiplied to both names
$(x*2)*(1/2) = 4$	since multiplication is commutative
$x*(2*(1/2)) = 4$	since multiplication is associative
$x*1 = 4$	by definition of an inverse element
$x = 4$	by definition of a neutral element

Figure 3. Solving an equation using the formal group definition from abstract algebra

Good and Bad Pre-calculus

Good pre-calculus shows that the number-formula, $T = 345 = 3*BB + 4*B + 5*1 = 3*x^2 + 4*x + 5$, has as special cases the formulas for constant linear, exponential, elastic, or accelerated change: $T = b*x+c$, $T = a*n^x$, $T = a*x^n$, and $T = a*x^2 + b*x + c$. It uses ‘parallel wording’ by calling root and logarithm ‘factor-finder’ and ‘factor-counter’ also. It introduces integral calculus with blending problems adding piecewise constant per-numbers, as e.g. 2kg at 3 $/kg plus 4kg at 5$/kg. It includes modeling examples from STEM areas (Science, Technology, Engineering, Mathematics)

Bad pre-calculus introduces linear and exponential functions as examples of a homomorphism satisfying the condition f(*x*#*y*) = f(*x*)$f(*y*). It includes modeling from classical word problems only.

Good and Bad Calculus

Good calculus begins with primary school calculus, adding two blocks next-to each other. It also includes middle school calculus adding piecewise constant per-numbers, to be carried on as high school calculus adding locally constant per-numbers.

It motivates the epsilon-delta definition of constancy as a way to formalize the three forms of constancy: global, piecewise and locally. It shows series with single changes and total changes calculated to realize that many single changes sum up as one single change, calculated as the difference between the end- and start-values since all the middle terms disappear.

This motivates the introduction of differential calculus as the ability to rewrite a block *h**dx* as a difference *dy, dy/dx = h*; and where the changes of block with sides f and g leads on to the fundamental formula of differential calculus, (f*g)'/(f*g)= f'/f + g'/g, giving (x^n)'/x^n = n*1/x, or (x^n)' = n*x^(n-1).

Bad calculus introduces differential calculus before integral calculus that is defined as anti-differentiation where the area under *h* is a primitive to *h*; and it introduces the epsilon-delta criterion without grounding it in different kinds of constancy.

Good and Bad Modeling

Good modeling is quantitative literature or number-stories coming in three genres as in word stories: Fact, fiction and fiddle. Fact and fiction are stories about factual and fictional things and actions. Fiddle is nonsense like 'This sentence is false' that is true if false, and vice versa.

Fact models, also called 'since-then' or 'room' models, quantify quantities and predict predictable quantities: "What is the area of the walls in this room?". Since the prediction is what is observed, fact models can be trusted. Fiction models, also called 'if-then' or 'rate' models, quantify quantities but predict unpredictable quantities: "My debt is gone in 5 years at this rate!". Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based on alternative assumptions. Fiddle models, also called 'then-what' or 'risk' models, quantify qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" Fiddle models should be rejected asking for a word description instead of a number description. (Tarp, 2017).

Bad modeling does not distinguish between the three genres but sees all models as approximations.

Good and Bad Geometry

Good geometry lets trigonometry precede plane geometry that is integrated with coordinate geometry to let algebra and geometry go hand in hand to allow formulas predict geometrical intersection points.

Bad geometry lets plane geometry precede coordinate geometry that precedes trigonometry.

Evil Mathematics

Evil mathematics talks about something existing only inside classrooms. Fractions as numbers and adding fractions without units are two examples. The tradition presents fractions as rational numbers, defined as equivalence classes in a set product created by the equivalence relation R, where (*a*,*b*) R (*c*,*d*) if *a**d = b**c*. Grounded in double-counting in two units, fractions are per-numbers double-counted in the same unit, as e.g. 3$ per 5$ or 3 per 5 or 3/5. Both are operators needing a number to become a number. Both must be multiplied to unit-numbers before adding, i.e. they add by their areas as in integral calculus.

Shortening or enlarging fractions is not evil mathematics. They could be called ‘footnote mathematics’ since they deal with operator algebra seldom appearing outside classrooms. They deal with re-counting numbers by adding or removing common units: to shorten, 4/6 it is re-counted as 2 2s over 3 2s giving 2/3. To be enlarged, both take on the same unit so that 2/3 = 2 4s over 3 4s = 8/12.Educating teachers, it is evil to silence the choices made in mathematics education. Instead, teachers should be informed about the available alternatives without hiding them in an orthodox tradition. Especially the difference between good and bad mathematics should be part of a teacher education.

Good and Bad Education

When children become teenagers, their identity work begins: ‘Who am I; and what can I do?’ So good education sees its goal as allowing teenagers to uncover end develop their personal talent through daily lessons in self-chosen practical or theoretical half-year blocks with teachers having only one subject and praising the students for their talent or for their courage to try out something unknown. Bad education sees its goal as selecting the best students for offices in the private or public sector. It uses fixed classes forcing teenagers to follow their age-group despite the biological fact that girls are two years ahead in mental development.

Good and Bad Research

Good research searches for truth about things that exist. It poses a question and chooses a methodology to transform reliable data into valid statements. Or it uses methodical skepticism to unmask choice masked as nature. Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question. With these three research genres, peer-review only works inside the same genre.

Conclusion and Recommendation

This paper used difference-research to look for different ways to the outside goal of mathematics education, mastery of Many. By meeting Many outside the present self-referring set-based tradition three ways were found, a good, and a bad, and an evil. Good mathematics respects the original tasks in Algebra and Geometry, to reunite Many and to measure earth. By identifying a hidden alternative, good mathematics creates a paradigm shift (Kuhn, 1962) that opens up a vast field for new research seeing mathematics as a many-matics, i.e. as a natural science about Many (cf. Tarp, 2018).

In short, we need to examine what happens if we allow children to keep and develop the quantitative competence they bring to school, two-dimensional block-numbers to be recounted and double-counted before being added on-top or next-to; and reported with full number-language sentences including both a subject that exists, and a verb, and a predicate that may be different.

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04. Addition-free STEM-based Math for Migrants

A curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to soon become STEM pre-teachers and pre-engineers. Multiplication formulas expressing recounting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and recount totals by bundling in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. A recount formula that expresses proportionality when changing units is a core prediction formula in all STEM subjects.

Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise has funding, see e.g. the Swedish National Centre for Mathematics Education. Yet, despite extra research and funding, and despite being warned against the possible irrelevance of a growing research industry (Tarp, 2004), decreasing Swedish PISA results caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change" since "more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)."

To find an unorthodox solution to poor PISA performance we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this a STEM based core curriculum allowing young migrants to soon become STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to improve schools in Sweden and elsewhere.

Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now (p. 16)."

As to institutions, of which mathematics education is an example, Bauman talks about rational action "in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement** (p. 84)." Of which one example is saying that the goal of mathematics education is to learn mathematics since such a goal statement is obviously made meaningless by its self-reference.

The link between a goal and its means is also present in existentialist philosophy described by Sartre (2007) as holding that "Existences precedes essence (p. 20)". Likewise, Arendt (1963) points out that practicing a means blindly without reflecting on its goal might lead to practicing "the banality of evil". Which makes Bourdieu (1977) says that "All pedagogic action is, objectively, symbolic violence insofar as it is the imposition of a cultural arbitrary by an arbitrary power (p. 5)". This raises the question if mathematics and education is universal or chosen, more or less arbitrarily.

DIFFERENT KINDS OF EDUCATION

The International Commission on Mathematical Instruction, ICMI, named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". At its conference in Tsukuba, Japan, in November 2018 it became clear during plenary discussions that internationally there is little awareness of two different kinds of educational systems practiced from secondary school.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

Moreover, as a social institution involving monopolizing and individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject, and in their own classrooms.

To protect its republic against attack from its German speaking neighbors, France created elite schools with multi-year forced classes, called 'pris-pitals' by Foucault (1995) pointing out that it mixes power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

DIFFERENT KINDS OF mathematics

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, rhetoric and logic (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both being action-words rooted in the physical fact Many through their original meanings. This resonates with the primary goal of knowledge seeking and education, to be able to master the outside world through proper actions. And in Europe, Germanic countries taught counting and reckoning in primary school and algebra and geometry in the lower secondary school until about 50 years ago when they all were replaced by the setbased 'New Math' even if mathematics is a mere label and not an action-word. But the point was that by being setbased mathematics could become a self-referential 'meta-matics' needing no outside root. Instead it could define concepts top-down as examples of inside abstractions instead of bottom-up as abstractions from outside examples.

Russell objected by pointing to the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. In this way Russell shows that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false. Instead Russell proposed a type theory banning self-reference. However, mathematics ignored Russell's paradox and his type theory since it prevented fraction from being numbers by being defined from numbers.

Instead, setbased mathematics changed classical grounded mathe-matics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be recounted as 6 1s.

Although spreading around the world, the United States rejected the New Math by going 'back to basics'. So today three kinds of mathematics may be taught: a pre-setbased, a present setbased and a post-setbased version (Tarp, 2017).

The tradition of mathematics education

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (*a*+*b*) * (*a*-*b*) = *a*^2 - *b*^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (*f**g*)'/(*f**g*) = *f'/f* + *g'/g*. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6**x*^2d*x* = 2**x*^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

theorizing the success of math education research

When trying to theorize the low success of 50 years of mathematics education research, the first question must be what we mean by mathematics and education and research.

As to education, who needs it if they already know? So, we must ask: what is it that students do not know and must be educated in? Or in other words: what is the goal of mathematics education? Two answers present themselves, one pointing to on the outside existence rooting mathematics, the other to its inside institutionalized essence.

Giving precedence to inside essence over outside existence the answer is: of course, the goal of mathematics education is to teach mathematics as defined by mathematicians at the universities. Modern societies institutionalize the creation and mediation of knowledge as universities and schools. Here priority should be given to useful knowledge as mathematics; and of course, mathematics must be taught before it can be applied, else there is nothing to apply! However, although very useful, mathematics is at the same time very hard to learn as witnessed again and again by research, carefully and in detail describing students' learning problems. So, 50 years of mathematics education research has not been unsuccessful, on the contrary, it has been extremely successful in proving that, by its very nature, mathematics is indeed difficult. The 'essence precedes existence' stance is typically argued by university scholars as e.g. Bruner (1962), Skemp (1971), Freudenthal (1973), and Niss (1994).

Giving precedence to outside existence over inside essence the answer is: It is correct that research has demonstrated many learning difficulties. However, what has been taught is not an outside rooted mathe-matics, but an inside self-referring meta-matism as defined above. And, until now research has primarily studied the two contemporary versions of mathematics, the pre-setbased and the present setbased version whereas very little if any research has studied the post-setbased mathematics that gives precedence to existence over essence by accepting and developing the mastery of Many in the number-language that children develop before school.

Giving precedence to essence or existence makes a difference to math education.

In its pre-setbased version, mathematics presents digits as symbols, and numbers as a sequence of digits obeying a place value system. Once a counting sequence is established, addition is defined as counting on, after which the other operations are defined from addition. Fractions are seen as numbers.

In its present setbased version, mathematics uses the inside concept set for deriving other concepts. Here numbers describe the cardinality of a set, and an operation is a function from a set product into a set. Again, addition is taught as the first operation.

In its post-setbased version, mathematics presents digits as icons with as many sticks as they represent; and numbers always carry units as part of number-language sentences bridging the outside existence with inside essence, thus connecting outside blocks with inside bundles, T = 2 3s = 2B0 3s. Here operations are icons also, and here counting comes before adding to respect that counting involves taking away bundles by division to be stacked by multiplication, to be pulled away by subtraction to find unbundled ones. And here counting and recounting and double-counting precedes the two forms of addition, on-top and next-to. And here fractions are per-numbers, both being operators needing numbers to become numbers.

Likewise, the core concept ‘function’ is treated differently. Pre-setbased mathematics sees a function as a calculation containing specified and unspecified numbers. Present setbased mathematics sees a function as a subset of set product where first-component identity implies second-component identity. Post-setbased mathematics sees a function as a number-language sentence T = 2*3 relating an outside existing total with an inside chosen essence.

Choosing an ‘inside-outside’ view will make mathematics self-referring and difficult by its missing link to its outside roots. Whereas choosing an ‘outside-inside’ view will allow mathematics develop the language children use to assign numbers to outside things and actions, i.e. a number-language similar to the word-language.

Mathematics as the Grammar of the Number-Language

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: ”This is a chair”. As does the number-language assigning numbers instead: ”The 3 chairs each have 4 legs”, abbreviated to ”The total is 3 fours”, or ”T = 3 4s” or ”T = 3*4”. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb and just saying ”3*4 = 12”.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ”This is a chair” leads to a meta-sentence ”The word ‘is’ is an auxiliary verb”. Likewise, the sentence ”T = 3*4” leads to a meta-sentence ”The operation ‘*’ is commutative”.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring setbased form has turned mathematics into a grammar labeling its outside roots as ‘applications’, used as means to dim the impeding consequences of teaching a grammar before its language.

Before 1970, language was taught as an example of its grammar (Chomsky, 1965). Then a reaction emerged. In his book ‘Explorations in the function of language’ Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a less correct language to be used for communication about outside things and actions.

Time for a Linguistic Turn in the Number-Language also

Thus, in language teaching a new version of the linguistic turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language.

So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

Maybe the time has come to realize that the two statements ‘2+3 = 5’ and ‘2*3 = 6’ have a different truth status.

The former is a conditional truth depending on the units. But, with 3 as the unit, the latter is an unconditional truth since 2 3s may always be recounted as 6 1s.

In short, maybe it is time to look for a different outside-inside mathematics to replace the present tradition, inside-outside meta-matism? And to ask what kind of math grows from the mastery of Many that children develop through use and before school?

Difference research looks at mathematics education

To answer, we let Many open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). Using ‘Difference-research’ (Tarp, 2017) searching for hidden differences making a difference, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many creates a ‘count-before-add’ curriculum

Meeting Many, we ask “How many in Total?” To answer, we count by bundling to create a number-language sentence as e.g. T = 2 3s that contains a subject and a verb and a predicate as in a word-language sentence; and that connects the outside total T with its inside predicate 2 3s (Tarp, 2018b). Rearranging many 1s into one symbol with as many sticks or strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting by bundling and stacking:

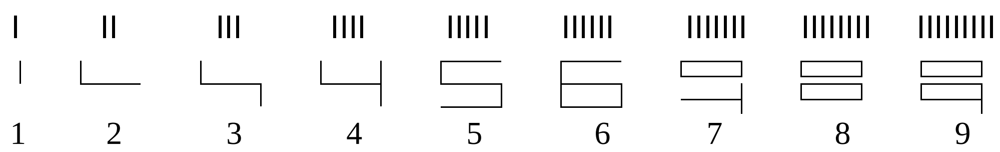

Figure 1: Digits as icons with as many sticks as they represent.

Holding 4 fingers together 2 by 2, a 3year-old will say ‘This is not 4, this is 2 2s’, thus describing what exists, bundles of 2s and 2 of them. This inspires ‘bundle-counting’, counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be recounted and double-counted before being processed by next-to and on-top addition, direct or reversed. Thus, a total *T* of 5 1s is recounted in 2s as *T* = 2 2s & 1; described by ‘bundle-writing’ as *T* = 2*B*1 2s; or by ‘decimal-writing’, *T* = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by ‘deficit-writing’, *T* = 3*B*-1 2s = 3.-1 2s = 3 bundles less 1 2s.

To bundle-count a total *T* we take away bundles *B* (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a rope pulling the block away).

A calculator thus predicts the result by a ‘recount formula’ $T = (T/B)*B$ saying that ‘from *T*, *T/B* times, *B*s can be taken away’: entering ‘5/2’ on a calculator gives ‘2.some’, and ‘5 – 2x2’ gives ‘1’, so *T* = 5 = 2*B*1 2s. The unbundled can be placed next-to the stack as .1 or on-top as ½ counted in 2s, thus rooting decimals and fractions.

The recount formula occurs all over science. With proportionality: $y = c*x$; in trigonometry as sine, cosine and tangent: $a = (a/c)*c = \sin A*c$ and $b = (b/c)*c = \cos A*c$ and $a = (a/b)*b = \tan A*b$; in coordinate geometry as line gradients: $\Delta y = \Delta y/\Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In economics, the recount formula is a price formula: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

Recounting in the Same Unit or in a Different Unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside ‘overload’; or as 3*B*-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33*B*6 = 28*B*56, so 336/7 = 4*B*8 = 48; or 336 = 35*B*-14, so 336/7 = 5*B*-2 = 48.

Recounting in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2*B*2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer, ‘2.some’, predicts that the unbundled singles come from taking away 2 5s, now asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s or 2 2/5 5s.

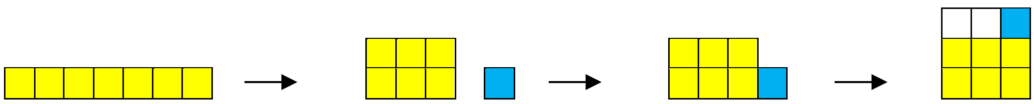

Figure 2: Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

Recounting from Icons to and from Tens

Recounting from icons to tens by asking e.g. ‘2 7s = ? tens’ is eased by using underloads: *T* = 2 7s = 2*7 = 2*(*B*-3) = 20-6 = 14; and T = 6 8s = 6*8 = (*B*-4)*(*B*-2) = *BB* - 4*B* - 2*B* -- 4*2 = 10*B* – 4*B* – 2*B* + 8 = 4*B*8 = 48. This makes sense since widening the base form t7 to ten will shorten the height from 6 to 4.8.

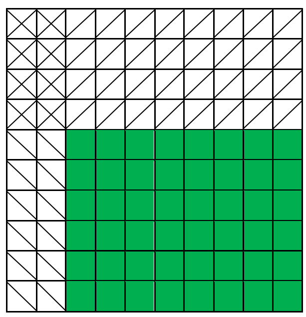

Figure 3: On an abacus 6 8s = 6*8 = (*B*-4)*(*B*-2) = 10*B* - 4*B* - 2*B* + 4 2s = 4*B*8 = 48.

Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer directly as 2*7 = 14, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation u*7 = 35. It is easily solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-Counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in the per-number 2 and use the per-number to bridge 2$ and 3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa: T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%. So 2/3 of 60 = 2$/3$ of 60$, where 60$ = (60/3)*3$ then gives (60/3)*2$ = 40$.

Double-Counting the Sides in a Block Creates Trigonometry

Halving a block by its diagonal allows mutual recounting of the sides, which creates trigonometry to precede plane and coordinate geometry: $a = (a/c)*c = \sin A*c$, and $a = (a/b)*b = \tan A*b$. Filling a circle with blocks shows that $\pi = n*\tan(180/n)$ for n large.

A short curriculum in addition-free mathematics

00. Playing with '1digit math' (Zybartas et al, 2005): Rearranging 3 cars into one 3-icon, etc. Recounting a total of ten fingers in bundles of e.g. 3s: T = 1Bundle7 = 2B4 = 3B1 = 4Bundle less 2 or 4B-2, and using both fingers and sticks and centi-cubes or LEGO bricks to experience algebra and geometry as always together, never apart. Recounting in a different unit when asking e.g. T = 2 3s = ?4s. Recounting to and from tens when asking e.g. T = 5 6s = ? tens, and T = 4B2 tens = ? 7s. Uniting blocks next-to and on-top when asking e.g. T = 2 3s & 4 5s = ? 8s; and T = 2 3s & 4 5s = ? 3s; and T = 2 3s & 4 5s = ? 5s. Splitting blocks next-to and on-top when asking e.g. T = 2 3s & ? 5s = 3 8s; and T = 2 3s & ? 5s = 7 3s; and T = 2 3s & ? 5s = 4 5s.

01. Until nine, many ones may be rearranged into one icon with as many sticks or strokes as it represents. As one bundle, ten needs no icon. So, a total typically consists of several countings: of unbundled ones, of bundles, of bundles of bundles, etc.

02. Parallel counting sequence stress the importance of bundling: 0Bundle1, 0*B*2, …, 0*B*9, 1*B*0, 1*B*1 etc.; or 0*B*1, 0*B*2, 0*B*3, 0*B*4, 0*B*5 or half Bundle, Bundle less 4, *B*-3, *B*-2, *B*-1, Bundle or 1*B*0, Bundle and 1 or 1*B*1, Bundle and 2 or 1*B*2, etc., thus rooting negative numbers. Here we mention that the Vikings used the words 'eleven' and twelve' as short for 'one-left' and 'two-left'. Using other bundles as units, ten fingers may be counted as 1*B*3 7s, 2*B*0 5s, 2*B*2 4s, 3*B*1 3s, 1*BB*0*B*1 3s, 5*B*0 2s, and 1*BBB*0*BB*1*B*0 2s. A Total of five fingers can be recounted in 2s in three ways, standard or with overload or underload: T = 2*B*1 2s = 1*B*3 2s = 3*B*-1 2s = 3 bundles less 1 2s.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. For prediction, a calculator uses a 'recount formula', $T = (T/B)*B$, saying that 'from T, T/B times, Bs can be taken away'.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', we enter '3*4/5' to ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2*B*2 5s. The unbundled can be placed next-to the bundles separated by a decimal point, or on-top counted in bundles, thus rooting decimals and fractions, T = 3 4s = 2*B*2 5s = 2.2 5s = 2 2/5 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation u*7 = 35, solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving

to opposite side with opposite calculation sign. Division is eased by using overloads or underloads: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48. As is multiplication: T = 4*78 = 4* 7*B*8 = 28*B*32 = 31*B*2 = 312.

06. Recounting from icons to tens by asking '2 7s = ? tens' is eased by underloads: *T* = 2*7 = 2*(*B*-3) = 20-6 = 14; 6*8 = (*B*-4)*(*B*-2) = *BB*-4*B*-2*B*--8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in two units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in the per-number: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Trigonometry can precede plane and coordinate geometry to show how, in a block halved by its diagonal, the sides can be mutually recounted as e.g. *a* = (*a/c*)*c = sin*A***c*.

Meeting Many in a STEM context

OECD (2015b) says: "In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth." STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H_2O, and carbon dioxide CO_2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH_4, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about 22.4*1000 liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, $p*V = n*R*T$, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per ‘carrier’ (charge-unit). Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law ‘Volt = Resistance*Ampere’, i.e., 220 = Resistance*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g. 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed *u* by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (*u**0,707)*0,71, solved as *u* = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

Adding addition to the curriculum

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being recounted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = $\Delta T/4$.

Integral calculus thus precedes differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions must be multiplied by the units before being added as the area under the per-number graph.

Using overloads and underloads eases addition and subtraction: $T = 23 + 49 = 2B3 + 4B9 = 6B12 = 7B2 = 72$; and $T = 56 – 27 = 5B6 – 2B7 = 3B\text{-}1 = 2B9 = 29$.

Moving in a coordinate system, distances add directly when parallel; and by squares when perpendicular. Re-counting the y-change in the x-change creates a linear change formula $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$, algebraically predicting geometrical intersection points, thus observing a ‘geometry & algebra, always together, never apart’ principle.

The number-formula $T = 456 = 4*B\text{^}2 + 5*B + 6*1$ shows the four ways to unite numbers offered by algebra meaning ‘reuniting’ in Arabic: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b):

Uniting/ splitting into	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** *T – a = n*	***T = a * n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫ a dn*** *dT/dn = a*	***T = a^n*** *loga(T) = n* *n√T = a*

Figure 4: An ‘Algebra-Square’ with the 4 and 5 ways to unite and split totals.

In its general form, the number formula $T = a*x\text{^}2 + b*x + c$ contains the different formulas for constant change: $T = a*x$ (proportionality), $T = a*x\text{^}2$ (acceleration), $T = a*x\text{^}c$ (elasticity) and $T = a*c\text{^}x$ (interest rate); as well as $T = a*x+b$ (linearity, or affinity, strictly).

As constant/changing, predictable change roots pre-calculus/calculus. Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Engineering: How Many Turns on a Steep Hill

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let *A* and *B* label the ground corners of the hillside. *C* labels the point where a road from *A* meets the edge for the first time, and *D* is vertically below *C* on ground level. We want to find the distance $BC = u$.

In the triangle *BCD*, the angle *B* is 30 degrees, and $BD = u*\cos(30)$. With Pythagoras we get $u\text{^}2 = CD\text{^}2 + BD\text{^}2 = CD\text{^}2 + u\text{^}2*\cos(30)\text{^}2$, or $CD\text{^}2 = u\text{^}2(1\text{-}\cos(30)\text{^}2) = u\text{^}2*\sin(30)\text{^}2$. In the triangle *ACD*, the angle *A* is 10 degrees, and $AD = AC*\cos(10)$. With Pythagoras we get $AC\text{^}2 = CD\text{^}2 + AD\text{^}2 = CD\text{^}2 + AC\text{^}2*\cos(10)\text{^}2$, or $CD\text{^}2 = AC\text{^}2(1\text{-}\cos(10)\text{^}2) = AC\text{^}2*\sin(10)\text{^}2$. In the triangle *ACB*, $AB = 1$ and $BC = u$, so with Pythagoras we get $AC\text{^}2 = 1\text{^}2 + u\text{^}2$, or $AC = \sqrt{(1+u\text{^}2)}$.

Consequently, $u\text{^}2*\sin(30)\text{^}2 = AC\text{^}2*\sin(10)\text{^}2$, or $u = AC*\sin(10)/\sin(30) = AC*r$,
or $u = \sqrt{(1+u\text{^}2)}*r$, or $u\text{^}2 = (1+u\text{^}2)*r\text{^}2$, or $u\text{^}2*(1\text{-}r\text{^}2) = r\text{^}2$, or $u\text{^}2 = r\text{^}2/(1\text{-}r\text{^}2) = 0.137$,
giving the distance $BC = u = \sqrt{0.137} = 0.37$.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Conclusion and recommendation

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a setbased top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and recounting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating.“ A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a).

Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing line-numbers with block-numbers. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

Thus, it is possible to solve core STEM problems without learning addition, that later should be introduced in both versions since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

So, as with another foreign language, why not learn the number-language through its use. And celebrate that core mathematics as proportionality, equations, per-numbers and calculus grow directly from the mastery of Many that children develop through use and before school? Let us see math, not as a goal in itself, but as an inside means to an outside goal that is reached better and by more with quantitative communication.

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05. Bundle-Counting Prevents & Cures Math Dislike

Inside-Outside Mathematics

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2, and where π = n*sin(180/n) for n large.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 - b^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)/(f*g) = f/f + g/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature.

Outside-Inside Mathematics

But, as with another foreign language, why not learn the number-language through its use? Is the goal of mathematics education to learn mathematics – or to learn how to master Many? Is math a goal in itself, or an inside means to an outside goal, that may be reached better and by more through quantitative communication? What math grows from the mastery of Many that children develop through use and before school?

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total *T* by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. *T* = 2 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', *T* = (*T/B*)**B*, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$;

in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c;$ in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking ‘3 4s = ? 5s’, sticks say 2*B*2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer ‘2.some’ predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer ‘2’ predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

05. Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

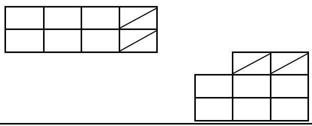
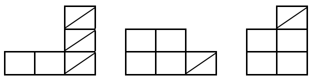
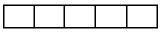
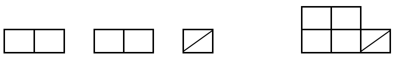
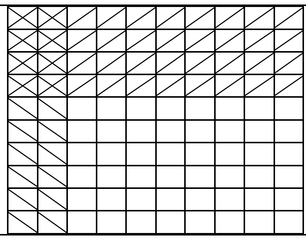
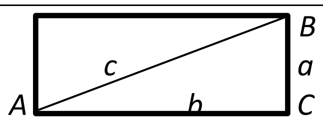
06. Recounting to tens by asking ‘2 7s = ? tens’ is eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$.

07. Double-counting a quantity in units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘*T* = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

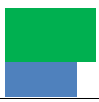
References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

06. Flexible Bundle-Numbers

respect & develop **Kids Own Math**	***Outside & Inside Math***	
Digits as ICONS III IIII IIIII	3 4 5 (drawn as icons)	**3 4 5**
Operations as ICONS	Push Lift Pull	**/ x –**
Count Fingers in 5s using BundleCounting & BundleNumbers	I●●●● II●●● III●● IIII● ~~IIIII~~ ~~IIIII~~ I●●●●	T = 0*B*1 = 1*B*-4 **5s** T = 0*B*2 = 1*B*-3 **5s** T = 0*B*3 = 1*B*-2 **5s** T = 0*B*4 = 1*B*-1 **5s** T = 1*B*0 = 5 T = 1*B*1 = 2*B*-4 **5s**
Unbundled creates Decimals & Fractions & Negative Numbers IIIIIIII ➔ ~~III~~ ~~III~~ II		T = 2*B*2 **3s** = 2.2 **3s** T = 2 2/3 **3s** T = 3*B*-1 3s = 3.-1 **3s** T = 1*BB* 0*B* -1 (T = *p*x^2 + q*x + r*)
ReCount in Same Unit creates Flexible Numbers ▌▌▌▌▌III ➔ 53	5: ~~IIII~~ I I I ~~IIII~~ ~~IIII~~ I ~~IIII~~ ~~IIII~~ ~~IIII~~ 	T = 1*B*3 *Overload* T = 2*B*1 *Standard* T = 3*B*-1 *Underload* T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 **tens**
Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38 7* 4*B*8 = 28*B*56 = 33*B*6 = 336 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48
ReCount in New Unit  *ReCount-Formula:*	5 = ? **2s**  T = (T/*B*) * ***B***	T = 5 = (5/2)*2 = ? = 2*B*1 **2s** 5/2 2.some 5 - 2*2 1
ReCount: Tens to Icons ▌▌▌IIIIII = ? **7s**	3*B*5 **tens** = *u**7	*u**7 = 35 = (35/7)*7 so *u* = 35/7
ReCount: Icons to Tens 6 **8s** = ? **tens** 		T = 6 **8s** = 6*8 = (***B***-4)*(***B***-2) = ***BB*** - 4***B*** - 2***B*** - - 8 = 10***B*** - 6***B*** + 8 = 4*B*8 = 4.8 **tens** = 48
DoubleCount gives PerNumbers	2$ per 3kg = 2$/3kg	T = 6$ = (6/2) * 2$ = (6/2) * 3kg = 9kg
Like Units: Fractions 5% of 40	5$/100$ of 40$	T = 40$ = (40/100)*100$ gives (40/100)*5$ = 2$
DoubleCount a Block halved by its Diagonal		*a = (a/c)*c* = sin *A*c* *a = (a/b)*b* = tan *A*b* π= *n**tan(180/*n*) *for n large*

Allan.Tarp@MATHeCADEMY.net MrAlTarp YouTube:

Add NextTo OnTop	T = 2 **3s** + 4 **5s** = 3*B*2 **8s** *Integration*  T = 2 **3s** + 4 **5s** = 1*B*1 **5s** + 4 **5s** = 5*B*1 **5s** *Proportionality* 	
MatheMatism	ADDING WITHOUT UNITS Digits or Fractions or Per-numbers	

Flexible Bundle-Numbers Respect & Develop Kids Own Math

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total T by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. T = 2 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total T of 5 1s is recounted in 2s as T = 1B3 2s = 2B1 2s = 3B-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, T = 5 = 2B1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', $T = (T/B)*B$, to predict that 'from T, T/B times, Bs can be taken away'. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c;$ in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', sticks say 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

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07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Next-to addition geometrically means adding by areas, so multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed.

09. On-top addition means using the recount-formula to get like units. Changing units is also called proportionality, or solving equations if reversed.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

07. Workshop in addition-free STEM-based math

Nature as Heavy Things in Motion in Time and Space

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

Matter and force and motion all represent different degrees of Many, thus calling for a science about Many. This is how mathematics arose in ancient Greece, so it should respect its root as a natural science by letting multiplication precede addition since the basic science formulas are multiplication formulas expressing 'per-numbers' coming from double-counting: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*sq.-meter = pressure*sq.-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat*kg; Δ momentum = (Δ momentum/sec)*sec = force*sec = impulse; Δ energy = (Δ energy/meter)*meter = force*meter = work; gram = (gram/mole)*mole = molar mass*mole; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp.

Counting in Icon-Bundles Allows Recounting in the Same and in a Different Unit

Meeting many, we observe that five ones may be recounted as one five-icon. Likewise, with the other digits; thus being, not symbols, but icons with as many strokes or sticks as they represent. 'Bundle-counting' in icon-bundles allows 'bundle-writing' where a total T of 5 1s is recounted in 2s as T = 1B3 2s = 2B1 2s = 3B-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, T = 5 = 2B1 2s = 2.1 2s = 2 ½ 2s.

Recounting in the same unit to create or remove over- or underloads eases operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross.

This creates a 'recount formula', $T = (T/B)*B$, saying that 'from T, T/B times, Bs can be taken away'. This formula predicts the result of recounting in another unit, called proportionality: Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the unbundled singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s recount in 5s as 2B2 5s or 2.2 5s or 2 2/5 5s.

This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

Recounting to and from Tens

Times tables ask '2 7s = ? tens', eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

References

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Workshop exercises in addition-free STEM-based math

STEM: Mathematics as one of the natural Sciences, applied in Technology and Engineering

Using 'Outside-Inside Math' allows outside science problems to be solved by inside math formulas.

Outside degrees of Many create inside number-icons with the number of strokes they represent. Outside counting-operations, occurring when bundles are pushed away, lifted and pulled away to find unbundled ones, create the operation-icons division, /, and multiplication, x, and subtraction -.

Once bundle-counted, recounting in different units (called proportionality) create a 'recount-formula', $T = (T/B)*B$, saying that 'from T, T/B times, Bs can be taken away'; occurring all over math and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$; in science as speed: $\Delta s = (\Delta s/\Delta t)*\Delta t = v*\Delta t$.

Asking '3 **4s** is how many **5s**?', outside sticks show that 3 **4s** becomes 2*B*2 **5s**: IIII IIII IIII -> VV II.

To predict inside, we enter '3*4/5' to ask a calculator 'from 3 **4s** we take away **5s'**. The answer '2.some' predicts that the unbundled ones come by taking away 2 **5s**. Now, asking '3*4 – 2*5' gives '2'. So, 3 **4s** = 2*B*2 **5s = 2.2 5s**.

3*4/5	2.some
3*4 – 2*5	2

Recounting a quantity in two different physical units gives a 'per-number' as e.g. 2m per 3sec, or 2m/3sec. To answer the question 'T = 6m = ?sec', we recount 6 in **2s** since the per-number is 2m/3sec: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec. Double-counting in the same unit creates fractions and %: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%. 5% of 40 = ?; T = 40 = (40/100)*100 gives (40/100)*5 = 2.

kg = (kg/cubic-meter)*cubic-meter = density*cub.-meter force = (force/square-meter)*sq.-meter = press.*sq.-meter meter = (meter/sec)*sec = velocity*sec energy = (energy/sec)*sec = Watt*sec energy = (energy/kg)*kg = heat*kg	Δ momentum = (Δ mom./sec)*sec = force*sec = impulse Δ energy = (Δ energy/meter)*meter = force*meter = work gram = (gram/mole)*mole = molar mass*mole energy/sec = (energy/charge)*(charge/sec), or Watt = Volt*Amp.

Science multiplication formulas expressing 'per-numbers' coming from double-counting

Five Ways to Solve Proportionality Questions

Inside recounting solves outside questions as “If 2m need 5sec, then 7m need ?sec; and 12sec gives ?m”

● Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: ‘2m takes 5s, 7m takes ?s’ to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: ‘5s gives 2m, 12s gives ?m’ to get to the answer (12*2)/5s = 4.8m.

Then, two new methods appeared, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

● Q1: 1m takes 5/2s, so 7m takes 7*(5/2) = 17.5s. Q2: 1s gives 2/5m, so 12s gives 12*(2/5) = 4.8m.

● Q1: 2/5 = 7/*x*, so 2**x* = 7*5, *x* = (7*5)/2 = 17.5. Q2: 2/5 = *x*/12, so 5**x* = 12*2, *x* = (12*2)/5 = 4.8.

● Alternatively, we may recount in the ‘per-number’ 2m/5s coming from ‘double-counting’ the total *T*. Q1: *T* = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: *T* = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● SET introduced modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function *f*(*x*) = *c***x* from the set of m-numbers to the set of s-numbers, having as domain *DM* = {*x*∈R | *x*>0}. Knowing that *f*(2) = 5, we set up the equation *f*(2) = *c**2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: *c**2 = 5, (*c**2)*½ = 5*½, *c**(2*½) = 5/2, *c**1 = 5/2, *c* = 5/2. With *f*(*x*) = 5/2**x*, the inverse function is *f*-1(*x*) = 2/5**x*. So with 7m, *f*(7) = 5/2*7 = 17.5s; and with 12s, *f*-1(12) = 2/5*12 = 4.8m.

Three different kinds of mathematics answering the question: What is a function?

pre-setcentric:	*a function is a calculation with specified and unspecified numbers.*
present setcentric:	*a function is a subset of a set-product where component identity transfers*.
post-setcentric:	*a function is a number-language sentence with a subject, a verb and a predicate.*

EXERCICES

E01. With sticks, transform many OUTSIDE ones into one INSIDE many-icon with as many strokes as it represents.

E02. Name fingers as 5s using BundleCounting & BundleNumbers: 0**B**1 = 1**B**-4, 0**B**2 = 1**B**-3, …**5s**

E03. Count 5 fingers in 2s using flexible bundle-numbers: T = 5 = 1**B**3 = 2**B**1 = 3**B**-1 **2s** (overload, standard, underload)

E04. Recount ten fingers in **4s**, **3s** and **2s**: T = ten = 1**B**6 = 2**B**2 = 3**B**-2 **4s**; T = 3**B**1 = 1**BB**1 = 1**BB** 0**B** 1 = 10.1 **3s**; T = 5**B**0 = 4**B**2 = 2**BB** 1**B** 0 = 1**BBB** 0**BB** 1**B** 0 **2s**. ReCount 7 fingers in **3s**: T = 7 = 2B1= 1BB-2 = 2.1 = 3.-2 = 2 1/3.

E05. Write traditional numbers as flexible BundleNumbers: T = 53 = 5**B**3 = 4**B**13 = 6**B**-7 tens

E06. Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92 6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38 7* 4**B**8 = 28**B**56 = 33**B**6 = 336 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

E07. With cubes, transform the three OUTSIDE parts of a counting process, PUSH & LIFT & PULL, into three INSIDE operation-icons: division & multiplication & subtraction.

Five counted in **2s**: I I I I I (push away **2s**) II II I (lift to stack) II/II I (pull to find unbundles ones) II/II I.

E08. OUTSIDE BundleCounting with icons as units may be predicted INSIDE by a **recount-formula T = (T/B)*B**, (from T, T/B times, take Bs away) using a full number-language sentence with a subject, a verb and a predicate.

OUTSIDE: T = I I I I I; T counted in **2s**: II II I; T – 2x2: = II II I ; INSIDE: $\begin{matrix} 5/2 & 2.\text{some} \\ 5-2\text{x}2 & 1 \end{matrix}$

<u>E09. Recount in a new unit to change units, predicted by the recount-formula</u>

OUTSIDE, use sticks or cubes to answer 3 **4s** = ? **5s**. INSIDE, the recount-formula predicts 3x4/5

<u>E10. Recount from tens to icons</u>

OUTSIDE, to answer the question '40 = ? **5s'**, on squared paper transform the block 4.0 **tens** to **5s**.

INSIDE, formulate an equation to be solved by recounting 40 in **5s**:

u*5 = 40 = (40/5)*5, so u = 40/5.

Notice that recounting gives the solution rule 'move to opposite side with opposite calculation sign'.

<u>E11. Recount from icons to tens</u>

OUTSIDE, to answer '3 **7s** = ? **tens'** on squared paper transform the block 3 **7s** to **tens**.

INSIDE: oops, with no ten-button on a calculator we can't use the recount-formula? Oh, we just multiply!

<u>E12. ReCounting in two physical units</u>

Recounting in two physical units gives a 'per-number' as e.g. 2m per 3sec, or 2m/3sec.

To answer the question 'T = 6m = ?sec', we just recount 6 in the per-number **2s**: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

<u>E13. Solving STEM proportionality heating problems with recounting</u>

With a heater giving 20 J in 30 sec, what does 40 sec give, and how many seconds is needed for 50J?

With 40 Joules melting 5kg, what will 60 Joules melt and what will 7 kg need?

With 3 degrees needs 50 Joules, what does 7 degrees need; and what does 70 Joules give?

With 4 deg. in 20kg needing 50 Joules, what does 9 deg. in 30 kg need? What does 70 Joules give in 40 kg?

<u>E14. Mutual ReCounting the sides in a block halved by its diagonal creates trigonometry</u>:

a = (a/b)*b = tanA*b

Draw a vertical tangent to a circle with radius r. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/radius with tangent of the angle on a calculator.

<u>E15. Engineering</u>

A 12x12 square ABCD has AB on the ground and is inclined 20 degrees. From B, a straight road is to be constructed intersecting the borderline AD in the point E, inclined 5 degrees. Find the length DE.

Hint: Show that if DE = 2, then the incline of the road is 3.2 degrees.

<u>E16. Traveling</u>

With 4 meters taking 5 seconds, what does 6 meters take; and what does 7 seconds give?

With distance d and speed v and time t related as d = v*t, what time is needed to go 20m with velocity 4m/s?

With distance d and time related as d = 5*t^2, what time is needed to go 30m?

Hint: Use that if p^2 < N < $(p+1)^2$, then √N ≈ $\frac{N+p^2}{2p}$

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	1**BB**10
10**B**0	10**B**1	10**B**2	10**B**3	10**B**4	10**B**5	10**B**6	10**B**7	10**B**8	10**B**9	10**B**10
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	9**B**10
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	8**B**10
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	7**B**10
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	6**B**10
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	5**B**10
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	4**B**10
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	3**B**10
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	2**B**10
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	1**B**10
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	0**B**10

CTRAS 2020 in China

Proposals for papers to the CTRAS 2020 conference, invitation to co-authorship

- STEM prevents a goal-displacement that makes mathematics a goal instead of a means. 1
- Stop teaching wrong numbers and operations, start guiding children's mastery of many . 2
- To support STEM, trigonometry and coordinate geometry should precede plane geometry. 3
- A fresh-start year10 (pre)calculus curriculum. 4
- To support STEM, calculus must teach adding bundle-numbers, per-numbers and fractions also. 5
- Conflicting grand theories create 2x3x2 different mathematics educations. 6
- Replacing STEAM with STEEM will include economics also. 7
- the power of per-numbers. 8
- Mixing design and difference research with experiential learning cycles allows creating classroom teaching for all students. 9
- From place value to bundle-bundles: units, decimals, fractions, negatives, proportionality, equations and calculus in grade one. 10
- Sociological imagination designs micro-curricula for experiential learning cycles . 11
- Concrete STEM subjects allow mathematics learning by modeling and peer-brain teaching. 12
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STEM prevents a goal-displacement that makes mathematics a goal instead of a means.

Asking what the purpose of mathematics education is, US and UK mathematics educators say "to learn school mathematics". Others say, "to learn set-based mathematics as defined by university mathematics." Focusing on competences leads to saying "to learn mathematical competences" or "to master mathematics". Seldom, if ever, is heard that the goal is "to master many" or "to develop the number-language that children bring to school."

Sociological imagination (Bauman, 1990) may prevent a goal displacement where a means becomes a goal instead. Historically, the Pythagoreans chose the word 'mathematics' meaning 'knowledge' in Greek as a common name for their knowledge about Many in space and time and by itself: astronomy, music, geometry and arithmetic. And today in North America, mathematics is still a common name for geometry and algebra, showing their outside goals in their original meanings, earth-measuring in Greek, and reuniting in Arabic. Integration and differentiation also name their tasks directly, to integrate small changes, and to differentiate a total change in small changes.

To avoid a goal displacement, mathematics must de-model (Tarp, 2019) its core ingredients: digits, operations, equations, fractions, functions etc. to allow primary school develop the flexible bundle-numbers children bring to school by teaching, not numbers to add, but numbering totals by counting, recounting and double-counting, where recounting 8 fingers in 2s as 8 = (8/2)*2 leads directly to the recount-formula $T = (T/B)*B$ with per-numbers that solve equations, that occur in most STEM-formulas typically predicting proportionality, and that become fractions when double-counting in the same unit.

Liberated from its goal displacement, mathematics education may have its own communicative turn as in the 1970s (Widdowson, 1978) such that from now on both the word- and the number-language are taught and learned through their use and not through their grammar, thus allowing all students to model outside quantities as to levels, change and distribution.

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Stop teaching wrong numbers and operations, start guiding children's mastery of many.

Learning means adapting the inside brain to outside nature and culture. Vygotsky prioritizes culture and wiser-brain teaching, Piaget nature and peer-brain learning.

Adapting to Many, children answer the question 'How many?' with bundle-numbers as T = 2 3s containing two digits: 3 is a quantity-number in space, also called a cardinal-number taking on positive integer values; 2 is a counting-number in time taking on also decimal, fractional, and negative values as T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

Quantity-numbers may add, and so may counting-numbers, but not in between. So, digits must be categorized before adding. Digits are not numbers but operators, needing a multiplier to become a number, T = 2 3s = 2*3, as seen when writing numbers fully as polynomials, as e.g. *T* = 345 = 3**B*^2 + 4**B* + 5*1

So, teaching digits as numbers is teaching wrong numbers. And bundle-numbers need not to be taught since children bring them to school, that should guide them to develop their number-language by learning that

- digits are icons with as many strokes as they represent.
- operations are icons also, rooted in the counting process: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.
- recounting 8 in 2s gives a recount-formula: 8 = (8/2)*2, or *T = (T/B)*B*, used to solve the equation *u**2 = 8 by recounting 8 in 2s to give the solution *u* = 8/2; thus solving most STEM-equations, typically predicting proportionality.

Later recounting between digit-units and tens leads to tables, and to equations when asking e.g. T = 4 6s = ? tens, and T = 42 = 4.2 tens = ? 7s.

So, childhood education should guide children develop the quantitative competence they bring to school using Kolb's experiential learning cycles.

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To support STEM, trigonometry and coordinate geometry should precede plane geometry

Halved by its diagonal *c*, a rectangle becomes a right triangle ABC with base *b* and height *h*. Using the recount-formula *T = (T/B)*B* coming from recounting 8 in 2s as 8 = (8/2)*2, mutual recounting gives trigonometry: *h = (h/c)*c* = sin*A*c*, *b = (b/c)*c* = cos*A*c*, *h = (h/b)*b* = tan*A*b*.

Splitting the diagonal in *c1* and *c2* by the triangle-height produces two triangles where cos*A* = *c1/b = b/c*, making *b^2 = c*c1*, and cos*B* = *c2/h = h/c*, making *h^2 = c*c2*, thus giving the Pythagoras rule *h^2 + b^2 = c^2*.

Finding √70 means squeezing 7 tens until becoming a square (8+*t*)^2 situated between 8^2 and 9^2. And having four parts as shown by two playing cards placed like an L: 8^2, and 8**t* twice, and *t*^2. Neglecting *t*^2, we get the equation 8**t* = (70-8^2)/2 = 3 = (3/8)*8, solved by recounting 8 in 3s, giving *t* = 3/8 = 0.375, so 8.375^2 = 70,14 = 70 approximately.

In a coordinate system, a circle with center in the origin and radius r gets the equation *x^2 + y^2 = r^2*, else *(Dx)^2 + (Dy)^2 = r^2*. In a horizontal right triangle, moving along the diagonal will change *x* and *y* with *Dx* and *Dy*. Recounting *Dy* in *Dx* gives *Dy = (Dy/Dx)*Dx = m*Dx* = tan*A*Dx* that allows drawing lines from tables.

Intersection points between lines are predicted by a linear equation solved by technology or by moving to opposite side with opposite sign.

Intersection points between lines and circles or parabolas are predicted by a quadratic equation *x^2 + b*x + c* = 0, solved by two L-placed playing cards showing that (*x*+*t*)^2 = *x^2 + 2*x*t + c + (t^2 – c)* where the first three terms disappear with *t* = *b*/2.

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A fresh-start year10 (pre)calculus curriculum.

Often precalculus suffers from lacking student knowledge. Three options exist: make mathematics non-mandatory, choose an application-based curriculum; or, to rebuild student self-confidence, design a fresh-start curriculum that also includes the core of calculus by presenting integral calculus first.

Writing a number out fully as a polynomial, e.g. T = 345 = 3*B^2 + 4*B + 5 shows the four ways to unite numbers, resonating with the Arabic meaning of the word algebra, to reunite: addition and multiplication unite changing and constant unit-numbers into totals; and next-to-block-addition (integration) and power unite changing and constant per-numbers, all having reverse operations that split totals into parts.

Addition, multiplication, and power are defined as counting-on, repeated addition and repeated multiplication. As reverse operations, x = 7-3 is defined as the number that added to 3 gives 7, thus solving the equation x+3 = 7 by moving to opposite side with opposite sign. Likewise, x = 7/3 solves x*3 = 7, the factor-finder (root) x = 3√7 solves x^3 = 7, and the factor-counter (logarithm) x = log3(7) solves 3^x = 7, again moving to opposite side with opposite sign.

Hidden brackets allow reducing a double calculation to a single: 2+3*x = 14 becomes 2+(3*x) = 14, solved by x = (14-2)/3. Next transposing letter-equations as T = *a+b*c^d* really boost self-pride.

Future behavior of 2set unit-number tables is predicted by linear, exponential, or power models assuming constant change-number, change-percent, or elasticity.

1-4set per-number speed tables are modeled with lines, parabolas and double-parabolas, allowing technology to calculate the distance covered, thus introducing integral calculus, that also occurs when adding per-numbers in mixture-problems, and when adding percent in cross tables generated by statistical questionnaires.

Trigonometry comes from mutual double-counting sides in a rectangle halved by its diagonal, and is used to model distances to far away points, bridges, roads on hillsides, motion down an incline, and jumps from a swing.

References

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To support STEM, calculus must teach adding bundle-numbers, per-numbers and fractions also.

Created to add locally constant per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier. Difference research searching for hidden differences finds that the answer is yes.

Integral calculus occurs in grade one when performing next-to addition of bundle-numbers as e.g. T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg, again with differential calculus coming from the reverse question: 2kg at 3$/kg + 4kg at ? $/kg = 6kg at 5 $/kg. Here the per-number graph is piecewise constant c, i.e. there exists a delta-interval so that for all positive epsilon, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus also added by their areas, and never without units.

In high school adding per-numbers occurs when the meters traveled with varying m/s speed P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P, is found by slicing the area thinly so that its change may be written as $dA = P*dx$ in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find $A' = dA/dx = P$.

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Conflicting grand theories create 2x3x2 different mathematics educations

As part of institutionalized education, mathematics falls under the focus of the three grand theories, philosophy, sociology and psychology, discussing different kinds of mathematics, of education and of learning; and recommending appropriate means to institutional goals. However, is the goal to master mathematics first, as a means to later master many; or to master many directly if mastering mathematics proves difficult?

As to learning, psychology sees coping coming from brains adapting to outside nature and culture, but which is more important? Vygotsky points to culture, mediated by a more knowledgeable wiser-brain, a teacher. Piaget points to nature, automatically creating inside schemata that accommodate if meeting outside resistance from nature or from peer-brain communication.

As to mathematics, philosophy has three conflicting views: Pre-modern mathematics is inspired by the Pythagoreans seeing mathematics as knowledge about Many in space and time and by itself as expressed in astronomy, music, geometry and arithmetic; and as part of the three basic Rs: reading, writing and 'rithmetic called reckoning in Germanic countries. Modern mathematics needs no outside examples for its concepts. Alternatively, postmodern scepticism sees mathematics as a number-language abstracting inside concepts from outside examples, and parallel to the word-language.

As to institutions, sociology recommends imagination to prevent a goal displacement making a means a goal instead. As to education, two conflicting views exits. One sees the student as a servant of the state forcing its population to choose between different multiyear tracks from upper secondary school, and forcing students back to start if changing track. One sees the state as a servant of the student by helping students to uncover and develop their personal talent in self-chosen half-year blocks after puberty.

So, two different learning forms, three different mathematics forms, and two different education forms create 2x3x2 different ways of conducting mathematics education.

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Replacing STEAM with STEEM will include economics also.

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of physical quantities. Statistics post-dicts unpredictable quantities by setting up probabilities for future behavior, using factual or fictitious numbers as median and fractals or average and deviation. Including economics and English in STEM opens the door to statistics also. Art may be an appetizer, but not a main course since geometry and algebra should be always together and never apart to play a core role in STEM.

Macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheet for simulations using different parameters.

Microeconomics describes equilibriums in the individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable cost, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determines the equilibrium price. Market surveys leads to statistics, as does insurance. In the households, spending comes from balancing income and transferals with saving and tax. In a bank, income come from simple and compound interest, from installment plans as well as risk taking. At a stock market, courses fluctuate. Governments must consider quadratic Laffer-curves describing a negative return of a tax-raise. To avoid units, factories use variations of Cobb-Douglas power elasticity production functions for modeling.

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The power of per-numbers

Uniting unit-numbers as 4$ and 5$, or per-numbers as 6$/kg and 7$/kg or 6% and 7%, we observe that addition and multiplication unite changing and constant unit-numbers into a total, and integration and power unite changing and constant per-numbers. Reversely, subtraction and division split a total into changing and constant unit-numbers, and integration and power split a total into changing and constant per-numbers.

Recounting 8 in 2s as 8 = (8/2)*2 creates a recount-formula *T = (T/B)*B*, saying 'From T, T/B times, T may be pushed away'; and used to change units when asking e.g. 2 6s = ? 3s, giving the prediction T = (2*6/3)*3 = 4*3 = 4 3s.

Recounting 8 in 2s also provides the solution *u* = 8/2 to the equations as *u**2 = 8 = (8/2)*2; thus solving most STEM-equations, since the recount-formula occurs all over. In proportionality, $y = c*x$; in coordinate geometry as line gradients, $\Delta y = \Delta y/\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In science as meter = (meter/second)*second = speed*second, etc. In economics as price formulas: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

With physical units, recounting gives per-numbers bridging the units. Thus 4$ per 5kg or 4/5 $/kg gives T = 15kg = (15/5)*5kg = (15/5)*4$ = 3$; and T = 16$ = (16/4)*4$ = (16/4)*5kg = 20kg. With like units, per-numbers become fractions.

Trigonometry occurs as per-numbers when mutually recounting sides in a rectangle halved by its diagonal, $a = (a/c)*c = \sin A*c$, etc.

Modeling mixtures as 2kg at 3$/kg + etc, unit-numbers add directly, but per-numbers *P* add by the area *A* under the per-number graph, found by slicing it thinly so that the change may be written as d*A* = *P**d*x* in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find the per-number *A*' = d*A*/d*x* = *P*.

References

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Mixing design and difference research with experiential learning cycles allows creating classroom teaching for all students

International tests show that not all students benefit from mathematics education. This poor-performance-problem raises a Cinderella question: is there a hidden difference that can make the Prince dance? If so, design research can create Kolb's experiential learning cycles to adapt a given micro-curriculum so that all students may benefit.

In primary school, difference research searching for hidden differences has identified several alternatives: Digits are icons. Numbers are double-numbers with bundles as units, e.g. T = 2 3s. Flexible bundle-numbers have over- and underloads, e.g. T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 tens, and ease operations as e.g. 329/ 7 = 32*B*9/ 7 = 28*B*49 /7 = 4*B*7 = 47, or 23*8 = 2*B*3*8 = 16*B*24 = 18*B*4 = 184.

Operations are icons also where division is a broom pushing away bundles, multiplication a lift stacking bundles, subtraction a rope pulling away stacks to find unbundled, and addition the two ways to unite stacks, on-top and next-to.

Changing units may be predicted by a recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, or, and used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically predicting proportionality: meter = (meter/sec)*sec = speed*sec.

In middle school, double-counting leads to per-numbers becoming fractions with like units, and adding by their areas as integral calculus. In algebra, factors are units placed outside a bracket. Trigonometry occurs when mutually double-counting sides in a rectangle halved by its diagonal.

In high-school, redefining inverse operations allows equations to be solved by moving to opposite side with opposite sign. And adding per-numbers by areas allows introducing integral calculus before differential calculus.

Designing and redesigning micro-curricula as experiential learning cycles allows teachers perform design research in their own classroom, to be reported as master projects first, and later perhaps as PhD projects including more details.

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From place value to bundle-bundles: units, decimals, fractions, negatives, proportionality, equations, and calculus in grade one

Traditionally, a multi-digit number as 2345 is presented top-down as an example of a place value notation counting ones, tens, hundreds, thousands, etc.; and seldom as four numberings of unbundled, bundles, bundle-bundles, bundle-bundle-bundles, etc., to provide a bottom-up understanding abstracted from concrete examples, which would introduce exponents in primary school as the number of bundle-repetitions. Counting ten fingers in 3s thus introduces bundle-bundles: T = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles, the unbundles singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles thus leading to fractions, T = 7 = 2 1/3 3s; or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may also be included in the counting sequence: 0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s may be written as a recount-formula: 8 = (8/2)*2, or $T = (T/B)*B$, used to solve the equation u*2 = 8 by recounting 8 in 2s to give the solution u = 8/2; thus solving most STEM-equations, typically expressing proportionality.

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

At the end of grade one, recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s.

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Sociological imagination designs micro-curricula for experiential learning cycles

Forced by peer review to focus on existing research, many education research articles fail to be validated in the classroom by observing if its educational goal is reached. However, the peer review crisis creates a need for a different research meeting its proper genre demands: reliable data and valid findings to a research question.

To help student brains adapt to the outside world, mathematics education must decide if its goal is to master inside mathematics as a means to later master outside quantity, thus risking what sociology calls a goal displacement (Bauman, 1990) where a means becomes a goal instead; or to master quantity directly if first mastering contemporary university mathematics becomes too difficult to many students.

Many curriculum reforms include competences. But again, we must ask: is the goal to obtain inside mathematical competence, or outside quantitative modeling competence?

A learning-by-doing curriculum calls for experiential learning cycles as described by Kolb's learning theory (Kolb, 1983) being adapted e.g. in the new Vietnamese curriculum; and containing cyclic phases. First micro-curriculum A is taught and validated if meeting its expected goals, next systematic observations gather reliable data as to which goals are met, and which are not, then reflections modifies the micro-curriculum into version B. Then plan B is taught, etc.

Combined with design research (Bakker, 2018), experiential learning cycles allows teachers to become action learners or action researchers in their own classroom reporting their work in master or PhD papers. To meet the genre demands of research, the data gathered must be reliable, and the findings must be tested for validity. In design research, reliability comes by making systematic observations through notes, interviews, questionnaires etc. And testing validity here means holding on to the strong parts of the actual micro-curriculum and changing the weak parts.

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Concrete STEM subjects allow mathematics learning by modeling and peer-brain teaching

Traditionally, mathematics is considered one of the core subjects in education because of the many 'applications of mathematics'. This phrasing leads directly to the view that "of course mathematics must be learned first before it can be applied by others". Consequently, mathematics teaches the operation order addition, subtraction, multiplication and division with cardinal numbers, later expanded to integers, rational and real numbers, again followed by expressions including also unspecified numbers.

Talking instead about outside roots leads to the opposite view that "of course, mathematics must be learned through its outside roots, also constituting its basic applications". This 'de-modeling' view resonates with the fact that historically, the Pythagoreans chose the name mathematics, meaning knowledge in Greeks, as a common label for their four areas of knowledge about Many in time and pace, in time, in space and by itself: astronomy, music, geometry and arithmetic. Later the Arabs added algebra with polynomial numbers created by systematic bundling. Here the outside roots are evident through the original meanings of geometry and algebra: earth-measuring and reuniting.

So, mathematics grew and may still grow from counting, recounting and double-counting bundles, and from applying science, describing forces pumping motion in and out of matter when having the same or opposite directions.

Working in groups with science applications allows students to learn through peer-brain teaching instead of through wiser-brain teaching. As to matter, tasks could be to find its mass, its center, its density, and the heat transfer under collision between visible macro-matter and invisible micro-matter, applied in steam power, or when placing ice-cubes in water.

As to motion, tasks could be to describe traveling with constant or changing speed horizontally or on an incline, vertical motion, projectile orbits; and circular motion, swings or see-saws on a market place. As well as how to use electrons to store or transport motion and information.

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The Simplicity of Mathematics Designing a STEM-based Core Curriculum for refugee camps

Numbers as 2345 evade the place value notation if seen as numbering unbundled, bundles, bundle-bundles, bundle-bundle-bundles. Here exponents occur as the number of bundling-repetitions, e.g. when counting ten fingers as T = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles in blocks, the unbundled singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles leading to fractions, T = 7 = 2 1/3 3s, or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may be included in the counting sequence: 0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s gives a recount-formula, 8 = (8/2)*2 or *T = (T/B)*B*, that changes unit from 1s to 2s (proportionality), that gives the equation u*2 = 8 the solution u = 8/2 (moving to opposite side with opposite sign), and that shows that per-numbers as 8/2 must be multiplied to areas before being added (integral calculus).

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

Recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s. Recounting in different units gives per-numbers bridging the units, becoming fractions with like units, and adding by areas. Mutually recounting sides in a block halved by its diagonal gives trigonometry.

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Calculation models: Fact, or fiction

As qualitative literature, also quantitative literature has the genres fact and fiction when modeling real world situations.

Fact is ‘since-then’ calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. ‘since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20’. Fact models can be trusted once the numbers and the formulas and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter.

Fiction is ‘if-then’ calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. ‘if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20’. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.

This paper looks at three infection models, the standard logistic model and two alternatives, one formulated as a differential equation, one as a difference equation.

The models all assume that the population change is proportional to the population itself thus giving a doubling factor that is assumed to decrease with the number of infected as the first model assumes, or with time as the two others assume.

However, where the standard model cannot be validated since infection data are difficult or impossible to achieve, reliable data from the number of hospital beds points to the other fiction models. Here, the scientific principle of simplicity, known as Occam's razor, points to the difference equation, easy to set up in a spreadsheet.

It may be simple but it provides important information: its high degree of elasticity recommends a gradual reduction of the two central factors, group size and meeting time, instead of a complete shutdown.

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Bundle- and Per-numbers Replacing the Number Line will Free the Magic of Numbers from its 'No-unit' Greenhouse

Outside, addition folds but multiplication holds, since factors are units while addition presupposes like units. This creates two paradigms in mathematics, an outside 'unit' paradigm, and an inside 'no-unit' paradigm making mathematics a semi-greenhouse. To make mathematics a true science with valid knowledge, we ask what mathematics can grow from bundle-numbers with units, being areas instead of points on a number line. Concretely constructed, digits become number-icons with as many sticks as they represent, and operations become counting-icons for pushing, lifting and pulling away bundles to be added next-to or on-top. Recounting 8 in 2s creates a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. By changing units, it occurs as proportionality formulas in science; it solves equations; and it shows that per-numbers and fractions, T/B, are not numbers, but operators needing numbers to become numbers. Fractions, decimals, and negative numbers describe how to see the unbundled. Recounting sides in a box halved by its diagonal allows trigonometry to precede plane and coordinate geometry. Once counted, total may add on-top after recounting makes the units the same; or next-to addition by adding areas as in integral calculus, which also occurs when adding per-numbers. So, mathematics created outside the 'no-units' greenhouse is the same as inside, only the order is different, and all is linked directly to outside things and actions making it easier to be applied. And, with multiplication preceding it, addition only occurs as integral calculus, unless inside brackets with like units.

Keywords: arithmetic, equation, proportionality, trigonometry, calculus

Chapter 01. The Two Paradigms in Mathematics Education

The necessity of numbers and calculations as social and individual tools makes them educational tasks in school. Writing the book 'Mathematics as an Educational Task', Freudenthal (1973) succeeded in giving university mathematics monopoly by claiming that mastering mathematics is the only way to later mastering Many. So, from day one in school, or even in preschool, children are forced to learn its foundation, the one-dimensional number line were 1+1 = 2, despite the fact that, with units, this is seldom the case.

But, numerous international tests together with more than 50 years of mathematics education research following the first International Congress on Mathematics Education, ICME, in 1969 have shown that mathematics is hard to learn. Consequently, more learning must come from more research mediated by more facilitators to more educated teachers so they can be more successful with more hard-working students helped by more advanced technology. In short, send more money.

Difference research (Tarp, 2018) instead asks the Cinderella question: are there hidden alternative ways to master Many that evades the hard-to-learn 'no-unit' mathematics? What if children already learn to master Many from adapting to it, can school then develop this into mastering a mathematics that may be different from the school version? So, we ask:

What mathematics can grow from the mastery of Many children develop before school?

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists in space and time: bundles of 2s, and 2 of them.

So, children adapt to Many by using 2-dimensional bundle-numbers with units. And they use full sentences as in the word-language with a subject, 'that', and a verb, 'is', and a predicate, '2 2s', which shortened transforms a number-language sentence into a formula 'T = 2 2s'.

However, the number line does not include units. Instead, school teaches a mathematics that is built upon the assumption that 1+1 = 2 unconditionally. And that thereby fails to meet the basic condition of a science: its statements must not be falsified outside. Which '1+1 = 2' typically is when including units: 1 week + 1 day = 8 days, 1 km + 1mm = 1 km, etc.

Thus, where school works with one-dimensional line numbers without units, children work with two-dimensional area numbers with units. So, there seems to be two paradigms in mathematics. The first is a ruling 'no-unit' paradigm that makes mathematics a semi-greenhouse since outside, addition folds and multiplication holds: 2x3 = 6 simply states that 2 3s can be recounted to 6 1s. The other paradigm is an opposite 'unit-paradigm' where numbers always carry units.

Inside the 'no-unit' greenhouse, outside sentences as T = 2 3s are shortened to 2, thus leaving out the subject, and the unit. This is called an inside modeling of the outside world. The reverse then is called an outside 'de-modeling' or reifying of an inside statement (Tarp, 2020).

As described by Kuhn (1962), to have a career, a researcher must work inside the ruling paradigm or 'truth regime' (Foucault, 1995) where it is more important to include library references than to perform laboratory testing. A paradigm shift first comes when somebody ignores the existing literature, and use outside validation as the only criteria for quality.

One answer to the above question is given by Tarp (2018, 2020). This answer will now be brought to a more detailed level to design micro-curricula to be tested in design research (Bakker, 2018). Because of space limitations, some micro-curricula designs are left out here.

Chapter 02. Counting in Time with Sequences

Before designing, we reflect on how a row with many sticks is worded by a counting sequence with different names until we reach the bundle, after which a reuse typically takes place when multi-counting the singles, the bundles, the bundles-of-bundles, etc., In the end we get a final total as T = 345 = 3 bundles-of-bundles & 4 bundles & 5 unbundled.

Many occurs in time and space. Repetitions in time may be represented in space by a row of sticks. In space, a lot may be rearranged in a row that is transformed into a total by repeating pushing away one item at a time, and at the same time wording the actual total.

Counting outside the 'no-units' greenhouse we always include units when describing what exists. So, we say '0 bundle 1' instead of just '1'.

Number-names become flexible when allowing 'overloads' as 'twenty-nine, twenty-ten, twenty-eleven'. And, when allowing 'underloads' counting 'bundle-less2, bundle-less1, bundle' instead of '8, 9, ten'.

Overloads and underloads allow flexible bundle-numbers with units to outside de-model inside unit-free numbers:

T = 38 = 3Bundle8 = 2Bundle18 = 4Bundle less2, or short

$T = 38 = 3B8 = 2B18 = 4B\text{-}2$, and

$T = 347 = 3BB4B7 = 2BB14B7 = 1BB23B17$, or

$T = 347 = 3BB4B7 = 3BB5B\text{-}3 = 4BB\text{-}6B7$.

Bundling in tens, it takes a while before meeting the bundle-of-bundles, *BB*. But, counting in 3s will allow meeting 9 as 1 *BB* since ten fingers then become a total of

T = ten = 3Bundle1 = 3*B*1 = 1*BB*1 3s, or more precisely

T = ten = 1*BB*0*B*1 3s, modeled inside the 'no-unit' greenhouse as 101.

Including the units thus makes place values needless, or easier to understand.

Based upon the above reflections we may now formulate a micro-curriculum to be tested.

Micro Curriculum 02. Different Counting Sequences Using Flexible Bundle-numbers

The goal is to see inside numbers as short models for what exists outside, totals of unbundled singles, bundles, bundle-of-bundles, etc.

The means is to count a total in flexible bundle-numbers using different bundle-sizes; and to outside de-model inside shortened numbers.

Exemplary guiding tasks may be: "Count five fingers in 2s in different ways using overloads and underloads", "Count ten fingers in 5s", "Count ten fingers in 3s".

Material may be the ten fingers (or twelve if including the arms), the finger-parts, sticks, stones, snap-cubes, strokes on a paper, and a western, Chinese or Japanese abacus.

Chapter 03. Counting in Space with Icons for Digits and Bundling

Before designing, we reflect on how in space, four ones may be rearranged as one four-icon called a 4-digit. And that the same is almost the case with the other digits also. So basically, a digit is an icon with as many sticks or strokes as it represents if written less sloppy (Tarp, 2018).

Bundling in tens, ten has no icon, since ten 1s is 1 ten-bundle and no unbundled:

T = ten = 1Bundle0 tens = 1*B*0 tens, modeled inside the 'no-unit' greenhouse as 10.

Once created, digits may be used when recounting a total, *T*, in a new unit, e.g.,

T = 8 = 4 2s, or *T* = 8 = 2 4s, or *T* = 9 = 3 3s

In time, we recount a total by bundling or grouping, i.e., by repeating pushing away bundles or groups. But which is the better word to use, to bundle or to group?

Humans may be grouped according to different qualities naming the groups: males, females, etc. Repeating grouping groups according to other qualities as, e.g., age will give a smaller number of members in the sub-groups.

Sticks do not have individual qualities to name different groups. Instead, bundling allows a bundle of 1s to be transformed into 1 bundle that becomes a unit for repeated counting; where again a bundle of bundles may be transformed into a new unit, 1 bundle-of-bundles containing more members than the bundle. So 'bundle' may be preferred to grouping since it reflects more precisely what exists and happens in space and time outside the 'no-unit' greenhouse.

We observe that we recount a total of 8 ones in 2s by from 8 pushing away bundles of 2s four times. This may be written with an uphill stroke iconizing a broom pushing away the bundles, called division.

By showing that 8/2 = 4, a calculator predicts that "from 8, we push 2, max 4 times".

Using snap-cubes, we can now stack the 4 twos as a 4-by-2 stack or box. This may be written with a cross iconizing a lift lifting away the 2-bundles 4 times, called multiplication.

By showing that 4 x 2 = 8, a calculator predicts that "4 times lifting 2s gives 8".

However, since the 4 came when pushing 2s away from 8, we may now write 8 = 4 x 2 as 8 = (8/2) x 2, or using letters for unspecified numbers

T = (*T/B*) x *B* saying: “From *T, T* push away *B* times we lift *B.*“

Since this ‘recount formula’ changes the units, it is also called a proportionality formula, a core formula used all over mathematics, science and economics.

We observe that 1 is left unbundled if recounting 9 in 2s. The action ‘from 9, pull away 4 2s’ may be written with a horizontal stroke iconizing a rope pulling away the stack, called subtraction.

By showing that 9 – 4x2 = 1, a calculator predicts: “From 9, pull 4 2s, leaves 1”.

Placed on-top of the stack, the unbundled is described as an underload, *T* = 9 = 4*B*1 2s = 5*B*-1 2s.

Or, it may be separated by the bundles by a decimal point,

T = 9 = 4*B*1 2s = 4.1 2s.

Or, recounted in 2s as 1 = (1/2) x 2, it may be seen as a part or fraction of a full bundle,

T = 9 = 4 ½ 2s.

Based upon the above reflections we may now formulate a micro-curriculum to be tested.

Micro Curriculum 03. Iconizing Digits and Recounting

The goal is to see digits as number-icons with as many sticks as they represent; and to see the operations division, multiplication, and subtraction as operation-icons for pushing, lifting and pulling away bundles; and to see that when changing units, the result may be predicted by a recount formula saying that a total T contains T/B units B.

The means is to rearrange many sticks, cars or dolls into 1 icon; and to recount a row of snap-cubes using a playing card to push away bundles to be lifted into a stack, that is pulled away by a rope or a rubber band; and to let a calculator predict the recounting result before carrying it out.

Exemplary guiding tasks may be: “There seems to be four sticks in the symbol for four so that digits are not symbols as are letters, but icons with as many sticks as they represent. Does this apply to the other digits also?”, “Recount first 8, then 9, then ten fingers or sticks or snap-cubes in 2s, 3s and 4s”, “Predict and test the result of recounting 2 3s in 4s; 4 2s in 3s; 3 4s in 2s. Likewise with 3 4s in 5s.” Material is the ten fingers (or twelve if including the arms), the finger-parts, sticks, stones, snap-cubes, strokes on a paper, a western, Chinese or Japanese abacus, a folding ruler.

Chapter 04. Recounting in and from Tens Solves Equations and Introduce Algebra

Before designing, we reflect on recounting between icons and tens.

Recounting from tens to icons means asking, e.g., “How many 2s in 8?”.

Using *u* for the unknown number, this may be written as an equation “*u* x 2 = 8”.

But, since 8 can be recounted in 2s as 8 = (8/2) x 2, we see that *u* = 8/2.

So, the equation *u* x 2 = 8 is solved by *u* = 8/2, i.e., by moving the known number to the opposite side with the opposite calculation sign. After solving an equation, the answer must be tested in the original equation:

With *u* = 8/2 = 4, *u* x 2 = 4 x 2 = 8 as expected.

The ‘opposite side & sign’ method resonates with the formal definition for division inside the ‘no-unit’ greenhouse. Here 8/2 is what multiplied with 2 gives 8: if 8/2 = *u* then *u* x 2 = 8.

Later we will see that it resonates with the formal definition for other operations, so basic equations are solved by moving to opposite side with opposite calculation sign:

$u + 2 = 8$	$u \times 2 = 8$	u ^2 = 8	2^u = 8
$u = 8 – 2$	$u = \frac{8}{2}$	$u = \sqrt[2]{8}$	$u = \log_2 8$

Figure 1. Solving equations by moving to opposite side with opposite calculation sign

Recounting from icons to tens means asking, e.g., “How many tens are 6 8s?”.

With no ten-button on a calculator, we cannot use the recount formula. But multiplication gives the result directly, only without units and decimal point: *T* = 6 8s = 6 x 8 = 48 = 4.8 tens

On a peg-board or a squared paper we may set up a ten-by-ten square with 6 and 8 on the sides written with underloads as *B*-4 and *B*-2. We see that the 6 x 8 stack is left when from the *B* x *B* stack we pull away a 4 x *B*, and a *B* x 2 stack, and add the 4 x 2 stack pulled away twice.

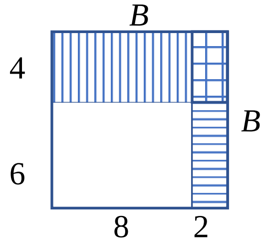
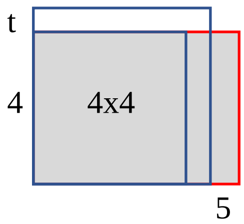
	*T* = 6 x 8 = (*B*-4) x (*B*-2) = *BB* – 2*B* – 4*B* + 8 = 4*B*8 = 48	$T = \binom{1B-4}{1B-2}$ = 1*BB* – 2*B* – 4*B* + 8 = 10*B* – 6*B* + 8 = 4*B*8 = 48	$\sqrt{20}$ 

Figure 2. Multiplying Numbers as Binomials; and squaring a 4x5 stack.

Multiplying two ‘less-numbers’ horizontally thus creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles. A short rule is: multiplying with less-numbers, subtract their sum and add their product (to 100).

A stack changes form with the unit, so to hold the same total, increasing the base will decrease the height, and vice versa.

A special form is a square. Wanting to square a 4-by-5 stack, its side is called $\sqrt{20}$, using lines to iconize the square. To find $\sqrt{20}$ we see that removing the 4-square leaves 20 – 4x4 = 4 shared by the two 4 x *t* stacks in a 4 + *t* square, giving *t* = 0.5. A little less since we neglect the *t*-square in the upper right corner. A calculator predicts that $\sqrt{20}$ = 4.472.

Chapter 05. Recounting in Physical Units Creates Per-numbers

Before designing, we reflect on recounting in physical units.

Inside the ‘no-unit’ greenhouse, shifting physical units is seen as an outside application of proportionality where division allows finding the unit cost to answer the two question types: “With 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?”

Outside the ‘no-unit’ greenhouse, recounting a total that is already counted in one physical unit creates a ‘per-number’ as 3$/2kg, used to easily shift units by recounting:

T = 6 kg = (6/2) x 2 kg = (6/2) x 3 $ = 9$; and

T = 15$ = (15/3) x 3$ = (15/3) x 2kg = 10 kg.

Alternatively, we may recount the units:

$ = ($/kg) x kg = (3/2) x 6 = 9; and

kg = (kg/$) x $ = (2/3) x 15 = 10.

Per-numbers occur all over science and mathematics:

meter = (meter/sec) x sec = speed x sec,

kg = (kg/cubic-meter) x cubic-meter = density x cubic-meter;

energy = (energy/sec) x sec = Watt x sec;

y-change = (*y*-change/*x*-change) x *x*-change = slope x *x*-change, or $\Delta y = (\Delta y/\Delta x) \times \Delta x$

With like units, per-numbers become fractions: 1$/4$ = ¼.

Inside, the 'no-unit' greenhouse teaches fractions as division: 1/4 of 12 = 12/4. Outside, a fraction is a per-number counting both the part and the total, ¼ is 1-part per 4-total.

Chapter 06. Trigonometry Recounts the Sides in a Box Halved by its Diagonal

Before designing, we reflect on recounting the sides in a box halved by its diagonal.

Inside the 'no-unit' greenhouse, trigonometry is taught after plane and coordinate geometry, and with heavy emphasis on trigonometric identities.

Outside, trigonometry are per-numbers coming from recounting the sides in a height-x-base box halved by a diagonal cut, which produces two like right triangles with horizontal base-side, *b*, a vertical height-side, *h*, and a slant cut-side, *c*.

Recounting the height in the base we get

height = (height / base) x base = tangent *A* x base, shortened to

h = (*h* / *b*) x *b* = tan *A* x *b* = tan *A* *b*s, thus giving the formula

tangent *A* = height / base, or tan *A* = *h*/*b*.

Using the words 'run' and 'rise' for 'base' and 'height', we get the formula:

tan *A* = rise / run, giving the steepness or slope of the diagonal.

The word 'tangent' is used since the height will be a tangent in a circle with centre in *A*, and with the base as its radius.

Increasing the angle will increase the tangent-number. A calculator has a tangent-button to show the relation between the two: tan 30 = 0.577, and reversely, tan-1(2/3) = 33.7.

Tangent gives a circumference formula since a circle contains many right triangles:

In an *h*-by-*r* half-box, *h* recounts in *r* as *h* = (*h/r*) x *r* = tan *A* x *r*.

A half circle is 180 degrees that split in 100 small parts as

180 = (180/100) x 100 = 1.8 100s = 100 1.8s.

With *A* as 1.8 degrees, the circle and the tangent, *h*, are almost identical.

So, half the circumference is

½*C* = 100 x *h* = 100 x tan 1.8 x *r* = 100 x tan (180/100) x *r* = 3.1426 x *r*

Calling the circumference for 2 x π x *r*, we get a formula for the number π:

π = tan (180/*n*) x *n*, for *n* sufficiently large.

Chapter 07. Once Counted and Recounted, Total can Add Next-to or On-top

Before designing, we reflect on how to add stacks horizontally and vertically.

Inside the 'no-unit' greenhouse, numbers are seen as placed along a one-dimensional number line with addition defined as counting on. Outside, numbers carry units and become 2-dimensional stacks with areas that add next-to or on-top.

Adding 2 3s and 4 5s next-to as 8s means adding or integrating the areas, also called integral calculus. Adding 2 3s and 4 5s on-top, the units must be the same by squeezing one or both stacks, i.e., by recounting one or both.

Again, the recount formula predicts the result on a calculator by entering

(2 x 3 + 4 x 5)/*B*, where *B* can be 3 or 5 or 8.

We see that when adding stacks, multiplication comes before addition.

Adding 20% to 30$, we have two units with the per-number 30$ per 100%. Adding 20% to 100% gives 120% that recounts in 100s as

120% = (120/100) x 100% = (120/100) x 30 $ = 120% x 30$.

So, we add 20% by multiplying with 120%, also called the index-number 120.

Reversing next-to and on-top addition, we may ask "2 3s and how many 5s total 4 8s?", leading to the equation 2 x 3 + *u* x 5 = 4 x 8

To find the answer, we pull away the 2 3s from the total *T* before recounting in 5s,

$u = (T – 2 \text{ x } 3) / 5 = \Delta T / 5$

The answer is predicted by asking a calculator: (4 x 8 – 2 x 3) / 5.

Here we used a difference to calculate the change in the total before recounting. The combination of subtraction and division is called differential calculus. That subtraction comes before division is natural since differentiating stacks is the opposite of integrating them where multiplication comes before addition.

Chapter 08. Subtracting and Adding Numbers with like Units

Before designing, we reflect on how to subtract and add numbers with like units.

Inside the 'no-unit' greenhouse, 1digit numbers are drawn serial next-to to find the result by counting on from 6 or 9. Outside, they are drawn parallel on-top showing that

T = 6 + 9 = 2*B*3 6s = 2*B*-3 9s, that both recounts as 1*B*5 tens

2 6s + 3 = 2 x 6 + 3 = 2 x (*B*-4) + 3 = 2*B*-8 + 3 = 12 + 3 = 15, and

2 9s – 3 = 2 x 9 – 3 = 2 x (*B*-1) – 3 = 2*B*-2 – 3 = 18 – 3 = 15

Or, directly with flexible bundle-numbers, *T* = 6 + 9 = *B*-4 + *B*-1 = 2*B*-5 = 15

However, subtraction before addition will allow using the hands to show the result after predicting it with a calculator.

Observing with fingers or snap-cubes that 5 – 3 = 2, we also observe that 5 = 2 + 3, so that 5 = (5 – 3) + 3. With unspecified numbers this gives a 'restack formula' *T* = (*T*–*B*) + *B* saying "From a total *T*, *T*–*B* is left when pulling away *B* to place next-to".

This formula solves addition-equations. Asking "3 added to what gives" 5 may be formulated as an equation, *u* + 3 = 5. Taking away from 5 the 3 we added will give the answer, *u* = 5 – 3, which can also be seen when restacking 5:

u + 3 = 5 = (5 – 3) + 3, gives *u* = 5 – 3.

Inside the ‘no-unit’ greenhouse, a combination of 3 digits is seen as one number obeying a place value principle where the places describe the number of ones, tens, hundreds etc. Seldom, if ever, ten is called ‘bundle’, and hundred is called ‘bundle-of-bundles’.

When adding, numbers are placed in columns, and carrying or borrowing may occur.

Outside, a combination of 3 digits is seen as 3 numberings of unbundled singles, bundles and bundle-of bundles as shown when including the units in the number:

T = 456 = 4*BB*5*B*6, or more formally as a polynomial:

T = 456 = 4 x *B*^2 + 5 x *B* + 6, with *B* = 10

The flexibility of bundle-numbers is handy with subtraction may give an underload:

T = 41 – 18 = 4*B*1 – 1*B*8 = 3*B*-7 = 2*B*3

Overloads may prove handy also:

T = 18 + 23 = 1*B*8 + 2*B*3 = 3*B*11 = 4*B*1, and

T = 336 / 7 = 33*B*6 / 7 = 28*B*56 / 7 = 4*B*8 = 48

T = 2 x 37 = 2 x 3*B*7 = 6*B*14 = 7*B*4

T = 34 x 56 = 3*B*4 x 5*B*6 = 15*BB* (18+20)*B* 24

= 15*BB* 38*B* 24 = 15*BB* 40*B* 4 =19*BB* 0*B* 4 =1904

Or, with stacks that may be extended to any multidigit numbers

	3*B*	4	**34 x 56**
	15*BB*	20*B*	5*B*
	18*B*	24	6
15*BB*	38*B*	24	
15*BB*	40*B*	4	
19*BB*	0*B*	4	
			1904

Figure 3. Multiplying Numbers as Binomials

So, outside flexible bundle-numbers do not need inside carrying or place values.

Chapter 09. Adding Per-Numbers and Fractions

Before designing, we reflect on how to add per-numbers and fractions.

Inside the ‘no-unit’ greenhouse, the concept ‘per-number’ is not accepted; and fractions are added without units, e.g., 1/2 + 2/3 is 7/6.

Outside, this hold only if the fractions are taken of the same total. It does not hold if the units are the same since 1/2 of a pie plus 2/3 of a pie total 7/12 of two pies; and since 1/2 of 2 apples plus 2/3 of 3 apples total 1+2 of 2+3 apples, i.e., 3/5 of the 5 apples, and not 7 apples of 6 as the ‘no-unit’ greenhouse teaches.

Outside, fractions are seen as per-numbers coming from recounting in the same unit. And when adding per-numbers we need to be careful with the units.

Asking “What is 2kg at 3$/kg plus 4kg at 5$/kg?” we observe that the unit-numbers 2 and 4 add directly to 6, whereas the per-numbers must be multiplied to unit-numbers before adding. And since multiplication creates areas, per-numbers add by their areas under the per-number-graph, i.e., as integral calculus.

And again, the opposite of integral calculus is called differential calculus asking “2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg?” As before, we subtract the $-number we added before recounting the rest in kg, p = (6 x 5 – 2 x 3) / 4 = Δ$ / Δkg.

Chapter 10. Change by Adding or by Multiplying

Before designing, we reflect on how to change a number by adding or by multiplying.

Inside the ‘no-unit’ greenhouse, change by adding or multiplying is called arithmetic and geometric sequences that are added as series.

Outside, we see that adding 3$/day to 2$ gives a total of T = 2 + 3 x n after n days. The general formula, $T = b + a$ x n, is called change by adding, or linear change.

Also, we see that adding 3%/day to 2$ gives a total of T = 2 x 103%^n after n days since adding 3% means multiplying with 103%. The general formula, $T = b$ x a^n = b x (1+r)^n, is called change by multiplying or exponential change.

Combining the two gives a simple formula for saving, $A/a = R/r$. Here a and r is the per-day input, and A and R is the final output, where 1+R = (1+r)^n.

Reversing change by adding gives an equation as 100 = 20 + 5 x u, easily solved by restacking and recounting:

100 = (100-20) + 20, so u x 5 = 100 – 20 = 80 = (80/5) x 5, so u = 80/5 = 16.

Reversing change by multiplying gives two equations. In the equation 20 = u^5, we need a factor u of which 5 gives 20, predicted by the factor-finding root $\sqrt[5]{20}$ = 1.82. In 20 = 5^u we need the number u of 5-factors that gives 20, predicted by the factor-counting logarithm $\log_5(20)$ = 1.86.

We now know all the ways to unite parts into a total, and to split a total in parts, the ‘Algebra-square’ (Tarp, 2018):

Operations **unite**/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a \times n}$ $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int a \times dn}$ $\frac{dT}{dn} = a$	$\boldsymbol{T = a^{\wedge}n}$ $\sqrt[n]{T} = a \qquad n = \log_a T$

Figure 4. The Ways to Unite Parts into a Total, and to Split a Total into Parts.

Chapter 11. Conclusion, Finally a Communicative Turn in Mathematics Education

It goes without saying that a total must be counted before being added. Where small totals may be glanced directly in space, larger totals need to be counted in time by pushing away 1s, i.e., by dividing by 1. Here the recount formula gives, e.g., 8 = (8/1) x 1 = 8 x 1, which shows that the total’s space-number is the same as its counted time-number.

But when counting in 1s, we never meet the bundles-of-bundles since the square, 1 1s, is still 1, whereas the square, 2 2s, is 4. And we may not see that in reality we bundle in tens, and ten-tens, etc., since we may never push away tens, the real unit in our number system as expressed by the full number-formula, the polynomial,

T = 345 = 3 Bundle-bundles + 4 Bundles + 5 = $3B^2+4B+5$.

Whereas, if counting in 3s and using the name '1 Bundle 0', for 3, we meet 9 as the bundle-of-bundles square, which may inspire us to use the same name for hundred when counting in tens.

So, we need to count in at least 2s to see the nature of outside bundle-counting. 1 is not a prime unit, as the other prime units that cannot split into new prime units.

The original question may now be answered in the following way: Children count in bundles, which gives the recount formula, $T = (T/B)xB$, as a direct key to core mathematics.

By changing units, it creates per-numbers that add by integrating their areas as in calculus, and that opens up to proportionality and linearity, and thus to countless STEM formulas, and to solving their equations. It makes trigonometry precede plane and coordinate geometry. It shows that per-numbers and fractions are not numbers, but operators needing a number to become a number. It shows that counting before adding makes the two basic counting operations, push and stack, division and multiplication, precede subtraction and addition that is ambiguous with its two options, next-to and on-top. It shows our four basic operations as icons for the outside counting actions: push, lift, pull and unite. It shows that negative numbers, decimals and fractions are different ways to count the unbundled. But, most importantly, it shows the power of formulas as predicting number-language sentences making us master many in nature and in society, and in time and space.

So, building on the mastery of Many developed when adapting to many before school, a 'counting before adding' curriculum allows children to outside master the same mathematics as is taught with great difficulties inside the 'no-units' paradigm's greenhouse seeing its very foundation, 1+1=2, fold when meeting units. So once tested, flexible bundle-numbers with units may also fold the myth "math is hard, and needs more funding." Meaning that we can finally have a communicative turn in number-language education as the foreign language education had in the 1970's (Widdowson, 1978).

Feynman famously asked: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words?"

Certainly, the recount formula is a candidate. So why not enlighten humans about it instead of forcing them inside the darkness of a greenhouse without units.

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CTRAS 2022 online

From STEAM to STEEM to include also economics

Le Thai Bao Thien Trung, Ho Chi Minh City University of Education; Allan Tarp, MATHeCADEMY.net

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of physical quantities. Statistics post-dicts unpredictable quantities by setting up probabilities for future behavior, using factual or fictitious numbers as median and fractals or average and deviation. Including economics so STEM becomes STEEM opens the door to statistics also. Art may be an appetizer, but not a main course since geometry and algebra should be always together and never apart to play a core role in STEEM.

Macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheet for simulations using different parameters.

Microeconomics describes equilibriums in the individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable cost, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determines the equilibrium price. Market surveys leads to statistics, as does insurance. In the households, spending comes from balancing income and transferals with saving and tax. In a bank, income come from simple and compound interest, from installment plans as well as risk taking. At a stock market, courses fluctuate. Governments must consider quadratic Laffer-curves describing a negative return of a tax-raise. To avoid units, factories use variations of Cobb-Douglas power elasticity production functions for modeling.

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CTRAS 2023 online

AI and Difference Research in Math Education,
https://youtu.be/4EPqjz8evd4.

Online math opens for a communicative turn in number language education.
https://youtu.be/36tan-gGjJg.

CTRAS 2024 online

Proposal: From a colonized to a decolonized mathematics, from 8 to 2 competences, from non-unit to unit-numbers

Keywords: Decolonization, mathematics, competence, curriculum, numeracy.

Asking a 3year-old child "How many years next time?" shows a need to decolonize mathematics. The child reacts to 4 fingers held together 2 by 2 "That is not 4, that is 2 twos". Adults only see inside essence, four, but the child sees outside existence, bundles of 2s in space as units when counted in time. Comparing the claims, "1+2 = 3" and "3x4 = 12", we see that without units, outside examples as "1week+2days = 9days" falsify the first claim. The second claim includes the unit by predicting that 3x4 as 3 4s outside may be re-counted as 1.2 tens. So, multiplication makes mathematics a natural science that becomes decolonized once the colonization by non-unit numbers has ended.

To decolonize a colonized non-unit mathematics with 8 competences, only 2 competences are needed, count and add: counting and re-counting bring outside totals inside to be added or split depending on how they occur, as like or unlike unit or per-numbers. A counting sequence in 3s always include the units: unbundled, bundles, and bundle-of-bundles that become squares on a Bundle-Bundle Pegboard, a BBBoard, where unit-numbers are tiles. Counting before adding changes both the order and the identity of the operations. Power is in bundle-bundles, division is a broom pushing away bundles, multiplication a lift stacking them, subtraction a rope pulling away the stack to find unbundled that placed on-top of the stack becomes decimals, fractions, or negatives, e.g., 9 = 4B1 = 4 ½ = 5B-1 2s. And addition shows the two ways to add stacks, next-to by areas as integral calculus, or on-top after the units are made like by the linearity of a recount-formula showing that when re-counting 8 in 2s, 8 = 4 x 2 = (8/2) x 2, or T = (T/B) x B with T and B for the total and the bundle. Recounting ten-bundles in digit-bundles creates equations solved by moving to opposite side with opposite sign: ux2 = 12 = (12/2) x 2, so u = 12/2. Recounting digits in tens gives the tables and early algebra on a BBBoard: 6x7 = (B-4) x (B-3) = 10B-4B-3B+4x3 (taken away twice) = 3B12 = 4B2 = 42. Recounting 6 4s as a bundle-bundle creates square roots and solves quadratics. Recounting $ in kg creates a per-number, e.g., 4$ per 5kg that bridges the units, becoming fractions with like units, and that multiplied to $ add as areas, i.e., as integral calculus. Probably all have learned this core math after primary school. Secondary school then is for the communicative turn using the number-language to write literature in its three genres, fact, fiction, and fake, as in the word-language.

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24. The 8th ICMI-East Asia Conference on Mathematics Education 2018

Theme of the Conference

“Flexibility in Mathematics Education” has been chosen as the theme of the conference. Flexibility is highly related to creativity, multiplicity, and adaptation. In the current era, rapid changes in economy, environment and society have been facilitated by the rapid development of technology and engineering. Flexibility in mathematical thinking, problem solving, teaching methods, evaluation, teacher education and mathematics education research is a key to empowering learners, teachers, educators and researchers to tackle the complexity and uncertainty, and to giving them the capacity and motive to change in the innovative era.

The Topic Study Group themes are:

TSG 1: Flexibility in Mathematics Curriculum and Materials

TSG 2: Flexibility in Mathematics Classroom Practices

TSG 3: Flexibility in Mathematics Assessment

TSG 4: Flexibility in Mathematics Teacher Education and Development

TSG 5: Flexibility in the Use of ICT in Mathematics

TSG 6: Flexibility in the Use of Language and Discourse in Mathematics

TSG 7: Flexibility in Mathematics Learning

Sixteen Proposals:

The Simplicity of Mathematics Revealing a Core Curriculum (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask ‘How many in Total?’. To answer, we count and add. First we take away bundles, thus rooting division; then we stack the bundles, thus rooting multiplication; then we move the stack away to look for singles, thus rooting subtraction; finally we answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

A calculator predicts the result by the recount-formula T = (T/B)*B saying ‘from T, T/B times, B can be taken away’, thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Recounting in another unit roots proportionality. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or totals add next-to, thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, again rooting proportionality, where per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, again rooting integration.

Now in a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Looking at constant and variable predictable change roots pre-calculus and calculus; and looking at unpredictable change roots statistics to post-dict the behavior of numbers by a mean and a deviation, again allowing probability to predict, not numbers but intervals. (Tarp, 2017)

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A STEM-based Math Core-Curriculum for migrants (TSG 01)

Seeing 'Mastery of Many' as the outside goal, we can construct a core math curriculum based upon exemplary situations of Many in a STEM context, having a positive effect on learners with a non-standard background (Han et al, 2014), thus allowing young male migrants to help their original countries as pre-teachers or pre-engineers.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells, and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is cup-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or variable. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, a core STEM-based curriculum could be about formulas controlling cycling water cycles (Tarp, 2017).

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50 years of Sterile Mathematics Education Research, Why? (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

PISA scores still are low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation’s need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

Academic articles can be written at a master-level exemplifying existing theories, or at a research-level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying ‘teach as little and as much as possible’.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So to answer the question ‘How to improve mathematics education research’, first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger (1962), asking ‘what is ‘is’?’

It turns out that, instead of mathematics, schools teach ‘meta-matism’ combining ‘meta-matics’, defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and ‘mathe-matism’ true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching 1/2 + 2/3 = 7/6.

So, instead of meta-matism, teach mathematics as ‘many-math’, a natural science about Many, in self-chosen half-year blocks.

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The Center of Math Education: Its Sentences or its Subjects? (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences. However, a controversy exists as to what is the center of mathematics education, the predicate 2*3 or the subject T.

Seeing reproducing textbook knowledge as the goal, Vygotsky points to good teaching as the best means and recommends teaching as much as possible. Seeing individual sentences about the outside fact Many as the goal, Piaget points to good guidance as the best means and recommends teaching as little as possible.

Thus, where a Vygotsky class follows a textbook strictly, a Piaget class brings the subject of its sentences to the class to allow the learner to create individual sentences to be adapted through sharing, thus respecting Many as the outside goal of mathematics. Which resonates with Heidegger (1962) saying: In a sentence, the subject exists, but the rest might be gossip.

Flexibility in a primary classroom thus means using full sentences where the total exists as sticks and where the predicate can be flexible by using cup-counting to count inside bundles and outside singles, e.g. T = IIIIIII = III III I = 2]1 3s or T = 1]4 3s = 3]-2 3s if allowing overloads and underloads outside the cup; which becomes useful when multiplying, T = 5* 67 = 5* 6]7 = 30]35 = 33]5 = 335; and when dividing: T = 335 /5 = 33]5 /5 = 30]35 /5 = 6]7 = 67.

The 'geometry and algebra, always together, never apart' principle allows learners to develop a flexible double number-concept, seeing the total T = 2*3 geometrically as number-block with 2 3s, that may or may not be recounted as 6 1s. Recounting 4*5 = 2 tens says that doubling its width, a block of 4 5s must halve its height to keep the total unchanged. Likewise, equations become tangible when recounting from tens to icons.

That totals must be counted and recounted before they add allows multiplication to precede addition.

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DoubleCounting roots Proportionality - and Fractions and Percentages as Per-Numbers (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

A school sees fractions as goals instead of means making a class stuck. Having heard about difference-research and per-numbers (Tarp, 2017), the teacher says: Time out. Next week, no fractions. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$.

To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

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Assessing Goals Instead of Means (TSG 03)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and compare totals; and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

Counting includes cup-counting to separate a total in bundles inside the cup and singles outside; and recounting in the same unit to create an outside overload or underload needed to ease operations, e.g. T = 4*56 = 4* 5]6 = 20]24 = 22]4 = 224.

Recounting in another unit, called proportionality, is predicted by a recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away', thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or next-to thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, becoming fractions if the units are the same. Since per-numbers and fractions are operators needing a number to become a number, they add by their areas, again rooting integration.

In a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points.

To avoid a goal displacement, assessment should test goals instead of means; and always use totals with units. With proportionality formulas in science as a core root for mathematics, several tasks should include per-numbers, e.g. taken from classical word problems. Numbers without units should be excluded, since adding numbers and fractions without units are examples of 'mathematism' true inside but seldom outside classrooms, where claims as 2+3 = 5 meet counterexamples as 2 weeks + 3 days = 1 7days. And where 1 red of 2 apples + 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as taught in school.

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The 2 Core Math Competences, Count & Add, in an e-learning Teacher Development (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3. Counting and double-counting in two units creates 4 number-types: variable and constant unit- and per-numbers that unite by addition, multiplication, integration and power.

That this simplicity typically is unknown to teachers created the MATHeCADEMY.net, teaching teachers to teach mathematics as 'ManyMath', a natural science about Many suing the CATS-approach: Count & Add in Time & Space. It is a virus academy saying: To learn mathematics, don't ask the instructor, ask Many. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How to count possibilities? How to predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? Quantitative Literature, what is that? Does it also have the 3 genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn. The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair.

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12 Theses not Taught in Teacher Education (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, mastering Many is the outside goal. As means, we iconize and bundle by digits, operations and formulas, becoming goals if forgetting the real goal.

1. Digits are icons with as many sticks as they represent.

2. A total T can be ‘cup-counted’ in the normal way or with an overload or underload: T = 5 = 2]1 2s = 1]3 2s = 3]-1 2s.

3. ‘Cup-writing’ makes operations easy: T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T = 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, ‘? 7s = 3 tens’, or ‘u*7 = 30 = (30/7)x7’, the answer u = 30/7 is found by ‘move to opposite side with opposite sign’.

9. Adding totals is ambiguous: On-top using proportionality, or next-to using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = □T/3.

11. Double-counting in different units gives ‘per-numbers’ as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a number to become a number, thus adding by their areas as integration.

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Difference-Research at Work in a Classroom (TSG 04)

The CTRAS (Classroom Teaching Research for All Students) wants all students to benefit. The 2017 conference contained example of classroom lessons. Difference-research (Tarp, 2017) looks for a different approach based upon outside goals to see if more students benefit. Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?'

The first task in a grade 5 class was to fill a 3x3 square with the numbers 1-9 so that they add to 15 horizontally, vertically and on both diagonals. Based upon the principle 'algebra & geometry, always together, never apart', the outside goal could be to give symmetry to both, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as 'Bundle less or plus' using 5 as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4. By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to 'add to zero'. Thus, each sum must contain 2 odd numbers, placed in the corners.

The second task in a grade 8 class was to give a geometrical proof of the Pythagoras Theorem. Here an outside goal could be to add travel-distances. If parallel, two distances add or subtract directly. If perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps, since 3^2 + 4^2 = 5^2.

The third task in a grade 3 class was to learn about and apply fractions. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting, statistics, splitting, per-numbers, changing. Double-counting bent and unbent fingers roots fractions as 2/5 of 5 and 2/5 of 10.

The fourth task in a grade 8 class was to find a formula connecting the number of angles to the angle sum in a polygon. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus, the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled from A to E. On the triangle, inserting an extra point P between A and B transforms the triangle ABC into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

A plenary address discussed decimal multiplication in a grade 5 class exemplified by 110*2.54. Here a difference is to see multiplication as shifting units. Here a total of 110 2.54s is to be recounted in tens. Factorizing will show how the ten-units can change place: T = 110*2.54 = 1.1*10*10*2.54 = 1.1*254 = 279.4

References

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Pre-schoolers and Migrants Predict Recounting by a Calculator (TSG 05)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer we count and add. Asking T = 7 = ? 3s, first we take away bundles, thus rooting division iconized as a broom wiping away the bundles; then we stack the bundles, thus rooting multiplication iconized as a lift stacking the bundles; then we move the stack away to look for unbundled singles, thus rooting subtraction iconized as a trace left by the stack; and finally, we answer with a number-language sentence, containing a subject and a verb and a predicate as does word-language sentences.

To have the calculator predict the result we enter '7/3'. The answer 2.some tells us that 2 times 3s can be taken away. To look for unbundled singles we stack the 2 3s as 2*3 to be removed, so we enter '7 – 2*3'. The answer 1 tells us that 7 can be counted in 3s as 2 3s and 1, written as T = 7 = 2 1/3 3s if the single is placed on-top of the stack counted in 3s, or as T = 7 = 2.1 3s if the single is placed next-to the stack as a stack of unbundled.

This shows that a natural number is decimal number with a unit where the decimal point separates the bundles from the unbundled.

A calculator thus predicts the result by the recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away'.

Recounting in tens means just multiplying. Recounting from tens to icons means asking 30 = ? 6s. Here we use the recount-formula to recount 30 in 6s, T = 30 = (30/6)*6 = 5*6. This shows, that an equation is solved by moving to the opposite side with opposite sign.

The totals 2 3s and 4 5s can add on-top as 3s or 5s, or next-to as 8s. Again, a calculator can predict the result: Entering (2*3+4*5)/8 gives 3.some and then (2*3+4*5) – 3*8 gives 2 so the prediction is T = 3.2 8s.

Also, the recount-formula can bridge units when double-counting has created a per-number as s4$/5kg. Here T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. Likewise with 18$.

References

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Mathematics as a Number-Language Grammar (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’ and ‘The total is 3x4 legs’, abbreviated to ‘T = 3x4’. Both are affected by the Heidegger (1962) warning: ‘In is-sentences, trust the subject but question the predicate’.

Both languages also have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3x4’ leads to a meta-sentence ‘’x’ is an operation’.

We master outside phenomena through actions, so learning a word-language means learning actions as how to listen and read and write and speak. Likewise, learning the number-language means learning actions as how to count and add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language. Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language.

Saying ‘the number-language is an application of mathematics’ implies that ‘of course mathematics must be taught and learned before it can be applied’. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

Instead school should follow the word-language and use full sentences ‘The total is 3 4s’ or ‘T = 3x4’. By saying ‘3x4’ only, school removes both the subject and the verb from number-language sentence, thus committing a goal displacement.

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Deconstructing the Vocabulary of Mathematics (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and 'The total is 3x4 legs', abbreviated to 'T = 3x4'.

Inspired by the Heidegger (1962) warning 'In is-sentences, trust the subject but question the predicate', Derrida (1991) to recommends deconstructing labels by destructing and reconstructing them inspired by the subject itself.

Thus Mathematics could be renamed to Many-matics, Many-math, Many-ology, or number-language. Geometry could be renamed to 'earth-measuring'; and algebra to 'reuniting numbers' according to its Arabic meaning.

Counting could split into its different forms: Cup-counting, using a cup for the bundles; re-counting to change the unit; and double-counting to bridge two units by a per-number.

In division, 'divided between 5' could be renamed to 'counted in 5s'; and 'to multiplied by 3' could be renamed 'to change the unit from 3s to tens' by reshaping the number block, widening the base and shorting the height.

Addition could split into on-top addition using proportionality to change the units, and next-to addition adding by areas as in integration.

Solving equations could be renamed to reversing calculations.

Fractions could be renamed to per-numbers coming from double-counting in the same unit.

Proportionality could be renamed changing units; and proportional could be renamed to 'the same except for units'.

Linear and exponential functions could be renamed change by adding and multiplying.

A function y = f(x) could be renamed to a formula or a 'number-language sentence'.

A root and a logarithm could be renamed to a factor-finder and a factor-counter.

Continuous could be renamed locally constant, and differentiable could be renamed locally linear

Integration could be renamed added by area.

In a right-angled triangle, the hypotenuse could be renamed the diagonal.

Finally, mathematical models could be named quantitative literature, having the same genres as qualitative literature, fact and fiction and fiddle (Tarp, 2017).

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Will Difference-Research Make a Difference? (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015). So maybe it is time for a different research approach, e.g. Difference-Research (Tarp, 2017).

Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?' The philosophical background is the Heidegger warning 'In is-sentences, trust the subject but question the rest since it might be gossip.'

Looking for outside goals to inside means presented as goals, we see:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: cup-counting in icon-bundles less than ten, recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2]1 2s = 1]3 2s = 3]-1 2s.

3. The tradition uses carrying. A difference is to use cup-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4]8 = 28]56 = 33]6 = 336. Likewise with division: T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

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Calculus in Primary and Middle and High School (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3 = 2 3s, seeing that natural numbers are block-numbers with units.

Once counted, totals can be added, but addition is not well-defined: Two totals T1 = 2 3s and T2 = 4 5s may add on-top or next-to as 8s: T1 + T2 = 2 3s + 4 5s = 3.2 8s. Thus next-to addition means adding areas by combining multiplication and addition, called integration.

Reversing next-to addition, we ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation or reversed integration.

'Double-counting' a total in two physical units creates 'per-numbers' as 4$/5kg, or fractions as 4$/5$ = 4/5 if the units are the same. Per-numbers and fractions are not numbers, but operators needing a number to become a number: Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other.

In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. This means adding many strips under a per-number graph, made easy by writing the strips as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus, and that integration should precede differentiation.

The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means that the difference can be made arbitrarily small. (Tarp, 2013)

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Curing Math Dislike With 1 Cup and 5 Sticks (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Meeting Many, we ask ‘How many in Total?’. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences, both affected by the Heidegger (1962) warning: ‘In is-sentences, trust the subject but question the predicate’. However, by neglecting the subject and presenting the predicate as the goal, the tradition creates widespread dislike in math classes especially with division. To get the class back on track, the total must be reintroduced physically and in the sentence.

A class is stuck in division and gives up on 237/5. Having heard about ‘1cup & 5 sticks’, the teacher says ‘Time out. Next week, no division. Instead we do cup-counting’. Teacher: ‘How many sticks?’ Class: ‘5.’ Teacher: ‘Correct, and how many 2s?’ Class: ‘2 2s and 1 left over’. Teacher: ‘Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?’ Class: ‘1inside-3outside; and 3inside-less1outside.’ Teacher: ‘Correct. A total can be counted in 3 ways. The normal way with 2inside-1outside. With overload as 1inside-3outside. With underload as 3inside-less1outside.’ Class: ‘OK’. Teacher. ‘Now 37 means 3inside-7outside if we count in tens. Try recounting 37 with overload and underload. Class: ‘2inside-17outside; and 4inside-less3outside.’

Teacher: ‘Now let us multiply 37 by 2, how much inside and outside?’ Class: 6inside-14outside. Or 7inside-4outside. Or 8inside-less6outside.’

Teacher: ‘Now to divide 78 by 3 we recount 7inside-8outside to 6inside-18outside. Dividing by 3 we get 2inside-6outside or 26. With 79 we get 1 leftover that still must be divided by 3. So 79/3 gives 28 and 1/3.’

Class: ‘So to divide 235 by 5 we recount 235 as 20 inside and 35 outside. Dividing by 5 we get 4 inside and 7 outside, or 47; With 237 we get 2 leftovers that still must be divided by 5. Thus 237/5 gives 47 and 2/5?’

Teacher: ‘Precisely. Now let us go back to multiplication and division and use cup-counting’.

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Quantitative Literature Also has 3 Genres: Fact and Fiction and Fiddle (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Humans communicate in languages: A word-language with sentences assigning words to things and actions; and a number-language with formulas assigning numbers or calculations to things and actions. ‘Word stories’ come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like ‘This sentence is false’. ‘Number stories’ are often called mathematical models. They come in the same three genres.

Fact models can be called a ‘since-then’ models or ‘room’ models. Fact models quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. The model’s prediction is what is observed, so fact models can be trusted when units are checked. Algebra’s four basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy.

Fiction models can be called ‘if-then’ models or ‘rate’ models. Fiction models quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves.

Fiddle models can be called ‘then-what’ models or ‘risk’ models. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

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Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (p. 3).

This may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference.

So, inspired by sociology we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?'

In short, could there be different kinds of mathematics? And could it be that among them, one is good, and one is bad, and one is evil? In other words, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17 days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Good and Bad and Evil Mathematics

The existence of three different versions of mathematics, many-matics and meta-matics and mathe-matism, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world. An example is T = 2*3 = 6 stating that a total of 2 3s can be re-counted as 6 1s. So good mathematics is tales about totals, and how to count and unite them.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming that 2+3 = 5, only valid if the units are the same, else meeting contradictions as 2weeks + 3days = 17days. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as claiming that 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3apples total 3reds of 5 apples and not 7reds of 6apples. So bad mathematics is tales about fractions as numbers.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a ‘Count-before-Adding’ Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to use as units when counting:

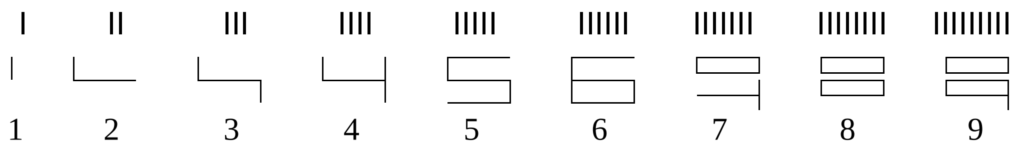

Figure 1. Digits as icons containing as many sticks as they represent

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total *T* we take away bundles *B* thus rooting and iconizing division as a broom wiping away the bundles. Stacking the bundles roots and iconizes multiplication as a lift stacking the bundles into a block. Moving the stack away to look for unbundled singles roots and iconizes subtraction as a trace left when dragging the block away. A calculator predicts the counting result by a 're-count formula' $T = (T/B)*B$ saying that 'from *T*, *T/B* times, *B* can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so *T* = 7 = 2*B*1 3s.

Placing the unbundled singles next-to or on-top of the stack of 3s roots decimals and fractions:

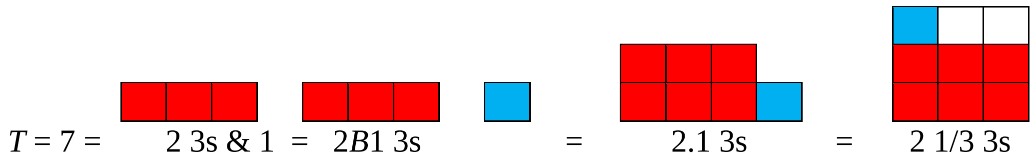

Figure 2. Re-counting a total of 7 in 3s, the unbundled single can be placed in three different ways

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, *T* = 42 = ? 7s = *u**7, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations, and differential calculus:

T = 2 3s + ? 4s = 5 7s gives differentiation: ? = (5*7 – 2*3)/4 = Δ*T*/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers *sine*, *cosine* and *tangent*. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the *y*-change in the *x*-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A Short Version of a Curriculum in Good Mathematics, Grounded Many-matics

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard, and with over- and underload): *T* = 2*B*1 5s = 1*B*3 5s = 3*B*-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use overload counting after 5: *T* = 4 7s = 4*7 = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 should be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, T = 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and T = 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, so T = 60$ = (60/3)*3$ gives (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sin$A$*c.

Good and Bad Mathematics

Today's tradition begins with arithmetic telling about line-numbers, processed by four basic operations, later extended with negative numbers and rational numbers and reel numbers. Algebra then repeats it all with letters instead. Geometry begins with plane geometry followed by coordinate geometry and trigonometry later. Functions are special set-products, and differential calculus precedes integral calculus.

In general, we see mathematics as truths about well-defined concepts. So we begin by discussing what can be meant by good and bad concepts.

Good and Bad Concepts

As an example, let us look at a core concept in mathematics, a calculation. To differentiate between y = 2*3 and y = 2*x, around 1750 Euler defined the concept 'function' as a calculation containing unspecified numbers. Later, around 1900, set-based mathematics defined a function as an example of a set-product where first component identity implies second component identity.

So where the former is a bottom-up definition of a concept as an abstraction from examples, the latter is a top-down definition of a concept as an example of an abstraction.

Since examples are in the world and since Russell warned that by its self-reference the set-concept is meaningless, we can label bottom-up and top-down definitions good and bad concepts respectively.

Good and Bad Numbers

Good numbers should reflect that our number-language describes a total as counted in bundles and expressing the result in a full sentence with subject and verb and predicate as in the word-language, as e.g. T = 2 3s. These are the numbers that children bring to school, two-dimensional block-numbers that contain three different number-types: a 'unit-number' for the size, a 'bundle-number' and a 'single-number' for the number of bundles and unbundled singles. Totals then are written in bundle- form or in decimal-form with a unit where a bundle-B or a decimal point separates the inside bundles from the outside singles, as e.g. T = 3B2 tens = 3.2 tens.

Good numbers are flexible to allow a total to be re-counted in a different unit; or in the same unit to create an overload or underload to make calculations easier, as e.g. T = 3*B*2 tens = 2*B*12 tens = 4*B*-8 tens. Good numbers are shown in two ways: an algebraic with bundles, and a geometrical with blocks. Good numbers also tell that eleven and twelve come from the Vikings saying 'one left' and 'two left'.

Bad numbers do not respect the children's own two-dimensional block-numbers by insisting on one-dimensional line-numbers be introduced as names along a line without practicing bundling. Numbers follow a place value system with different places for the ones, tens, hundreds, and thousands; but seldom renaming them as bundles, bundle of bundles, and bundles of bundles of bundles.

Good and Bad Counting

A good counting sequence includes bundles in the names, as e.g. 01, 02, …, Bundle, 1*B*1, etc.; or 0Bundle1, 0*B*2, etc. Another sequence respects the nearness of a bundle by saying 0*B*6, 1*B*less3, 1*B*-2, etc.

Good counting lets counting and re-counting and double-counting precede addition; and allows the re-count formula to predict the counting-result; and it presents the symbols for division, multiplication and subtraction as icons coming from the counting process, thus introducing the operations in the opposite order.

Bad counting neglects the different forms of counting by going directly to adding, thus not respecting that totals must be counted before they can be added.

Bad counting treats numbers as names thus hiding their bundle nature by a place value system. This leads some to count 'twenty-ten' instead of 'thirty', and to confuse 23 and 32.

Good and Bad Addition

Good addition waits until after totals have been counted and re-counted in the same and in a different unit, to and from tens, and double-counted in two units to create per-numbers bridging the units. Likewise, good addition respects its two forms: on-top rooting proportionality since changing the units might be need; and next-to rooting integral calculus by being added by the areas.

Bad addition claims it priority as the fundamental operation defining the others: multiplication as repeated addition, and subtraction and division as reversed addition and multiplication. It insists on being the first operation being taught. Numbers must be counted in tens. Therefore there is no need to change or mention the unit; nor is there a need to add next-to as twenties.

Bad addition does not respect that in block-numbers as T = 2*B*3 4s, the three digits add differently. Unit-numbers, as 4, only add if adding next-to. Bundle-numbers, as 2, only add if the units are the same; else re-counting must make them so. Single-numbers, as 3, always add, but might be re-counted because of an overload.

Good and Bad Subtraction

Good subtraction sees its sign as iconizing the trace left when dragging away a stack to look for unbundled singles, thus leading on to division as repeated subtraction moving bundles away. It does not mind taking too much away and leaving an underload, as in 3*B*2 - 1*B*5 = 2*B*-3.

Bad subtraction sees its sign as a mere symbol; and sees itself as reversed addition; and doesn't mind subtracting numbers without units.

Good and Bad Multiplication

Good multiplication sees its sign as iconizing a lift stacking bundles. It sees 5*7 as a block of 5 7s that may or may not be re-counted in tens as 3.5 tens or 35; and that has the width 7 and the height 5 that, if recounted in tens, must widen it width and consequently shorten its height. Thus, it always sees the last factor as the unit.

Good multiplication uses flexible numbers when re-counting in tens by multiplying, as e.g. *T* = 6*8 = 6*(ten-2) = (ten-4)*8 = (ten-4)*(ten-2). This allows reducing the ten by ten multiplication table to a five by five table.

Bad multiplication sees its sign as a mere symbol; and insists that all blocks must be re-counted in tens by saying that 5*7 IS 35. It insists that multiplication tables must be learned by heart.

Good and Bad Division

Good division sees its sign as iconizing a broom wiping away the 2s in *T* = 8/2. It sees 8/2 as 8 counted in 2s; and it finds it natural to be the first operation since when counting, bundling by division comes before stacking by multiplication and removing stacks by subtraction to look for unbundled singles.

Bad division sees its sign as a mere symbol; and teaches that 8/2 means 8 split between 2 instead of 8 counted in 2s. Bad division accepts to be last by saying that division is reversed multiplication; and insists that fractions cannot be introduced until after division.

Good and Bad Calculations

Good calculations use the re-count formula to allow a calculator to predict counting-results.

Bad calculations insist on using carrying so that the result comes out without overloads or underloads.

Good and Bad Proportionality

Good proportionality is introduced in grade 1 as re-counting in another unit predicted by the re-count formula. It is re-introduced when adding blocks on-top; and when double-counting in two units to create a per-number bridging the units by becoming a proportionality factor.

Bad proportionality is introduced in secondary school as an example of multiplicative thinking or of a linear function.

Good and Bad Equations

Good equations see equations as reversed calculations applying the opposite operations on the opposite side thus using the 'opposite side and sign' method in accordance with the definitions of opposite operations: 8-3 is the number *x* that added to 3 gives 8; thus if *x*+3 = 8 then *x* = 8-3. Likewise with the other operations.

Good equations sees equations as rooted in re-counting from tens to icons, as e.g. 40 = ? 8s, leading to an equation solved by re-counting 40 in 8s: *x**8 = 40 = (40/8)*8, thus *x* = 40/8 = 5.

Bad equations insist that the group definition of abstract algebra be used fully or partwise when solving an equation. It thus sees an equation as an open statement expressing identity between two number-names. The statements are transformed by identical operations aiming at neutralizing the numbers next to the unknown by applying commutative and associative laws.

2*x* = 8	an open statement about the identity of two number-names
(2*x*)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(*x**2)*(1/2) = 4	since multiplication is commutative
*x**(2*(1/2)) = 4	since multiplication is associative

x*1 = 4	by definition of an inverse element
x = 4	by definition of a neutral element

Figure 3. Solving an equation using the formal group definition from abstract algebra

Good and Bad Pre-calculus

Good pre-calculus shows that the number-formula, *T* = 345 = 3**BB* + 4**B* + 5*1 = 3**x*^2 + 4**x* + 5, has as special cases the formulas for constant linear, exponential, elastic, or accelerated change: *T = b*x+c*, *T = a*n^x*, *T = a*x^n*, and *T = a*x^2 + b*x + c*. It uses 'parallel wording' by calling root and logarithm 'factor-finder' and 'factor-counter' also. It introduces integral calculus with blending problems adding piecewise constant per-numbers, as e.g. 2kg at 3 $/kg plus 4kg at 5$/kg. It includes modeling examples from STEM areas (Science, Technology, Engineering, Mathematics)

Bad pre-calculus introduces linear and exponential functions as examples of a homomorphism satisfying the condition f(*x#y*) = f(*x*)$f(*y*). It includes modeling from classical word problems only.

Good and Bad Calculus

Good calculus begins with primary school calculus, adding two blocks next-to each other. It also includes middle school calculus adding piecewise constant per-numbers, to be carried on as high school calculus adding locally constant per-numbers.

It motivates the epsilon-delta definition of constancy as a way to formalize the three forms of constancy: global, piecewise and locally. It shows series with single changes and total changes calculated to realize that many single changes sum up as one single change, calculated as the difference between the end- and start-values since all the middle terms disappear.

This motivates the introduction of differential calculus as the ability to rewrite a block *h*dx* as a difference *dy, dy/dx = h;* and where the changes of block with sides f and g leads on to the fundamental formula of differential calculus, (f*g)'/(f*g)= f'/f + g'/g, giving (x^n)'/x^n = n*1/x, or (x^n)' = n*x^(n-1). Bad calculus introduces differential calculus before integral calculus that is defined as anti-differentiation where the area under *h* is a primitive to *h*; and it introduces the epsilon-delta criterion without grounding it in different kinds of constancy.

Good and Bad Modeling

Good modeling is quantitative literature or number-stories coming in three genres as in word stories: Fact, fiction and fiddle. Fact and fiction are stories about factual and fictional things and actions. Fiddle is nonsense like 'This sentence is false' that is true if false, and vice versa.

Fact models, also called 'since-then' or 'room' models, quantify quantities and predict predictable quantities: "What is the area of the walls in this room?". Since the prediction is what is observed, fact models can be trusted. Fiction models, also called 'if-then' or 'rate' models, quantify quantities but predict unpredictable quantities: "My debt is gone in 5 years at this rate!". Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based on alternative assumptions. Fiddle models, also called 'then-what' or 'risk' models, quantify qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" Fiddle models should be rejected asking for a word description instead of a number description. (Tarp, 2017).

Bad modeling does not distinguish between the three genres but sees all models as approximations.

Good and Bad Geometry

Good geometry lets trigonometry precede plane geometry that is integrated with coordinate geometry to let algebra and geometry go hand in hand to allow formulas predict geometrical intersection points.

Bad geometry lets plane geometry precede coordinate geometry that precedes trigonometry.

Evil Mathematics

Evil mathematics talks about something existing only inside classrooms. Fractions as numbers and adding fractions without units are two examples. The tradition presents fractions as rational numbers, defined as equivalence classes in a set product created by the equivalence relation R, where (a,b) R (c,d) if $a*d = b*c$.

Grounded in double-counting in two units, fractions are per-numbers double-counted in the same unit, as e.g. 3$ per 5$ or 3 per 5 or 3/5. Both are operators needing a number to become a number. Both must be multiplied to unit-numbers before adding, i.e. they add by their areas as in integral calculus.

Shortening or enlarging fractions is not evil mathematics. They could be called 'footnote mathematics' since they deal with operator algebra seldom appearing outside classrooms. They deal with re-counting numbers by adding or removing common units: to shorten, 4/6 it is re-counted as 2 2s over 3 2s giving 2/3. To be enlarged, both take on the same unit so that 2/3 = 2 4s over 3 4s = 8/12.

Educating teachers, it is evil to silence the choices made in mathematics education. Instead, teachers should be informed about the available alternatives without hiding them in an orthodox tradition. Especially the difference between good and bad mathematics should be part of a teacher education.

Good and Bad Education

When children become teenagers, their identity work begins: 'Who am I; and what can I do?' So good education sees its goal as allowing teenagers to uncover end develop their personal talent through daily lessons in self-chosen practical or theoretical half-year blocks with teachers having only one subject; and praising the students for their talent or for their courage to try out something unknown.

Bad education sees its goal as selecting the best students for offices in the private or public sector. It uses fixed classes forcing teenagers to follow their age-group despite the biological fact that girls are two years ahead in mental development.

Good and Bad Research

Good research searches for truth about things that exist. It poses a question and chooses a methodology to transform reliable data into valid statements. Or it uses methodical skepticism to unmask choice masked as nature.

Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.

With these three research genres, peer-review only works inside the same genre.

Conclusion and Recommendation

This paper used difference-research to look for different ways to the outside goal of mathematics education, mastery of Many. By meeting Many outside the present self-referring set-based tradition three ways were found, a good, and a bad, and an evil. Good mathematics respects the original tasks in Algebra and Geometry, to reunite Many and to measure earth. By identifying a hidden alternative, good mathematics creates a paradigm shift (Kuhn, 1962) that opens up a vast field for new research seeing mathematics as a many-matics, i.e. as a natural science about Many (cf. Tarp, 2018).

In short, we need to examine what happens if we allow children to keep and develop the quantitative competence they bring to school, two-dimensional block-numbers to be recounted and double-counted before being added on-top or next-to; and reported with full number-language sentences including both a subject that exists, and a verb, and a predicate that may be different.

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25. ICMI Study 24, School Mathematics Curriculum Reforms 2018

A twin curriculum since contemporary mathematics may block the road to its educational goal, mastery of many

Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.

POOR PISA PERFORMANCE DESPITE FIFTY YEARS OF RESEARCH

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 years since 1969. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to significantly below the OECD average in 2012, causing OECD (2015) to write the report 'Improving Schools in Sweden'. Likewise, math dislike seems to be widespread in high performing countries also. With mathematics and education as social institutions, grand theory may explain this 'irrelevance paradox', the apparent negative correlation between research and performance.

GRAND THEORY

Ancient Greece saw two forms of knowledge, 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Christianity eagerly took over metaphysical patronage and changed the academies into monasteries. The sophist skepticism was revived by Brahe and Newton, insisting that knowledge about nature comes from laboratory observations, not from library books (Russell, 1945).

Newton's discovery of a non-metaphysical changing will spurred the Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. Two republics arose, in the United States and in France. The US still has its first Republic, France its fifth, since its German-speaking neighbors tried to overthrow the French Republic again and again.

In North America, the sophist warning against hidden patronization lives on in American pragmatism and symbolic interactionism; and in Grounded Theory, the method of natural research resonating with Piaget's principles of natural learning. In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnosed cures, and education; all may hide patronizing choices presented as nature (Lyotard, 1984).

Within philosophy itself, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

As to what we say about the world, Foucault (1995) focuses on discourses about humans that, if labeled scientific, establish a 'truth regime'. In the first part of his work, he shows how a discourse disciplines itself by only accepting comments to already accepted comments. In the second part he shows how a discourse disciplines also its subject by locking humans up in a predicate prison of abnormalities from which they can only escape by accepting the diagnose and cure offered by the 'pastoral power' of the truth regime. Foucault thus sees a school as a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cell daily and accept the diagnose 'un-educated' to be cured by, of course, education as defined by the ruling truth regime.

Mathematics, stable until the arrival of SET

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a SET-derived 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

SET thus transformed classical grounded 'many-matics' into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

Proportionality illustrates the variety of mastery of Many and of quantitative competence

Proportionality is rooted in questions as “2kg costs 5$, what does 7kg cost; and what does 12$ buy?”

Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, the divide. So first we ask, Q1: ‘2kg cost 5$, 7kg cost ?$’ to get to the answer (7*5/2)$ = 17.5$. Then we ask, Q2: ‘5$ buys 2kg, 12$ buys ?kg’ to get to the answer (12*2)/5$ = 4.8kg.

Then, two new methods appeared, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$. Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg. Q1: 2/5 = 7/*x*, so 2**x* = 7*5, *x* = (7*5)/2 = 17.5. Q2: 2/5 = *x*/12, so 5**x* = 12*2, *x* = (12*2)/5 = 4.8.

SET chose modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function *f*(*x*) = *c***x* from the set of kg-numbers to the set of $-numbers, having as domain *DM* = {*x*∈R | *x*>0}. Knowing that *f*(2) = 5, we set up the equation *f*(2) = *c**2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: *c**2 = 5, (*c**2)*½ = 5*½, *c**(2*½) = 5/2, *c**1 = 5/2, *c* = 5/2. With *f*(*x*) = 5/2**x*, the inverse function is *f*-1(*x*) = 2/5**x*. So with 7kg, *f*(7) = 5/2*7 = 17.5$; an with 12$, *f*-1(12) = 2/5*12 = 4.8kg.

In the future, we simply ‘re-count’ in the ‘per-number’ 2kg/5$ coming from ‘double-counting’ the total *T*. Q1: *T* = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$; Q2: *T* = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

Grand theory looks at mathematics education

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB+4*B+5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, contemporary university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word ‘mathematics’ must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a pris-pital, a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a ‘the goal of math education is to learn math’ discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

Meeting Many, children bundle in block-numbers to count and share

How to master Many can be learned from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not four, that is two twos’, thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use block-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3 extra’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, placing them next-to each other, they typically say ‘2 7s and 3 extra’.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don’t mind writing a total of 7 using ‘bundle-writing’ as *T* = 7 = 1*B*5 = 2*B*3 = 3*B*1 = 4*B*~~1~~; or even as 1*BB*3 or 1*BB*1*B*1. Also, children love to count in 3s, 4s, and in hands.

Sharing 9 cakes, 4 children take one by turn saying they take 1 of each 4. Taking away 4s roots division as counting in 4s; and with 1 left they often say “let’s count it as 4”. Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. And they smile when seeing that entering ‘9/4’ allows a calculator to predict the sharing result as 2 1/4; and when seeing that entering ‘2*5/3’ will predict the result of sharing 2 5s between 3 children.

Children thus master sharing, taking parts and splitting into parts before division and counting- and splitting-fractions is taught; which they may like to learn before being forced to add without units.

So why not develop instead of rejecting the core mastery of Many that children bring to school?

A typical contemporary mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting ‘(ten and) 1 left’, ‘(ten and) 2 left’. Never children are asked to use full number-language sentences, *T* = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1*B*14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children’s existing mastery and quantitative competence.

A QUESTION GUIDED COUNTING CURRICULUM

The question guided re-enchantment curriculum in counting could be named ‘Mastering Many by counting, recounting and double-counting’. The design is inspired by Tarp (2018). It accepts that while eight competencies might be needed to learn university mathematics (Niss, 2003), only two are needed to master Many (Tarp, 2002), counting and uniting, motivating a twin curriculum. The corresponding pre-service or in-service teacher education can be found at the MATHeCADEMY.net. Remedial curricula for classes stuck in contemporary mathematics can be found in Tarp (2017).

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H". Follow-up activities could be counting things.

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported in a number-language sentence with subject, verb and predicate: *T* = 5 = 1Bundle3 2s = 2*B*1 2s = 3*B*-1 2s = 1*BB*1 2s, called an 'inside bundle-number' describing the 'outside block-number'. A western abacus shows this in 'outside geometry space-mode' with the 2 2s on the second and third bar and 1 on the first bar; or in 'inside algebra time-mode' with 2 on the second bar and 1 on the first bar. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 5/2 means 'from 5, wipe away bundles of 2s'. The calculator says '2.some', thus predicting it can be done 2 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles. By showing '5-2*2 = 1' the calculator indirectly predicts that a total of 5 can be recounted as 2B1 2s. An additional learning opportunity is to write and use the 'recount-formula' *T* = (*T*/*B*)**B* saying "From *T*, *T*/*B* times *B* can be taken away." This proportionality formula occurs all over mathematics and science. Follow-up activities could be counting cents: 7 2s is how many fives and tens? 8 5s is how many tens?

Q05, unbundled as decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the learning opportunity is to see that with blocks, the unbundled occur in three ways. Next-to the block as a block of its own, written as *T* = 7 = 2.1 3s, where a decimal point separates the bundles from the singles. Or on-top as a part of the bundle, written as *T* = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q06, prime or foldable units: "Which blocks can be folded?" Here the learning opportunity is to examine the stability of a block. The block *T* = 2 4s = 2*4 has 4 as the unit. Turning over gives *T* = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. So, a block of 3 2s cannot be folded, whereas a block of 3 4s can: *T* = 3 4s = 3 * (2*2) = (3*2) * 2. A number is called even if it can be written with 2 as the unit, else odd.

Q07, finding units: “What are possible units in *T* = 12?”. Here the learning opportunity is that units come from factorizing in prime units, *T* = 12 = 2*2*3. Follow-up activities could be other examples.

Q08, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. *T* = 3 4s = ? 5s, the recount-formula will say *T* = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘*T* = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter *u* for the unknown number, *u**8 = 24. This is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is *u* = 24/8 attained by moving 8 to the opposite side with the opposite sign. Follow-up activities could be other examples of recounting from tens to icons.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that if asking ‘T = 3 7s = ? tens’, the recount-formula cannot be used since the calculator has no ten-button. However, it is programmed to give the answer directly by using multiplication alone: *T* = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: *T* = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Follow-up activities could be other examples of recounting from icons to tens.

Q11, double-counting in two units: “How to double-count in two different units?” Here the learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To answer questions we just recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa, asking ‘?$ = 12kg’, the answer is T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Follow-up activities could be numerous other examples of double-counting in two different units since per-numbers and proportionality are core concepts.

Q12, double-counting in the same unit: “How to double-count in the same unit?” Here the learning opportunity is that when double-counted in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5; or percentages, 3 per hundred = 3/100 = 3%. Thus, to find a fraction or a percentage of a total, again we just recount in the per-number. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Follow-up activities could be other examples of double-counting in the same unit since fractions and percentages are core concepts.

Q13, recounting the sides in a block. “How to recount the sides of a block halved by its diagonal?” Here, in a block with base *b*, height *a*, and diagonal *c*, mutual recounting creates the trigonometric per-numbers: $a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$. Thus, rotating a line can be described by a per-number *a*/*b*, or as tan*A* per 1, allowing angles to be found from per-numbers. Follow-up activities could be other blocks e.g. from a folding ruler.

Q14, double-counting in STEM (Science, Technology, Engineering, Math) multiplication formulas with per-numbers coming from double-counting. Examples: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter) * square-meter = pressure * square-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg) * kg = heat * kg; gram = (gram/mole) * mole = molar mass * mole; Δ momentum = (Δ momentum/sec) * sec = force * sec; Δ energy = (Δ energy/ meter) *

meter = force * meter = work; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp; dollar = (dollar/hour)*hour = wage*hour.

Q15, navigating. "Avoid the rocks on a squared paper". Four rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock? Predict and measure the angles on the journey.

A QUESTION GUIDED UNITING CURRICULUM

The question guided re-enchantment curriculum in uniting could be named 'Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers'.

A general bundle-formula $T = a*x^2 + b*x + c$ is called a polynomial. It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Q21, next-to addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus. Follow-up activities could be other examples of next-to addition.

Q22, reversed next-to addition: "If *T*1 = 2 3s and *T*2 add next-to as *T* = 4 7s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Follow-up activities could be other examples of reversed next-to addition.

Q23, on-top addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Follow-up activities could be other examples of on-top addition.

Q24, reversed on-top addition: "If *T*1 = 2 3s and *T*2 as some 5s add to *T* = 4 5s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again is called differential calculus. An underload is removed by recounting. Follow-up activities could be other examples of reversed on-top addition.

Q25, adding tens: "With *T*1 = 23 and *T*2 = 48, what is *T*1+*T*2 when added as tens?" Again, recounting removes an overload: *T*1+*T*2 = 23 + 48 = 2*B*3 + 4*B*8 = 6*B*11 = 7*B*1 = 71; or *T* = 236 + 487 = 2*BB*3*B*6 + 4*BB*8*B*7 = 6*BB*11*B*13 = 6*BB*12*B*3 = 7*BB*2*B*3 = 723.

Q26, subtracting tens: "If *T*1 = 23 and *T*2 add to *T* = 71, what is *T*2?" Again, recounting removes an underload: *T*2 = 71 – 23 = 7*B*1 – 2*B*3 = 5*B*-2 = 4*B*8 = 48; or *T*2 = 956 – 487 = 9*BB*5*B*6 – 4*BB*8*B*7 = 5*BB*-3*B*-1 = 4*BB*7*B*-1 = 4*BB*6*B*9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, showing that -(-1) = +1.

Q27, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: *T* = 7*43 = 7*4*B*3 = 28*B*21 = 30*B*1 = 301; or 27*43 = 2*B*7*4*B*3 =8*BB*+6*B*+28*B*+21 =8*BB*34*B*21 =8*BB*36*B*1 = 11*BB*6*B*1 = 1161, solved geometrically in a 2x2 block.

Q28, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: *T* = 348 /6 = 3*BB*4*B*8 /6 = 34*B*8 /6 = 30*B*48 /6 = 5*B*8 = 58.

Q29, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform into unit-number by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining constancy by epsilon and delta.

Q30, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q31, finding common units: “Only add with like units, so how to add *T* = 4ab^2 + 6abc?”. Here units come from factorizing: *T* = 2*2**a***b***b* + 2*3**a***b***c* = 2**b**(2**a***b*) + 3**c**(2**a***b*) = 2*b*+3*c* 2*ab*s.

CONCLUSION

A math education curriculum must make a choice. Shall it teach the ontology or the epistemology of Many? Shall it mediate the contemporary university discourse where the set-concept has transformed classical bottom-up ‘many-matics’ into a self-referring top-down ‘meta-matism’; or shall it develop the mastery of Many already possessed by children? Shall it teach about numbers or how to number? To allow choosing between a mediating and a developing curriculum, we need an alternative to the present curriculum, unsuccessfully trying to mediate contemporary university mathematics. So, Luther has a point arguing that reaching a goal is not always helped by institutional patronization. Grand theory thus has an answer to the ‘irrelevance paradox’ of mathematics education research: Accepting the child’s own epistemology will avoid a goal displacement where a litany of self-referring university mathematics blocks the road to its outside educational goal, mastery of Many.

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A decolonized curriculum - but for which of the 3x2 kinds of math education

An essay on observations and reflections at the ICMI study 24 curriculum conference Dec. 2018

As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-‘setcentric’ mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.

Coherence and relevance in the school mathematics curriculum

The International Commission on Mathematical Instruction, ICMI, has named its 24th study “School mathematics Curriculum Reforms: Challenges, Changes and Opportunities”. Its discussion document has 5 themes among which theme B, “Analysing school mathematics curriculum for coherence and relevance” says that “All mathematics curricula set out the goals expected to be achieved in learning through the teaching of mathematics; and embed particular values, which may be explicit or implicit.” So, to analyze we use the verb ‘cohere’ and the predicate ‘relevant’ when asking: “to what does this curriculum cohere and to what is it relevant?” As to the meaning of the words ‘cohere’ and ‘relevant’ we may ask dictionaries. The Oxford Dictionaries (en.oxforddictionaries.com) writes that ‘to cohere’ means ‘to form a unified whole’ with its origin coming from Latin ‘cohaerere’, from co- ‘together’ + haerere ‘to stick’; and that ‘relevant’ means being ‘closely connected or appropriate to what is being done or considered.’

We see, that where ‘cohere’ relates to states, ‘relevant’ relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that ‘to cohere’ means ‘to hold together firmly as parts of the same mass’. As to synonyms for cohere, it lists: ‘accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.’ And as to antonyms, it lists: ‘differ (from), disagree (with).’

In the same dictionary, the word ‘relevant’ means ‘having significant and demonstrable bearing on the matter at hand’. As to synonyms for relevant, it lists: ‘applicable, apposite, apropos, germane, material, pertinent, pointed, relative.’ And as to antonyms, it lists: ‘extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.’

If we accept the verb ‘apply’ as having a meaning close to the predicate ‘relevant’, we can rephrase the above analysis question using verbs only: “to what does this curriculum cohere and apply?”

Seeing education metaphorically as bridging an individual start level for skills and knowledge to a common end level described by goals and values, we may now give a first definition of an ideal curriculum: “To apply to a learning process as relevant and useable, a curriculum coheres to the start and end levels for skills and knowledge.”

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?

- How to make the end level cohere to goals and values expressed by the society?
- How to make the end level cohere to goals and values expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

Goals and values expressed by the society

In her plenary address about the 'OECD 2030 Learning Framework', Taguma shared a vision:

> The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) And in an era characterised by a new explosion of scientific knowledge and a growing array of complex societal problems, it is appropriate that curricula should continue to evolve, perhaps in radical ways (p. 10).

Talking about learner agency, Taguma said:

> Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners' individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial. (p. 11)

By emphasizing learner's individual potentials, personalised learning environment and own learning projects and processes, Taguma seems to indicate that flexible half-year micro-curricula may cohere better with learners' future needs than rigid multi-year macro-curricula. As to specifics, numeracy is mentioned as one of the two parts of a solid foundation helping learners enable agency.

Different kinds of numeracy

Numeracy, however, is not that well defined. Oxford Dictionaries and Merriam-Webster agree on saying 'ability to understand and work with numbers'; whereas the private organization National Numeracy (nationalnumeracy.org.uk) says 'By numeracy we mean the ability to use mathematics in everyday life'.

The wish to show usage was also part of the Kilpatrick address, describing mathematics as bipolar:

> I want to stress that bipolarity because I think that's an important quality of the school curriculum and every teacher and every country has to deal with: how much attention do we give to the purer side of mathematics. The New Math thought that it should be entire but that didn't work really as well as people thought. So how much attention do we give to the pure part of mathematics and how much to the applications and how much do we engage together. Because it turns out if the applications are well-chosen and can be understood by the children then that helps them move toward the purer parts of the field. (p. 20)

After discussing some problems caused by applications in the curriculum, Kilpatrick concludes:

> If we stick with pure mathematics, with no application, what students cannot see, "when will I ever use this?", it's not surprising that they don't go onto take more mathematics. So, I think for self-preservation, mathematicians and mathematics educators should work on the question of: how do we orchestrate the curriculum so that applications play a good role? There is even is even a problem with the word applications, because it implies first you do the mathematics, then you apply it. And actually, it can go the other way. (p. 22)

So, discussing what came first, the hen or the egg, applications or mathematics, makes it problematic to define numeracy as the ability to apply mathematics since it gives mathematics a primacy and a monopoly as a prerequisite for numeracy. At the plenary afterwards discussion, I suggested using the word ‘re-rooting’ instead of ‘applying’ to indicate that from the beginning, mathematics was rooted in the outside world as shown by the original meanings of geometry and algebra: ‘to measure earth’ in Greek and ‘to reunite’ in Arabic.

Mathematics through history

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy, seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra. And in Europe, Germanic countries taught ‘reckoning’ in primary school and ‘arithmetic’ and ‘geometry’ in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here a wish for exactness and unity created a ‘setcentric’ ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox ‘this sentence is false’, being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

Setcentrism thus changed classical grounded ‘many-matics’ into a self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside a classroom where adding numbers without units as ‘1 + 2 IS 3’ meets counter-examples as e.g. 1week + 2days is 9days.

The introduction of the setcentric New Mathematics created different reactions. Inside the United States it was quickly abandoned with a ‘back-to-basics’ movement. Outside it was implemented at teacher education, and in schools where it gradually softened. However, it never retook its original form or name, despite, in contrast to ‘mathematics’, ‘reckon’ is an action-word better suited to the general aim of education, to teach humans to master the outside world through appropriate actions.

Different kinds of mathematics

So, a curriculum must choose between a pre-, a present, and a post-setcentric mathematics as illustrated by an example from McCallum’s plenary talk. After noting that “a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships” (p. 4), McCallum asked the audience: “What is the difference between 5/3 and 5÷3”.

Pre-setcentric mathematics will say that 5/3 is a number on the number-line reached by taking 5 steps of the length coming from dividing the unit in 3 parts; and that 5÷3 means 5 items shared between 3. Present setcentric mathematics will say that 5/3 is a rational number defined as an equivalence class in the product set of integers, created by the equivalence relation (a,b) eq. (c,d) if cross-multiplication holds, $a \times d = b \times c$; and, with 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5x1/3, i.e. the as the solution to the equation 3x*u* = 5, found by applying and thus legitimizing abstract algebra and group theory; thus finally saying goodbye to the Renaissance use of a vertical line to separate addends from subtrahends, and a horizontal line to separate multipliers from divisors. Post-setcentric mathematics (Tarp, 2018) sees setcentric mathematics as meta-matism hiding the original Greek meaning og mathematics as a science about Many. In this 'Many-math', 5/3 is a per-number coming from double-counting in different units (5$/3kg), becoming a fraction with like units (5$/3$ = 5/3). Here per-numbers and fractions are not numbers but operators needing a number to become a number (5/3 of 3 is 5, 5/3 of 6 is 10); and 5÷3 means 5 counted in 3s occurring in the 'recount-formula' recounting a total *T* in bundles of 3s as *T* = (*T*/3)x3, saying 'from *T*, *T*/3 times, 3 can be taken away'. This gives flexible numbers: *T* = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s, introduced in grade one where bundle-counting and re-counting in another unit precedes adding, and where recounting from tens to icons, *T* = 2.4 tens = ? 6s, leads to the equation *T* = *u*x6 = 24 = (24/6)x6 solved by recounting. In post-setcentric mathematics, per-numbers, fractions, ratios and proportionality melt together since double-counting in two units gives per-numbers as ratios, becoming fractions with like units. And here proportionality means changing units using the recount-formula to recount in the per-number: With 5$/3kg, "how much for 20$?" is found by re-counting 20 in 5s: *T* = 20$ = (20/5)x5$ = (20/5)x3kg = 12 kg. Likewise if asking "how much for 15 kg?"

Different kinds of education

As to education, from secondary school there is a choice between multi-year lines and half-year blocks. At the discussion after the Kilpatrick plenary session I made a comment about these two educational systems, which mas a lady from the United States say I was misinforming since in the states Calculus required a full year block. Together with other comments in the break, this made me realize that internationally there is little awareness of these two different kinds of educational systems. So here is another example of what the Greek sophists warned against, choice masked as nature.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

At the conference, the almost total absence of federal states as Germany, Canada, the United States and Russia seems to indicate that the problems reside with multi-year national curricula, becoming rigid traditions difficult to change. While federal competition or half-year blocks creates flexibility through an opportunity to try out different curricula.

Moreover, as a social institution involving individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlighten teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching one subject only in their own classrooms.

To protect its republic against its German speaking neighbors, France created elite schools, criticized today for exerting hidden patronization. Bourdieu thus calls education 'symbolic violence', and Foucault points out that a school is really a 'pris-pital' mixing power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings.

Different kinds of competences

As to competences, new to many curricula, there are at least three alternatives to choose among. The European Union recommends two basic competences, acquiring and applying, when saying that "Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge."

At the conference two alternative notions of competences were presented. In his plenary address, Niss recommended a matrix with 8 competences per concept (p. 73). In his paper, Tarp (pp. 317-324) acknowledged that 8 competences may be needed if the goal of mathematics education is to learn present setcentric university mathematics; but if the goal is to learn to master Many with post-setcentric mathematics, then only two competences are needed: counting and adding, rooting a twin curriculum teaching counting, recounting in different units and double-counting before adding.

Making the learning road more passable

Once a curriculum is chosen, the next question is to make its bridge between the start and end levels for skills and knowledge more passable. Here didactics and pedagogy come in; didactics as the captain choosing the way from the start to the end, typically presented as a textbook leaving it to pedagogy, the lieutenants, to take the learners through the different stages.

The didactical choices must answer general questions from grand theory. Thus, philosophy will ask: shall the curriculum follow the existentialist recommendation, that existence precedes essence? And psychology will ask: shall the curriculum follow Vygotsky mediating institutionalized essence, or Piaget arranging learning meetings with what exists in the outside world? And sociology will ask: on which ethical grounds are children and teenagers retained to be cured by institutionalized education?

Colonizing or decolonizing curricula

The conference contained two plenary panels, the first with contributors from France, China, The Philippines and Denmark, almost all from the northern hemisphere; the second with contributors from Chile, Australia, Lebanon and South Africa, almost all from the southern hemisphere. Where the first panel talked more about solutions, the second panel talked more about problems.

In the first panel, France and Denmark represented some of the world's most centralized states with war-time educational systems dating back to the Napoleon era, which in France created elite-schools to protect the young republic from the Germans, and in Germany created the Humboldt Bildung schools to end the French occupation by mediating nationalism, and to sort out the population elite for jobs as civil servants in the new central administration; both just replacing the blood-nobility with a knowledge-nobility as noted by Bourdieu. The Bildung system latter spread to most of Europe.

Not surprisingly, both countries see university mathematics as the goal of mathematics education ('mathematics is what mathematicians do'), despite the obvious self-reference avoided by instead formulating the goal as e.g. learning numerical competence, mastery of Many or number-language. Seeing mathematics as the goal, makes mathematics education an example of a goal displacement (Bauman) where a monopoly transforms a means into a goal. A monopoly that makes setcentric mathematics an example of what Habermas and Derrida would call a 'center-periphery colonization', to be decentered and decolonized by deconstruction.

Artigue from France thus advocated an anthropological theory of the didactic, ATD, (p. 43-44), with a 'didactic transposition process' containing four parts: scholarly knowledge (institutions producing and using the knowledge), knowledge to be taught (educational system, 'noosphere'), taught knowledge (classroom), and learned available knowledge (community of study).

> The theory of didactic transposition developed in the early 1980s to overcome the limitation of the prevalent vision at the time, seeing in the development of taught knowledge a simple process of elementarization of scholarly knowledge (Chevallard 1985). Beyond the well-known succession offered by this theory, which goes from the reference knowledge to the knowledge actually taught in classrooms (..), ecological concepts such as those of niche, habitat and trophic chain (Artaud 1997) are also essential in it.

Niss from Denmark described the Danish 'KOM Project' leading to eight mathematical competencies per mathematical topic (pp. 71-72).

> The KOM Project took its point of departure in the need for creating and adopting a general conceptualisation of mathematics that goes across and beyond educational levels and institutions. (..) We therefore decided to base our work on an attempt to define and characterise mathematical competence in an overarching sense that would pertain to and make sense in any mathematical context. Focusing - as a consequence of this approach - first and foremost on the *enactment* of mathematics means attributing, at first, a secondary role to mathematical content. We then came up with the following definition of mathematical competence: Possessing *mathematical competence* – mastering mathematics – is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve actual or potential mathematical challenges of any kind. In order to identify and characterise the fundamental constituents in mathematical competence, we introduced the notion of mathematical competencies: A *mathematical competency* is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve a certain kind of mathematical challenge.

Some of the consequences by being colonized by setcentrism was described in the second panel.

In his paper 'School Mathematics Reform in South Africa: A Curriculum for All and by All?' Volmink from South Africa Volmink writes (pp. 106-107):

> At the same time the educational measurement industry both locally and internationally has, with its narrow focus, taken the attention away from the things that matter and has led to a traditional approach of raising the knowledge level. South Africa performs very poorly on the TIMSS study. In the 2015 study South Africa was ranked 38th out of 39 countries at Grade 9 level for mathematics and 47th out of 48 countries for Grade 5 level numeracy. Also in the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), South Africa was placed 9th out of the 15 countries participating in Mathematics and Science – and these are countries which spend less on education and are not as wealthy as we are. South Africa has now developed its own Annual National Assessment (ANA) tests for Grades 3, 6 and 9. In the ANA of 2011 Grade 3 learners scored an average of 35% for literacy and 28% for numeracy while Grade 6 learners averaged 28% for literacy and 30% for numeracy.

After thanking for the opportunity to participate in a cooperative effort on the search of better education for boys, girls and young people around the world, Oteiza from Chile talked about 'The Gap Factor' creating social and economic differences. A slide with the distribution of raw scores at PSU mathematics by type of school roughly showed that out of 80 points, the median scores were 40 and 20 for private and public schools respectively. In his paper, Oteiza writes (pp. 81-83):

> Results, in national tests, show that students attending public schools, close to de 85% of school population, are not fulfilling those standards. How does mathematical school curriculum contribute to this gap? How might mathematical curriculum be a factor in the reduction of these differences? (..) There is tremendous and extremely valuable talent diversity. Can we justify the existence of only one curriculum and only one way to evaluate it through standardized tests? (..) There is a fundamental role played by researchers, and research and development centers and institutions. (..) How do the questions that originate in the classroom reach a research center or a graduate program? "*Publish or perish*" has led our researchers to publish in prestigious international journals, but, are the problems and local questions addressed by those publications?"

The Gap Factor is also addressed in a paper by Hoyos from Mexico (pp. 258-259):

> The PISA 2009 had 6 performance levels (from level 1 to level 6). In the global mathematics scale, level 6 is the highest and level 1 is the lowest. (..) It is to notice that, in PISA 2009, 21.8% of Mexican students do not reach level 1, and, in PISA 2015, the percentage of the same level is a little bit higher (25.6%). In other words, the percentage of Mexican students that in PISA 2009 are below level 2 (i.e., attaining the level 1 or zero) was 51%, and this percentage is 57% in PISA 2015, evidencing then an increment of Mexican students in the poor levels of performance. According to the INEE, students at levels 1 or cero are susceptible to experiment serious difficulties in using mathematics and benefiting from new educational opportunities throughout its life. Therefore, the challenges of an adequate educational attention to this population are huge, even more if it is also considered that approximately another fourth of the total Mexican population (33.3 million) are children under 15 years of age, a population in priority of attention".

As a comment to Volminks remark "Another reason for its lack of efficacy was the sense of scepticism and even distrust about the notion of People's Mathematics as a poor substitute for the "real mathematics"" (p. 104), and inspired by the sociological Centre–Periphery Model for colonizing, by post-colonial studies, and by Habermas' notion of rationalization and colonization of the lifeworld by the instrumental rationality of bureaucracies, I formulated the following question in the afterwards discussion: "As former colonies you might ask: Has colonizing stopped, or is it still taking place? Is there an outside central mathematics that is still colonizing the mind? What happens to what could be called local math, street math, ethno-math or the child's own math?"

Conclusion and recommendations

Designing a curriculum for mathematics education involves several choices. First pre-, present and post-setcentric mathematics together with multi-year lines and half-year blocks constitute 3x2 different kinds of mathematics education. Combined with three different ways of seeing competences, this offers a total of 18 different ways in which to perform mathematics education at each of the three educational levels, primary and secondary and tertiary, which may even be divided into parts.

Once chosen, institutional rigidity may hinder curriculum changes. So, to avoid the ethical issues of forcing cures from self-referring diagnoses upon children and teenagers in need of guidance instead of cures, the absence of participants from federal states might be taken as an advice to replace the national multi-year macro-curriculum with regional half-year micro-curricula. At the same time, adopting the post version of setcentric mathematics will make the curriculum coherent with the mastery of Many that children bring to school, and relevant to learning the quantitative competence and numeracy desired by society.

And, as Derrida says in an essay called 'Ellipsis' in 'Writing and Difference': "Why would one mourn for the centre? Is not the centre, the absence of play and difference, another name for death?"

Postscript: Many-math, a post-setcentric Mathematics for all

As post-setcentric mathematics, Many-math, can provide numeracy for all by celebrating the simplicity of mathematics occurring when recounting the ten fingers in bundles of 3s:

T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Or, if seeing 3 bundles of 3s as 1 bundle of bundles,

T = ten = 1BB0B1 3s = 1*B^2 + 0*B + 1 3s, or T = ten = 1BB1B-2 3s = 1*B^2 + 1*B – 2 3s.

This number-formula shows that a number is really a multi-numbering of singles, bundles, bundles of bundles etc. represented geometrically by parallel block-numbers with units. Also, it shows the four ways to unite: on-top addition, multiplication, power and next-to addition, also called integration.

Which are precisely the four ways to unite constant and changing unit- and per-numbers numbers into totals as seen by including the units; each with a reverse way to split totals.

Thus, addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the Algebra Square', also showing that equations are solved by moving to the opposite side with opposite signs.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** *T – n = a*	***T = a*n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫a*dn*** *dT/dn = a*	***T = a^n*** *n√T = a logaT = n*

An unbundled single can be placed on-top of the block counted in 3s as T = 1 = 1/3 3s, or next-to the block as a block of its own written as T = 1 = .1 3s Writing T = ten = 3 1/3 3s = 3.1 3s = 4.-2 3s thus introduces fractions and decimals and negative numbers together with counting.

The importance of bundling as the unit is emphasized by counting: 1, 2, 3, 4, 5, 6 or bundle less 4, 7 or B-3, 8 or B-2, 9 or B-1, ten or 1 bundle naught, 1B1, …, 1B5, 2B-4, 2B-3, 2B-2, 2B-1, 2B naught.

This resonates with 'Viking-counting': 1, 2, 3, 4, hand, and1, and2, and3, less2, less1, half, 1left, 2left. Here '1left' and '2left' still exist as 'eleven' and 'twelve', and 'half' when saying 'half-tree', 'half-four' and 'half-five' instead of 50, 70 and 90 in Danish, counting in scores; as did Lincoln in his Gettysburg address: "Four scores and seven years ago …"

Counting means wiping away bundles (called division iconized as a broom) to be stacked (called multiplication iconized as a lift) to be removed to find unbundled singles (called subtraction iconized as a horizontal trace).

Thus, counting means postponing adding and introducing the operations in the opposite order of the tradition, and with new meanings: 7/3 means 7 counted in 3s, 2x3 means stacking 3s 2 times. Addition has two forms, on-top needing recounting to make the units like, and next-to adding areas, i.e. integral calculus. Reversed they create equations and differential calculus.

The recount-formula, T = (T/B)*B, appears all over mathematics and science as proportionality or linearity formula:

- Change unit, T = (T/B)*B, e.g. T = 8 = (8/2)*2 = 4*2 = 4 2s
- Proportionality, $ = ($/kg)*kg = price*kg
- Trigonometry, *a* = *(a/c)*c* = sin*A*c*, a = *(a/b)*b* = tan*A*b*, *b* = *(b/c)*c* = cos*A*c*
- STEM-formulas, meter = (meter/sec)*sec = speed*sec, kg = (kg/m^3)*m^3 = density*m^3
- Coordinate geometry, $\Delta y = (\Delta y/\Delta x)*\Delta x = m*\Delta x$
- Differential calculus, d*y* = (d*y*/d*x*)*d*x* = *y*' * d*x*

The number-formula also contains the formulas for constant change: *T = b*x* (proportional), *T = b*x + c* (linear), *T = a*x^n* (elastic), *T = a*n^x* (exponential), *T = a*x^2 + b*x + c* (accelerated).

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics 'post-dicting' what we cannot be 'pre-dicted'.

The general curriculum choices of post-setcentric mathematics

Making the curriculum bridge cohere with the individual start levels in a class is obtained by always beginning with the number-formula, and with recounting tens in icons less than ten, e.g. T = 4.2 tens = ? 7s, or u*7 = 42 = (42/7)*7, thus solving equations by moving to opposite side with opposite sign. And by always using full number-language sentences with a subject, a verb and a predicate as in the word language, e.g. T = 2*3. This also makes the bridge cohere to previous and following bridges.

Making the end level cohere to goals and values expressed by the society and by the learners is obtained by choosing mastery as the end goal, not of the inside self-referring setcentric construction of contemporary university mathematics, but of the outside universal physical reality, Many.

Making the bridge passable is obtained by choosing Piagetian psychology instead of Vygotskyan.

Flexible numbers make teachers follow

Changing a curriculum raises the question: will the teachers follow? Here, seeing the advantage of flexible numbers makes teachers interested in learning more about post-setcentric mathematics:

Typically, division creates problems to students, e.g. 336/7. With flexible numbers a total of 336 can be recounted with an overload as T = 336 = 33B6 = 28B56, so 336/7 = 28B56 /7 = 4B8 = 48; or with an underload as T = 336 = 33B6 = 35B-14, so 336/7 = 35B-14 /7 = 5B-2 = 48.

Flexible numbers ease all operations:

T = 48*7 = 4B8*7 = 28B56 = 33B6 = 336
T = 92 – 28 = 9B2 – 2B8 = 7B-6 = 6B4 = 64
T = 54 + 28 = 5B4 + 2B8 = 7B12 = 8B2 = 82

To learn more about flexible numbers, a group of teachers can go to the MATHeCADEMY.net designed to teach teachers to teach MatheMatics as ManyMatics, a natural science about Many, to watch some of its YouTube videos. Next, the group can try out the "Free 1day Skype Teacher Seminar: Cure Math Dislike by ReCounting" where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally, and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows after the coffee break.

To learn more, a group of eight teachers can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for secondary school. For modelling, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where the 8 teachers form 2 teams of 4, choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers. The material mediates learning by experimenting with the subject in number-language sentences, i.e. the total T. Thus, the material is self-instructing, saying "When in doubt, ask the subject, not the instructor".

The material for primary and secondary school has a short question-and-answer format. The question could be: "How to count Many? How to recount 8 in 3s? How to count in standard bundles?" The corresponding answers would be: "By bundling and stacking the total T, predicted by T = (T/B)*B. So, T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s = 3.-1 3s. Bundling bundles gives multiple blocks, a polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4*B^2 + 5*B + 6*1."

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The same Mathematics Curriculum for Different Students

To offer mathematics to all students, parallel tracks often occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out e.g. calculus; or they might contain a different kind of mathematics believed to be more relevant to students by including more applications. However, an opportunity presents itself for designing the same curriculum for all students no matter which track they may choose. As number-language, why not let mathematics follow the communicative turn that took place in language education in the 1970s by prioritizing its connection to the outside world higher than its inside connection to its grammar? We will consider examples of all three curricula options. A communicative turn also offers new curricula opportunities for primary and middle school that may make obsolete the need for parallel curricula in high school.

01. A need for curricula for all students

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing the Organisation for Economic Co-operation and Development (OECD, 2015) to write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’:

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)

Other countries also experience declining PISA results; and in high performing countries not all students are doing well.

02. Addressing the need

By saying “All students should study mathematics in each of the four years that they are enrolled in high school.“ the US National Council of Teachers of Mathematics (2000, p. 18) has addressed the need for curricula for all students in their publication ‘Principles and Standards for School Mathematics’. In the overview the Council writes

> We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding (..) In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors. (..) everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (p. 1)

In this way the Council points out that it is important to master ‘mathematical competence’, i.e. to understand and do mathematics to solve problems creatively and to compute fluently. This will benefit the personal life, the workplace, as well as further study leading to productive futures.

Consequently, the Council has included in the publication a curriculum that “is mathematically rich providing students with opportunities to learn important mathematical concepts and procedures with understanding”. This in order to “provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.”

The publication includes a set of standards: “The Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten to grade 12.” The five standards

> present goals in the mathematical content areas of number and operations, algebra , geometry, measurement and data analysis and probability. (..) Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century” (p. 2)

In the chapter ‘Number and operations’, the Council writes

> Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense. Students with number sense naturally decompose numbers (..) For example, children in the lower elementary grades can learn that numbers can be decomposed and thought about in many different ways - that 24 is 2 tens and 4 ones and also two sets of 12. (p. 7)

In the chapter ‘The Curriculum Principle’, the Council writes

> A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades (..) for teachers at each level to know what mathematics their students have already studied and will study in future grades. (p. 3, 4)

All in all, the Council points to the necessity of designing a curriculum that is relevant in students’ ‘personal life, in the workplace and in further study’ and that is coherent at the same time to allow teachers to know ‘what mathematics their students have already studied and will study in future grades’.

03. Coherence and relevance

So, in their publication, the National Council of Teachers of Mathematics stresses the importance of coherence and relevance. To allow teachers follow a prescribed curriculum effectively, and to allow students build upon what they already know, it must be ‘well articulated across the grades’. And, to have importance for students a curriculum must be relevant by supplying them with ‘the basic skills and understandings that students will need to function effectively in the twenty-first century’.

With ‘cohere’ as a verb and ‘relevant’ as a predicate we can ask: “to what does this curriculum cohere, and to what is it relevant?” As to the meaning of the words ‘cohere’ and ‘relevant’ we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that ‘to cohere’ means ‘to form a unified whole’ with its origin coming from Latin ‘cohaerere’, from co- ‘together’ + haerere ‘to stick’; and that ‘relevant’ means being ‘closely connected or appropriate to what is being done or considered.’

We see, that where ‘cohere’ relates to states, ‘relevant’ relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that ‘to cohere’ means ‘to hold together firmly as parts of the same mass’. As to synonyms for cohere, it lists: ‘accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.’ And as to antonyms, it lists: ‘differ (from), disagree (with).’

In the same dictionary, the word ‘relevant’ means ‘having significant and demonstrable bearing on the matter at hand’. As to synonyms for relevant, it lists: ‘applicable, apposite, apropos, germane, material, pertinent, pointed, relative.’ And as to antonyms, it lists: ‘extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.’

If we accept the verb ‘apply’ as having a meaning close to the predicate ‘relevant’, we can rephrase the above analysis question using verbs only: “to what does this curriculum cohere and apply?”

Metaphorically, we may see education as increasing skills and knowledge by bridging individual start levels to a common end level described by institutional goals. So, we may now give a first definition of an ideal curriculum: “To apply to a learning process as relevant, a curriculum coheres to the individual start levels and to the end goal, which again coheres with the need expressed by the society funding the educational institution.”

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals expressed by the society?
- How to make the end level cohere to goals expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

04. Parallel tracks to the main curriculum, examples

In their publication chapter Grades 9 through 12, the National Council of Teachers of Mathematics discusses to the possibility to introduce parallel courses in the high school.

> In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.
>
> Because students’ interests and inspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of careers and educational options. They should experience the interplay of algebra, geometry, statistics, probability and discrete mathematics.
>
> High school mathematics builds on the skills and understandings developed in the lower grades. (..) High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and take other mathematics courses. In both situations, students can choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics. (p. 18-19)

The Council thus emphasizes the importance of studying ‘mathematics in each of the four years that they are enrolled in high school’. This the council sees as feasible if implementing one or more of three options allowing students to ‘study mathematics that extends beyond the material expected of all students’. Some students may want to study ‘material that extends the foundational material in depth or sophistication’. Others may want to take additional courses cohering to the college level, especially calculus. Others may want to take additional courses relevant to their daily life or a workplace.

So, as to a parallel track to the traditional curriculum, the National Council of Teachers of Mathematics suggests that including a different kind of mathematics might be an option, e.g. finite mathematics. In the US this idea was taken up by the Consortium for Mathematics and its Applications (COMAP) working out a material including a textbook and a series of television shows to show ‘mathematics at work in today’s world’. Part of this material was also included in a parallel curriculum in Portugal called ‘Mathematics Applied to the Social Sciences’ (MACS) offering to Portuguese students also to study mathematics in each of their high school years, as the National Council of Teachers of Mathematics recommends.

For all Practical Purposes, Introduction to Contemporary Mathematics

In the US, the Consortium for Mathematics and its Applications (COMAP) has worked out a material called ‘ For all practical purposes’ (COMAP, 1988). In its preface, the material presents itself as

> (..) an introductory mathematics course for students in the liberal arts or other nontechnical curricula. The course consists of twenty-six half-hour television shows, the textbook, and this Telecourse guide. This series shows mathematics at work in today’s world. (..) For all practical purposes aims to develop conceptual understanding of the tools and language of mathematics and the ability to reason using them. We expect most students will have completed elementary algebra and some geometry in high school. We do not assume students will be pursuing additional courses in mathematics, at least none beyond the introductory level. (p. iii)

As to content, the material has five parts (p. v - vi)

Part one focuses on graph theory and linear programming illustrated with network as scheduling and planning. It includes an overview show and four additional shows called street smarts: street networks; trains, planes and critical paths; juggling machines: scheduling problems; juicy problems: linear programming.

Part two deals with statistics and probability illustrated with collecting and deducing from data. It includes an overview show and four additional shows called behind the headlines: collecting data; picture this: organizing data; place your bets: probability; confident conclusions: statistical inference.

Part three focuses on social choice, fair division and game theory illustrated by different voting systems and conflict handling. It includes an overview show and four additional shows called the impossible dream: election theory; more equal than others: weighted voting; zero-sum games: games of conflict; prisoner’s dilemma: games of partial conflict.

Part four focuses on using geometry, the classical conic sections, shapes for tiling a surface, geometric growth in finance in and in population, and measurement. It includes an overview show and four additional shows called how big is too big: scale and form; it grows and grows: populations; stand up conic: conic sections; it started in Greece: measurement.

Part five focuses on computer algorithms. It includes an overview show and four additional shows called rules of the games: algorithms; counting by two’s: numerical representation; creating a cde: encoding information; moving picture show: computer graphics.

The video sections are available on YouTube.

A Portuguese Parallel High School Curriculum

Portugal followed up on the COMAP initiative. In his paper called “Secondary mathematics for the social sciences” (Silva, 2018), Jaime Silva describes how the initiative inspired an innovative two-year curriculum for the Portuguese upper secondary school.

As to the background, Silva writes

> There are two recurring debates about the mathematics curriculum in secondary schools, especially in countries like Portugal where compulsory education goes till the 12th grade. First, should all students study mathematics (not necessarily the same) or should the curriculum leave a part of the students “happy” without the mathematics “torture”? Second, should all students study the same “classic” mathematics, around ideas from differential and integral calculus with some Geometry and some Statistics?
>
> When the 2001 revision (in great part in application today) of the Portuguese Secondary School curriculum was made (involving the 10th, 11th and 12th grades) different kinds of courses were introduced for the different tracks (but not for all of them) that traditionally existed. Mathematics A is for the Science and Technology track and for the Economics track and is a compulsory course. Mathematics B is for the Arts track and is an optional course. Mathematics Applied to the Social Sciences (MACS) is for the Social Sciences track and is an optional course. The Languages track was left without mathematics or science. Later the last two tracks were merged and the MACS course remained optional for the new merged track. The technological or professional tracks could have Mathematics B, Mathematics for the Arts or Modules of Mathematics (3 to 10 to be chosen from 16 different modules, depending on the professions). (p. 309)

As to the result of debating a reform in Portugal, Silva writes

> When, in 2001, there was a possibility to introduce a new Mathematics course for the “Social Sciences” track, for the 10th and 11th grade students, there were some discussions of what could be offered. The model chosen was inspired in the course “For All Practical Purposes” (COMAP, 2000) developed by COMAP because it “uses both contemporary and classic examples to help students appreciate the use of math in their everyday lives”. As a consequence, a set of independent chapters, each one with some specific purpose, was chosen for this syllabus, that included 2 years of study, with 4.5 hours of classes per week (normally 3 classes of 90 minutes each). The topics chosen were: 10th grade Decision Methods: Election Methods, Apportionment, Fair Division; Mathematical Models: Financial models, Population models Statistics (regression); 11th grade Graph models, Probability models, Statistics (inference). (p. 310)

As to the goal of the curriculum, Silva writes

> The stated goal of this course is for the students to have *“significant mathematical experiences that allow them to appreciate adequately the importance of the mathematical approaches in their future activities”*. This means that the main goal is not to master specific mathematical concepts, but it is really to give students a new perspective on the real world with mathematics, and to change the students view of the importance that mathematical tools will have in their future life. In this course it is expected that the students study simple real situations in a form as complete as possible, and *“develop the skills to formulate and solve mathematically problems and develop the skill to communicate mathematical ideas (students should be able to write and read texts with mathematical content describing concrete situations)”.* (p. 310)

As to the present state of the curriculum, Silva writes

> After 15 years there is no thorough evaluation of how the course is run in practice in the schools, or which is the real impact on the further education or activities of the students that studied “Mathematics Applied to the Social Sciences”. In Portugal there is no institution in charge of this type of work and evaluations are done on a case by case basis. All Secondary Schools need to do selfevaluations but normally just compare internal statistics to national ones to see where they are in the national scene. In

> the reports consulted there was no special mention to the MACS course and so we have the impression that the MACS course entered the normal Portuguese routine in Secondary School. (p. 315)

To offer mathematics to all students, parallel tracks thus may occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out e.g. calculus; or they might contain a different kind of mathematics believed to be more relevant to students by including more applications.

Parallel tracks all build on the assumption that mastery of mathematics must precede mastery of Many in contrast to existentialist philosophy holding that existence must precede essence. So, letting mastery of Many precede mastery of mathematics presents an opportunity for designing the same curriculum for all students no matter which track they may choose.

05. Precalculus, typically the last mandatory curriculum

This chapter looks at the part of a mathematics curriculum called precalculus, typically being the first part that is described in a parallel curriculum since it contains operations as root and logarithm that is not considered part of a basic mathematics algebra curriculum. First, we look at an example of a traditional precalculus curriculum. The we ask what could be an ideal precalculus curriculum and illustrates it with two examples. In the next chapter, we look at a special case, a Danish precalculus curriculum that has served both as a parallel and a serial curriculum during the last 50 years.

A Traditional Precalculus Course

An example of a traditional precalculus course is found in the Research and Education Association book precalculus (Woodward, 2010). The book has ten chapters. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

An Ideal Precalculus Curriculum

In their publication, the National Council of Teachers of Mathematics writes “High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)”

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they have from the lower grades?

Mathe-matics, Meta-matics, and Mathe-matism

Furthermore, what kind of mathematics has been taught? Was it ‘grounded mathematics’ abstracted bottom-up from its outside roots, or ‘ungrounded mathematics’ or ‘meta-matics’ exemplified top-down from inside abstractions, maybe becoming ‘meta-matism’ if mixed with ‘mathe-matism’ (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept ‘function’, Euler saw it as a bottom-up abstracted name for ‘standby calculations’ containing specified and unspecified numbers. Later meta-matics defined a function top-down as an example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may be seen as a number-language sentence containing a subject, a verb and a predicate allowing its value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that $(a,b) \sim (c,d)$ if cross multiplication holds, $a*d = b*c$. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group concepts applying the associative and commutative law as well as the neutral element and inverse elements thus using five steps to solve the equation $2*u = 6$, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2.

$2*u = 6$, so $(2*u)*½ = 6*½$, so $(u*2)*½ = 3$, so $u*(2*½) = 3$, so $u*1 = 3$, so $u = 3$.

However the equation $2*u = 6$ can also be seen as recounting 6 in 2s using the recount-formula ‘T = (T/B)*B’ present all over mathematics as the proportionality formula thus solved in one step:

$2*u = 6 = (6/2)*2$, giving $u = 6/2 = 3$.

Thus, a lack of skills and understanding may be caused by being taught meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology. Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction about an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

Restarting from Scratch with Grounded Outside-Inside Mathematics

Let students see how outside degrees of Many are iconized by inside digits with as many strokes as it represents, five strokes in the 5-icon etc. Let students see that after nine we count by bundling creating icons for the counting operations as well, where division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see that an outside block of 2 3s becomes an inside calculation 2*3 and vice versa. Let students see the commutative law by turning *and* $a*b$ block, and see the distributive law by splitting a into c and d, and see the associative law by turning an $a*b*c$ box.

Let students see that both the word- and the number-language use full sentences with a subject, a verb, and an object or predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2*3’

Let students enjoy flexible bundle-numbers where decimals and fractions negative and numbers are created to describe the unbundle ones placed next-to or on-top of the block, thus allowing 5 to be recounted in 3s as T = 5 = 1B2 = 1.2 B = 1 2/3 B = 2B-1.

Let student see, that recounting in other units may be predicted by the recount-formula 'T = (T/B)*B' saying "From the total T, T/B times, B may be pushed away". Let students see that where the recount-formula in primary school recounts 6 in 2s as 6 = (6/2)*2 = 3*2, in secondary school the same task is formulated as solving the equation u*2 = 6 as u*2 = 6 = (6/2)*2 giving u = 6/2, thus moving 2 to the opposite side with the opposite calculation sign.

Let students see the power of 'flexible bundle-numbers' when the inside multiplication 7*8 = (B-3)*(B-2) = BB-2B-3B+6 = 5B6 = 56 may be illustrated on an outside ten by ten block, thus showing that of course minus times minus must give plus since the 2*3 corner has been subtracted twice.

Let students see that recounting in two units create per-numbers as 2$ per 3kg, or 2$/3kg. To bridge the units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and asking '9kg = ?$' we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

And, that double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Thus finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

And, that double-counting sides in a block halved by its diagonal creates trigonometry: $a = (a/c)*c = \sin A * c$; $b = (b/c)*c = \cos A * c$; $a = (a/b)*b = \tan A * b$. With a circle filled from the inside by right triangles, this also allows phi to be found from a formula: circumference/diameter = $\pi \approx n*\tan(180/n)$ for n large.

And, how recounting and double-counting physical units create per-numbers and proportionality all over STEM, Science, Technology, Engineering and mathematics: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter.

Also, let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation formula. Writing '$y = f(x)$' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying $f(x) = 2 + x$ will give $y = 2+3 = 5$ if $x = 3$, and $y = 2+4 = 6$ if $x = 4$.

Algebra and Geometry, Always Together, Never Apart

Let students enjoy the power and beauty of integrating algebra and geometry.

First, let students enjoy seeing that multiplication creates blocks with areas where 3*7 is 3 7s that may be algebraically recounted in tens as 2.1 tens. Or, that may be geometrically transformed to a square u^2 giving the algebraic equation u^2 = 21, creating root as the reverse calculation to power, $u = \sqrt{21}$. Which may be found approximately by locating the nearest number p below u, here p = 4, so that u^2 = (4+t)^2 = 4^2 + 2*4*t + t^2 = 21.

Neglecting t^2 since t is less than 1, we get 4^2 + 2*4*t = 21, which gives $t = \frac{21 - 4\text{^}2}{4*2}$, or $t = \frac{N - p\text{^}2}{p*2}$, if p is the nearest number below u, where u^2 = N.

As an approximation, we then get $\sqrt{N} \approx p + t = p + \frac{N - p\text{^}2}{p*2}$; or $\sqrt{N} \approx \frac{N + p\text{^}2}{p*2}$, if p^2 < N < (p+1)^2

Then let students enjoy the power and beauty of predicting where a line geometrically intersects lines or circles or parabolas by algebraically solving two equations with two unknowns, also predicted by a computer software.

A Number Seen as a Multiple Numbering

Let students see the number 456 as what it really is, not three ordered digits obeying a place-value system, but three numberings of bundles-of-bundles, bundles, and unbundled ones as expressed in the number-formula, formally called a polynomial: T = 456 = 4*B^2 + 5*B + 6*1, with B = ten.

Let students see that a number-formula contains the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, students see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Let students see calculations as predictions, where 2+3 predicts what happens when counting on 3 times from 2; where 2*5 predicts what happens when adding 2$ 5 times; where 1.02^5 predicts what happens when adding 2% 5 times; and where adding the areas 2*3 + 4*5 predicts how to add the per-numbers when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Reversed Calculation Moving Numbers to Opposite Side

Let students see the subtraction '5-3' as the unknown number *u* that added with 3 gives 5, *u*+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite sign with opposite calculation sign'; a rule that applies also to the other reversed operations:

● the division *u* = 5/3 is the number *u* that multiplied with 3 gives 5, *u**3 = 5

● the root *u* = 3√5 is the factor *u* that applied 3 times gives 5, *u*^3 = 5, making root a 'factor-finder'

● the logarithm *u* = log3(5) is the number *u* of 3-factors that gives 5, 3^*u* = 5, making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to single calculations by unhiding 'hidden bracket' where 2+3*4 = 2+(3*4) since with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s. This will prevent solving the equation 2+3**u* = 14 as 5**u* = 14 with *u* = 14/5, by allowing the hidden bracket to be shown: 2+3**u* = 14, so 2+(3**u*) = 14, so 3**u* = 14-2, so *u* = (14-2)/3, so *u* = 4 to be verified by testing: 2+3**u* = 2+(3**u*) = 2+(3*4) = 2+12 = 14.

Let students enjoy singing a ‘Hymn to Equations’: “Equations are the best we know, they’re solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don’t want to stop!”

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = *a+b*c*, T = *a-b*c*, T = *a+b/c*, T = *a-b/c*, T = *(a+b)/c*, T = *(a-b)/c*, etc. ; as well as formulas as e.g. T = *a*b^c*, T = *a/b^c*, T = *a+b^c*, T = *(a-b)^c*, T = *(a*b)^c*, T = *(a/b)^c*, etc.

Let student place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas *u*^2, *b*^2/4, and *b*/2**u* twice if the cards have the lengths *u* and *u*+*b*/2. Which means that (*u* + *b*/2)^2 = *u*^2 + *b***u* + *b*^2/4. So, with a quadratic equation saying *u*^2 + *b***u* + *c* = 0, the first two terms disappear by adding and subtracting *c*:

(*u* + *b*/2)^2 = *u*^2 + *b***u* + *b*^2/4 = (*u*^2 + *b***u* + *c*) + *b*^2/4 – *c* = 0 + *b*^2/4 – *c* = *b*^2/4 – *c*.

Now, moving to opposite side with opposite calculation sign, we get the solution

(*u* + *b*/2)^2 = *b*^2/4 – *c*

$u + b/2 = \pm\sqrt{b\text{^}2/4 - c}$

$u = -b/2 \pm\sqrt{b\text{^}2/4 - c}$

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, containing also the formulas for constant change: T = *b*x* (proportional), T = *b*x + c* (linear), T = *a*x^n* (elastic), T = *a*n^x* (exponential), T = *a*x^2 + b*x + c* (accelerated).

If not constant, numbers change: constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to ‘post-dict’ what we cannot ‘pre-dict’.

Combining linear and exponential change by *n* times depositing *a*$ to an interest rate *r*%, we get a saving *A*$ predicted by a simple formula, *A/a = R/r*, where the total interest rate *R* is predicted by the formula 1+*R* = (1+*r*)^*n*. Such a saving may be used to neutralize a debt, *Do*, that in the same period has changed to *D* = *Do**(1+*R*).

The formula and the proof are both elegant: in a bank, an account contains the amount *a/r*. A second account receives the interest amount from the first account, *r*a/r = a*, and its own interest amount, thus containing a saving *A* that is the total interest amount *R*a/r*, which gives *A/a = R/r*.

Precalculus Deals with Constant Change

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers creating the operations integration and its reverse, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power and its two inverse operations, root and logarithm.

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$ changing it to 300*1.07 $. Adding 7% *n* times thus changes 300$ to T = 300*1.07^*n* $, leading to the formula for change with a constant change-factor, also called exponential change, T = *b*a^n*. Reversing the question, this formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percent that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor, a, that applied ten times gives the change-factor 1.70. So here the job is ‘factor-finding’ which leads to defining the tenth root of 1.70, i.e. 10√1.70, as predicting the factor, a, that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is “5.4% is the percent that added ten times will give a total change-percent 70%.”

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation *a*^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, *a* = 10√1.70. This resonates with the ‘to opposite side with opposite calculation sign’ method that also solved the equations *a*+3 = 7 by *a* = 7-3, and *a**3 = 20 by *a* = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^*n* = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^*n* = 1.70 then *n* = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e. an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^*n* = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, *n* = log1.07(1.70). Again, this resonates with the ‘to opposite side with opposite calculation sign’ method.

An ideal precalculus curriculum could ‘de-model’ the constant percent change exponential formula T = *b*a^n* to outside real-world problems as a capital in a bank, or as a population increasing or radioactive atoms decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in *n*.

An example is the area of a square T = *s*^2 where a 1% change in the side *s* will give a 2% change in the square, approximately: With To = *s*^2, T1 = (*s**1.01)^2 = *s*^2*1.01^2 = *s*^2* 1.0201 = To*1.0201.

Statistics Deals with Unpredictable Change

Once mastery of constant change-percent is established, it is possible to look at time series in statistical tables asking e.g. “How has the unemployment changed over a ten-year period?” Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = *b+a*n*. The other describes the average yearly change-percent by using a constant change-percent formula, T = *b*a^n*.

These average numbers would allow setting up a forecast predicting the yearly numbers in the ten-year period, if the numbers were predictable. However, they are not, so instead of predicting the number with a formula, we might ‘post-dict’ the numbers using statistics dealing with unpredictable numbers, but still trying to predict a plausible interval by describing the unpredictable random change by nonfictional numbers, median and quartiles, or by fictional numbers, mean and standard deviation.

Calculus Deals with Adding Per-Numbers by their Areas

Likewise, real-world phenomena as unemployment occur in both time and space, so unemployment may also change in space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status, also called contingency tables or crosstabs. The unit-numbers lead to percent-numbers within each of the categories.

To find the total employment percent, the single percent-numbers do not add, they must be multiplied back to unit-numbers to find the total percent. However, once you multiply you create an area, and adding per-numbers by areas is what calculus is about, thus here introduced in a natural way through double-tables from statistical materials, thus introducing integral calculus before differential.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percent-numbers must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5+4.5 unemployed = 9.5 unemployed among 100 persons, i.e. a total of 9.5% unemployment, also called the weighted average. Later, this may be renamed to Bayes formula for conditional probability.

Calculus as adding per-numbers by their areas may also be introduced through mixture problems asking e.g. '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before being added, thus adding by their areas.

Modeling in Precalculus Illustrates Quantitative Literature

Furthermore, the entry of graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum as well as introducing 'quantitative literature' having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2002).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from $x = 0$ to $x = x$ under a constant per-number graph $y = 1$ is $A = x$; and the area A under a constant changing per-number graph $y = x$ is $A = ½*x$^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph $y = x$^2 is $A = 1/3*x$^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips $y*\Delta x$, the total area $A = \Sigma\, y*\Delta x$ is always changed by the last strip. Consequently, $\Delta A = y*\Delta x$, or $\Delta A/\Delta x = y$, or $dA/dx = y$, or $A' = y$ for very small changes.

Reversing the above calculations then shows that if *A* = *x*, then *y* = *A*' = *x*' = 1; and that if *A* = ½**x*^2, then *y* = *A*' = (½**x*^2)' = *x*; and that if *A* = 1/3**x*^3, then *y* = *A*' = (1/3**x*^3)' = *x*^2.

This suggest that to find the area under the per-number graph *y* = *x*^2 over the distance from *x* = 1 to *x* = 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

This makes sense since adding many small strips means adding many small changes, which gives just one final change since all the in-between end- and start-values cancel out:

$$\int_1^3 y * dx = \int_1^3 dA = \Delta_1^3 \text{A} = \Delta_1^3 \left(\frac{1}{3} * x^3\right) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$$

On the calculus course we just leave out the area by renaming it to a 'primitive' or an 'antiderivative' when writing

$$\int_1^3 y * dx = \int_1^3 x^2 * dx = \Delta_1^3 \left(\frac{1}{3} * x^3\right) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$$

A graphing calculator show that this suggestion holds. So, finding areas under per-number graphs not only allows adding per-numbers, it also gives a grounded and natural introduction to integral and differential calculus where integration precedes differentiation just as additions precedes subtraction.

From the outside, regression allows giving a practical introduction to calculus by analysing a road trip where the per-number speed is measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what the maximum speed was, and what distance is covered in total and in the different intervals.

Another example of regression is the project 'Population versus food' looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by *Y*1= 815*1.013^*x* and the food by *Y*2= 300 + 30*x*. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

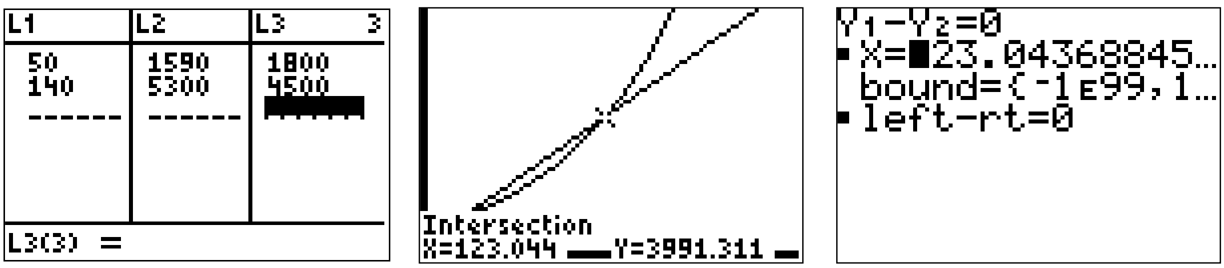

Figure 02. A Malthusian model of population and food levels

A Literature-based Compendium

An example of a literature-based precalculus curriculum is described in a paper called 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2009) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as *Y*1 and *Y*2 and equations were solved by 'solve $Y1$-$Y2 = 0$'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

Modeling STEM Situations

With the increased educational interest in STEM, modeling also allows including science-problems as e.g. the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, precalculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 - T1$. Using a spreadsheet, it is fun to see the behavior of a total that changes with a constant number or a constant percent, as well as with a decreasing number or a decreasing percent, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

So, by focusing on uniting and splitting into constant per-numbers, the ideal precalculus curriculum has constant change-percent as its core. This will cohere with a previous curriculum on constant change-number or linearity; as well as with the following curriculum on calculus focusing on uniting and splitting into locally constant per-numbers, thus dealing with local linearity. Likewise, such a precalculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students' daily life as participants in civil society where tables presented in the media are frequent.

Two Curriculum Examples Inspired by an Ideal Precalculus Curriculum

An example of a curriculum inspired by the above outline was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X^2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

Later, around year 2000, another version was designed but not tested. The curriculum goal was stated as: 'the students develop their number-language so they can participate in social practices involving quantitative descriptions of change and shape.' The curriculum means included

1. Numbers and calculations. Quantities and qualities. Number-language, word-language, meta-language. Unit-numbers and per-numbers, and how to calculate their totals. Equations as predicting statements. Forwards and reverse calculations.

2. Change calculations. Change measuring change with change-number and change-percent and index-number. Calculation rules for the change of a sum, a product and a ratio.

3. Constant change. Change with a constant change-number. Change with a constant change-percent. Change with both.

4. Unpredictable change. Fractals, mean and deviation, 95% confidence interval. Binomial distribution approximated by a normal distribution.

Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X^2 test.

5. Trigonometry. Dividing and measuring earth. Calculation the sides and angles in a triangle.

06. Precalculus in the Danish parallel high school, a case study

In the post-war era, the Organization for Economic Co-operation and Development (OECD) called for increasing the population knowledge level, e.g. by offering a second chance to take a high school degree giving entrance to tertiary education. In Denmark in 1966, this resulted in creating a two-year education called 'Higher preparation exam' as a parallel to the traditional high school. Two levels of two-years mathematics courses were included, a basic precalculus course for those who did not choose the calculus course.

The 1966 Curriculum

The precalculus curriculum came from leaving out small parts of the calculus curriculum, thus being an example of a reduced curriculum.

The goal of the calculus course stated it should 'supply students with knowledge about basic mathematical thinking and about applications in other subject areas, thus providing them with prerequisites for carrying through tertiary education needing mathematics.'

The goal of the precalculus course was reduced to 'supplying students with an impression of mathematical thinking and method and to mediate mathematical knowledge useful also to other subject areas.'

So, where the calculus curriculum has to cohere and be relevant to tertiary education needing mathematics, the precalculus course is a parallel curriculum meant to be relevant to the students themselves and to other high school subjects.

The content of the precalculus curriculum had five sections.

The first section contained basic concepts from set theory as sets, subsets, complementary set, union, intersection, product, difference. The function concept. Mapping into an on a different set, one-to one mapping, inverse mapping (inverse function), composite mappings. The calculus curriculum added nothing here.

Section two contained concepts from abstract algebra: Composition rules. The associative law. The commutative law. Neutral element. Inverse element. The group concept with examples. Rules for operations on real numbers. Numeric value. Here the calculus curriculum added the distributive law, the concept of a ring and a field, the ring of whole numbers as well as quotient classes. The calculus curriculum added nothing here.

Section three contained equations and inequalities. Examples on open statements in one or two variables. Equations and inequalities of degree one and two with one unknown. Equations and inequalities with the unknown placed inside a square root or a numeric sign. Simple examples of Equations and inequalities of degree one and two with two unknowns. Graphical illustration. The calculus curriculum added nothing here.

Section four contained basic functions. The linear function in one variable. A piecewise linear function. The second-degree polynomial. The logarithm function with base ten, the logarithmic scale, the calculator stick, the use of logarithm tables. Trigonometric functions, tables with functions values. Calculations on a right-angled triangle using trigonometric functions. Here the calculus curriculum added rational functions in one variable, exponential functions, and the addition formulas and logarithmic formulas in trigonometry.

Section five contained combinatorics. The multiplication principle. Permutations and combinations. Here the calculus curriculum added probability theory, probability field, and examples of probability based upon combinatorics.

Finally, the calculus curriculum added a section about calculus.

The new set-based mathematics coming into education around 1960 inspired the 1966 precalculus curriculum thus cohering with the university mathematics at that time, but it was not especially relevant to the students. Many had difficulties understanding it and they often complained about seeing no reason for learning it or why it was taught.

In my own class, I presented it as a legal game where we were educating us to become lawyers that could convince a jury that we were using lawful methods to solving equations in one of two different methods by referring to the relevant paragraphs in the law. The first method was the traditional one used at that time way by moving numbers to the opposite side with opposite calculation sign, now legitimized by the theorem that in a group the equation a*u = b has as a solution a^-1*b. The second method was a new way with many small steps where, for each step, you have to refer to laws for associativity, and commutativity etc.; and, where a group contained exactly the paragraphs needed to use this method. Once seen that way, the students found it easy but boring. However, they accepted since they needed the exam to go on, and we typically finished the course in half time allowing time for writing a script for a movie to be presented at the annual gala party.

So, all in all, the 1963 curriculum was coherent with the next step, calculus, and with the university math view at that time, set-based; but it was mostly irrelevant to the students.

The 1974 Curriculum

The student rebellion in 1968 asked for relevance in education, which led to a second precalculus in 1974 revision. Here the goal was stated as ‘giving the students a mathematical knowledge that could be useful to other subjects and to their daily life, as well as an impression of mathematical methods thinking’. Now the curriculum structure was changed from a parallel one to a serial one where all students took the precalculus course and some chose to continue with the calculus course afterwards just specifying in its curriculum what was needed to be added.

The 1974 precalculus curriculum now had four sections.

The first section contained concepts from set theory and logic and combinatorics. Set, subset; solution set to an open statement, examples on solving simple equations and inequalities in one variable; the multiplication principle, combinations.

Section two contained the function concepts: Domain, function value, range; injective function; monotony intervals; inverse function, composite function.

Section three contained special functions; graphical illustration. A linear function, a piecewise linear function, an exponential function; examples of functions defined by tables; coordinate system, logarithmic paper.

Section four contained descriptive statistics. Observations described by numbers; frequency and their distribution and cumulated distribution; graphical illustration; statistical descriptors.

Section five described probability and statistics. A random experiment, outcome space, probability function, probability field; sampling; binomial distribution; binomial testing with zero hypothesis, critical set, significance level, single and double-sided test, failure of first degree.

Section six was called 'Free lessons'. 20m lessons are to be used for studying details in one of the above sections, or together with one or more other school subjects to work with an area applying mathematics.

The second 1974 curriculum thus maintains a basis of set-theory but leaves out the abstract algebra. As to functions, it replaces the second-degree polynomial with the exponential function. Here trigonometry is excluded to be included in the calculus curriculum.

The combinatorics section is to great extent replaced by descriptive statics.

Finally, the section has been added with quite detailed probability theory and testing theory within statistics.

All in all, the coherence with the university set-based mathematics has been softened by leaving out abstract algebra and second-degree polynomial. Instead of introducing a first-degree polynomial together with a second-degree polynomial, the former now is introduced as a linear function together with the exponential function allowing modelling outside change with both a constant change-number and a constant change-percent.

This makes the curriculum more relevant to the students individually as well as to other high school subjects as required by the goal statement.

The quite detailed section on testing theory was supposed to make the curriculum more relevant to students but the degree of detail make it fail to do so by drowning in quite abstract concepts.

The 1990 Curriculum

As the years passed on it was observed that the free hours were used on trigonometry, and on savings and instalments, the first cohering with the following calculus course, the latter highly relevant to many students, and at the same time combining linear and exponential change, the core of the curriculum. This let to designing an alternative curriculum around 1990 to choose instead of the standard curriculum if wanted.

The 1990 curriculum did not change the goal but included the following subjects

1) Numbers, integers, rational and real numbers together with their calculation rules. Number sets. Calculations with power and root.

2) Calculations including percent and interest rates: Average percent, index number, weighed average. Simple and compound interest, saving and installments.

3) Geometry and trigonometry. Similar triangles. Right triangles. Calculations on sides and angles.

4) Functions. The function concept, domain, functional values, range, monotony. Various ways to define a function. Elementary functions as linear, piecewise linear and exponential growth and decay. Coordinate system. Examples of simple equations and inequalities including the functions mentioned above.

5) Probability and statistics. A stochastic experiment. Discrete stochastic variables, probability distribution, mean value, binomial distribution, observation sets described graphically, representation by statistical descriptors, examples of a normal distribution, normal distribution paper.

6) Calculation aids. Pocket calculator, formulas, tables, semi logarithmic paper, normal distribution paper.

The 2005 Curriculum

Then a major reform of the Danish upper secondary high school was planned for 2005. As to precalculus, it was inspired by the entry of graphing calculators and computer assisted systems allowing regression to transform tables into formulas, thus allowing realistic modeling to be included.

Now the goal defined the competences students should acquire:

The students can

- handle simple formulas and translate between symbolic and natural language and use symbolic language to solve simple problems with a mathematical content.
- apply simple statistical models for describing a given data set, pose questions based upon the model and sense what kind of answers are to be expected and knows how to formulate conclusion in a clear language.
- apply relations between variables to model a given data set, can make forecasts, and can reflect on them and their domain of relevance
- describe geometrical models and solve geometrical problems
- produce simple mathematical reasoning
- demonstrate knowledge about mathematical methods, applications of mathematics, and examples of cooperation between mathematics and other sciences, as well as its cultural and historical development
- apply information technology for solving mathematical problems

The means include

- The hierarchy of operations, solving equations graphically and with simple analytical methods, calculating percent and interest rates, absolute and relative change
- Formulas describing direct and inverse proportionality as well as linear, exponential ad power relations between variables

- Simple statistical methods for handling data sets, graphical representation of statistical materials, simple statistical descriptors
- Ratios in similar triangles and trigonometry used for calculations in arbitrary triangles.
- *xy*-plot of data sets together with characteristics of linear, exponential and power relations, the use of regression.
- Additional activities for 25 lessons are examples of mathematical reasoning and proofs, modeling authentic data sets, examples of historical mathematics.

The 2017 Curriculum

Then in 2017 e new reform was made to inspire more students to continue with the calculus level by moving some subjects to the precalculus level:

- interpreting the slope of a tangent as a growth rate in a mathematical model
- combinatorics, basic probability theory and symmetrical probability space
- the function concept and characteristics of linear, exponential and power functions and their graphs
- graphical handling of a quadratic function, and the logarithm functions and their characteristics
- graphical determination of a tangent, and monotony intervals, as well as finding extrema values in a closed interval
- prime characteristics at mathematical models and simple modelling using the functions above alone or in combination.

Relevance and Coherence

The 1966 had internal coherence with the previous and following curriculum, but with the emphasis on abstract algebra, there was little external coherence. It was indirectly relevant to students wanting later to take a calculus course but only little relevant to the daily life of students.

The 1972 curriculum took the consequence and changed from a parallel curriculum to a serial curriculum so that it had internal coherence to the calculus curriculum, and by replacing quadratics with exponential functions, it obtained an external relevance to change calculations with a constant change-number or a constant change-percent. Also, including a considerable amount of probability gave coherence to eternal testing situations, however these were not part of student daily life, so they didn't add to the relevance for students. However, including the free lessons allowed the students to choose areas that they found relevant, in this case interest rates and saving and installment calculations as well as trigonometry.

The 1990 curriculum was inspired by this and re-included trigonometry and interest rates while at the same time reducing probability a little.

The 2005 reform was informed by the occurrence of competence concept as well as the advances in calculation technology. Her the function concept was replaced by variables to make it cohere more with external applications in science and economics and daily life. Now the probability was gone, so this curriculum showed coherence and relevance to external appliers and to the student's daily life as well for other school subjects. It was close to the ideal precalculus curriculum.

The 2017 reform was inspired by the wish to motive more to continue with a calculus course, so part of this was moved down to the precalculus level, making the two levels cohere better, however the things imported had little relevance to the students' daily life.

07. Per-numbers connect Fractions and Proportionality and Calculus and Equations

In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word 'percent' to also talk about other 'per-numbers' as e.g. 'per-five' thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

Mathematics is Hard, or is it?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature.

That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969.

Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Witnessing poor PISA performance, Denmark has lowered the passing limit at the final exam is to around 15% and 20 % in lower and upper secondary school.

Other countries also witness poor PISA performance. And high-ranking countries admit they have a high percentage of low scoring students.

As to finding the cause, Kilpatrick, Swafford, and Findell (2001, p. 36) points out that "what is actually taught in classrooms is strongly influenced by the available textbooks". Personally, working ethnographically in schools in Denmark and abroad, listening to teachers and students confirms the picture that textbooks are followed quite strictly.

So, it seems only natural to look at what is currently being discussed in textbook research e.g. by looking at the Third International Conference on Mathematics Textbook Research and Development, ICMT3, in Germany.

The ICMT3 Conference

The September 2019 ICMT3 conference consisted of 4 keynote addresses, 15 symposium papers, 2 workshops, 40 oral presentations and 13 posters.

The name 'fraction' occurred 212 times in the proceedings, and one of the keynotes addressed the problems students have when asked to find 3/5 of 2/4.

As to fractions, Ripoll and Garcia de Souza writes that "The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School's classrooms and textbooks." (Rezat et al, 2019, p. 131). Looking at 13 Brazilian textbooks from 4th to 7th grade they conclude that

> The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like "Two given fractions *a/b* and *c/d* are equivalent if and only if *ad = bc*." In most cases only a partial equivalence criterion is presented, like "Two fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number."

The authors thus take it that fractions should obey the New Math 'set-centrism' (Derrida, 1991) by saying: in a set-product of integers, a fraction is an equivalence class created by the equivalence relation statin that *a/b ~c/d* if *a*d = b*c*; and thus neglect the pre-setcentric version mentioned above where a fraction keeps its value by being expanded or shortened; as well as the post-setcentric version seeing a fractions as an example of a per-number, described later in this paper.

Confirming in the afterwards discussion that fractions are introduced by the part-whole model, an argument was made that if a fraction is defined as a part of a whole then a fraction must always be a fraction of something; thus being an operator needing a number to become a number, and not a number in itself.

Of course, in a 30 minutes presentation there is little time to discuss the nature of fractions thoroughly, so this question needs to be addressed in more details.

Also addressing middle school problems, Watanabe writes that "Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn." (p. 353)

And, talking about proportionality, Memis and Yanik writes that "Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks." (p. 245)

Textbooks must follow curricula, and middle school problems were also mentioned at the International Commission on Mathematical Instruction Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, in Japan November 2018. Here in his plenary talk, McCallum after noting that "a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships" challenged the audience by asking "What is the difference between 5/3 and 5÷3?" (ICMI, 2018, p. 4).

So, this paper will focus on these challenges by asking: "Is there a hidden different way to see and teach core middle school concepts as fractions, quotients, ratios, rates and proportionality?" A question that might be answered answer by Difference-research (Tarp, 2018) using sociological imagination (Mills, 1959) to search for differences making a difference by asking two questions: 'Can this be different – and will the difference make a difference?'

Different Ways of Seeing Fractions

In a typical curriculum using a 'part-whole' approach, fractions are introduced after division has been taught as sharing a whole in equal parts: 8/4 is 8 split in 4 parts or 8 split by 4.

Representing the whole geometrically as a bar or a circle, dividing in 4 parts creates 4 pieces each called ¼ of the total. Assigning numbers to the whole allows finding ¼ of e.g. 8 by the division 8/2. Then the fraction ¾ means taking ¼ three times, so that taking ¾ of 8 involves two calculations, first 8/4 as 2, then 3*2 as 6, so that ¾ of 8 is 8/4*3, later reformulated to one calculation, 8*¾, multiplying the integer 8 with the rational number ¾.

However, in the ‘part-whole’ approach a fraction is a fraction of something, thus introducing a fraction as an operator needing a number to become a number.

This becomes problematic when the fraction later is claimed to be a point on a number line, i.e. a number in its own right, a rational number, defined by set-centrism as an equivalence class in a set-product as described above.

Furthermore, set-centrism is problematic in itself by making mathematics a self-referring ‘Meta-matics’, defined from above as examples from abstractions instead of from below as abstractions from examples.

And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

To avoid self-reference Russell introduced a type-theory allowing reference only to lower degree types. Consequently, fractions could not be numbers since they refer to numbers in their setcentric definition.

Neglecting the Russell paradox by defining fractions as rational numbers leads to additional educational questions: When are two fractions equal? How to shorten or expand a fraction? What is a fraction of a fraction? Which of two fractions is the bigger? How to add fractions? Etc.

Fraction later leads on to percentages, the special fractions having 100 as the denominator, which leads to the three percentage questions coming from the part-whole formula defining a fraction, fraction = part/whole.

Seeing fractions as, not numbers, but operators still raises the first three questions whereas the two latter are meaningless since the answer depends on what whole they are taken of as seen by ‘the fraction paradox’ where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes, and never 7 cokes of 6 bottles.

Adding numbers without units may be called ‘mathe-matism’, true inside but seldom outside the classroom. And strangely enough the two latter questions are only asked with fractions and seldom with percentages.

Ratios and Rates

When introduced, ratios are often connected to fractions by saying that splitting a total in the ratio 2:3 means splitting it in 2/5 and 3/5.

Where fractions and ratios typically are introduced without units, rates include units when talking e.g. about speed as the ratio between the meter-number and the second-number, speed = 2m/3s.

Per-numbers Occur when Double-counting a Total in two Units

The question “What is 2/3 of 12?” is typically rephrased as “What is 2 of 3 taken from 12?” Seldom it is rephrased as “What is 2 per 3 of 12?”. Even if the word ‘per’ occurs in many connections, meter per second, per hundred, etc.

When we rephrase “taking 30% of 400” as “taking 30 per 100 of 400”, why don’t we rephrase “taking 3/5 of 400” as “taking 3 per 5 of 400”?

In short, why don’t we rephrase 3/5 both as ‘3 of 5’ and as ‘3 per 5’?

In his conference paper, Tarp (p. 332) introduces per-numbers and recounting:

An additional learning opportunity is to write and use the ‘recount-formula’ T = (T/B)*B, saying “From T,T/B times B can be taken away”, to predict counting and recounting examples. (..) Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Of course, you might argue that we cannot write ‘6$ = 9kg’ since the units are not the same. But then again, we write ‘2 meter = 200 centimeter’ even if the units are different, and we are allowed to do so since the bridge between the two units is the per-number 1m/100cm. Likewise, we should be allowed to write ‘6$ = 9kg’ since the bridge between the two units for now is the per-number 2$/3kg.

The difference is that the per-number between meter and centimeter is globally valid, whereas the per-number between kilogram and dollar is only locally valid. Still, it has validity as long as you are talking about the same outside total.

The interesting thing is that by including units, per-numbers connects fractions and proportionality. And that by including units, the recount-formula gives an introduction to fractions saying that 1/3 is ‘1 counted in 3s’: 1 = (1/3)*3 = 1/3 3s.

Fractions as Per-numbers

With per-numbers coming from double-counting the same total in two units, we see that when double-counting in the same unit, the unit cancels out and we get a ratio between two numbers without units, a fraction as e.g. 3$/8$ = 3/8.

Reversely, inside fractions without units may be ‘de-modeled’ outside by adding new units, e.g. ‘good’ and ‘total’ transforming 3/8 to 3g/8t. This allows per-numbers and recounting to be used when solving the three fraction questions:

“What is 3/4 of 60?”, and “20 is what of 60?”, and “20 is 2/3 of what?”

Asking “What is ¾ of 60” means asking “What is 3 per 4 of 60”, or de-modeled with units, “What is 3g per 4t of 60t”,

Of course, 60t is not 4t, but 60 can be recounted in 4s by the recount-formula, 60t = (60/4)*4t = (60/4)*3g = 45g, giving the inside answer “¾ of 60 is 45”.

Asking “20 is which fraction of 60” means asking “What fraction is 20 per 60”, or with units, “Which per-number is 20g per 60t”, giving the answer directly as 20g/60t or 20/60 g/t. Here we might look for a common unit in 20 and 60 to cancel out, e.g. 20, giving 20/ 60 = 1 20s/3 20s = 1/3. This allows transforming the outside answer “20 per 60 is 1 per 3” to the inside answer “20 is 1/3 of 60”.

Asking “20 is 2/3 of what” means asking “20 is 2 per 3 of what”, or with units, “20g is 2g per 3t of which total”. Of course, 20g is not 2g, but 20 can be recounted in 2s by the recount-formula, 20g = (20/2)*2g = (20/2)*3t = 30t. This allows transforming the outside answer “20 is 2 per 3 of 30” to “20 is 2/3 of 30.”

Expanding and Shortening Fractions

With fractions as per-numbers coming from double counting in the same unit that has cancelled out, we are always free to add a common unit to both numbers.

Using numbers as units will expand the fraction:

2/3 = 2 7s/3 7s = 2*7/3*7 = 14/21

Reversely, if both numbers contain a common unit, this will cancel out:

14/21= 2*7/3*7 =2 7s/ 3 7s = 2/3

Taking Fractions of Fractions, the Per-number Way

One of the keynotes pointed out that to understand that 6/20 is the answer to the question “What is 3/5 of 2/4?” we must draw a rectangle with 4 columns if which 2 are yellow, and with 5 rows of which 3 are blue. Then 6 double-colored squares out of a total of 20 squares gives an understanding that 3/5 of 2/4 is 6/20, which also comes from multiplying the numerators and the denominators.

Seeing fractions as per-numbers the question “What is 3/5 of 2/4?” translates into “What is 3 per 5 of 2 per 4”. Knowing that using per-numbers to bridge two units involves recounting them in the per-number which again involves division, we might begin with a number that is easily recounted in 4s and 5s, e.g. 4*5 = 20, and reformulate the question to “3 per 5 of 2 per 4 is what per 20?”.

To find 2 per 4 of 20 means finding 2g per 4t of 20t, so we recount 20 in 4s:

20t = (20/4)*4t = (20/4)*2g = 10g, so 2 per 4 of 20 is 10.

To find 3 per 5 of 10 means finding 3g per 5t of 10t, so we recount 10 in 5s:

10t = (10/5)*5t = (10/5)*3g = 6g, so 3 per 5 of 10 is 6

Thus, we can conclude that 3 per 5 of 2 per 4 is the same as 6 per 20, or, with fractions, that 3/5 of 2/4 is 6/20, again coming from multiplying the numerators and the denominators.

Of course, we could discuss, which method gives a better understanding, but we might never reach an answer, given the many different understandings of the word ‘understanding’

Direct and Inverse Proportionality

Using a coordinate system with decimal numbers comes natural if bundle-writing totals in tens so e.g. T = 26 becomes T = 2.6 tens. This allows fixing a 3x5 box in the corner with the base and the height on the *x*- and *y*-axes. The recount-formula T = (T/B)*B then shows a total T as a box with base *x* = B and height *y* = T/B.

To keep the total unchanged, increasing the base will decrease the height (and vice versa) making the upper right corner create a curve called a hyperbola with the formula height = T/base, or *y* = T/*x*, showing inverse proportionality.

In a 3x5 box, the raise of the diagonal is the per-number 3/5. Expanding or shortening the per-number by adding or removing extra units will make the diagonal longer or shorter without changing direction. This will make the upper right corner move along a line with the formula 3/5 = height/base = *y*/*x*, or y = 3/5**x*, showing direct proportionality.

Adding Fractions, the Per-number Way

Adding per-numbers occurs in mixture problems asking e.g. “What is 2kg at 3$/kg plus 4kg at 5$/kg?”. We see that the unit-numbers 2 and 4 add directly, whereas the per-numbers cannot add before multiplication changes them to unit-numbers. However, multiplication creates the areas 2*3 and 4*5, which gives the answer: 2kg at 3$/kg + 4kg at 5$/kg gives (2+4)kg at (2*3+4*5)/(2+4)$/kg.

So we see that per-numbers add by the areas under the per-number graph in a coordinate system with the kg-numbers and the per-numbers on the axes.

But adding area under a graph is what integral calculus is all about. Only here, the per-number graph is piecewise constant, where the velocity graph in a free fall, is not piecewise, but locally constant, which means that the total area comes from adding up very many small area-strips.

This may be done by observing that the total area always changes with the last area-strip thus creating a change equation $\Delta A = p*\Delta x$, which motivates differential calculus to answer questions as $dA/dx = p$, thus finding the area formula that differentiated gives the give per-number formula p, e.g. $d/dx\ (x^2) = 2*x$.

Interchanging epsilon and delta to change piecewise constancy to local may be postponed to high school, that would benefit considerably by a middle school introduction of integral calculus as adding locally constant per-numbers by the area under the per-number graph, using differential calculus to find the area in a quicker way than asking a computer to add numerous small area-strips.

Solving Proportionality Equations by Recounting

Reformulating the recount-formula from T = (T/B)*B to T = c*B shows that with an unknown number u it may turn into an equation as 8 = u*2 asking how to recount 8 in 2s, which of course is found by the recount-formula, $u*2 = 8 = (8/2)*2$, thus providing the equation $u*2 = 8$ with the solution $u = 8/2$ obtained by isolating the unknown by moving a number to the opposite side with the opposite sign.

This resonates with the formal definition of division saying that 8/2 is the number u that multiplied by 2 gives 8: if $u*2 = 8$ then $u = 8/2$.

Set-centrism of course prefers applying and legitimizing all concepts from abstract algebra's group theory (commutativity, associativity, neutral element and inverse element) to perform a series of reformulations of the original equation: $2*u = 8$, so $(2*u)*½ = 8*½$, so $(u*2)*½ = 4$, so $u*(2*½) = 4$, so $u*1 = 4$, so $u = 4$.

Seven Ways to Solve the two Proportionality Questions

The need to change units has mad the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

● Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.

● Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: $2/5 = 7/x$, where cross-multiplication gives $2*x = 7*5$, which gives $x = (7*5)/2 = 17.5$. Q2: $2/5 = x/12$, where cross-multiplication gives $5*x = 12*2$, which gives $x = (12*2)/5 = 4.8$.

● Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.

● Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.

● Modeling a linear function $f(x) = c*x$, with $f(2) = 5$, $f(7) = ?$, and $f(x) = 12$.

A Case: Peter, about to Peter Out of Teaching

As a new middle school teacher, Peter is looking forward to introducing fractions to his first-year class coming directly from primary school where the four basic operations have been taught so that Peter can build upon division when introducing fractions in the traditional way. However, Peter is shocked when seeing many students with low division performance, and some even showing dislike when division is mentioned. So, Peter soon is faced with a class divided in two, a part that follows his introduction of fractions, and a part that transfers their low performance or dislike from divisions to fractions.

The following year seeing his new class behaving in the same way, Peter is about to give up teaching when a colleague introduces him to a different approach where division is used for bundle-counting instead of sharing called 'Recounting fingers with flexible bundle-numbers'. The colleague also recommends some YouTube videos to watch and some material to download from the MATHeCADEMY.net to try it yourself.

Inspired by this, Peter designs a micro-curriculum for his class aiming at introducing the class to bundle-counting leading to the recount-formula leading to double-counting in two units leading to per-numbers having fractions as the special case with like units.

"Welcome class, this week we will not talk about fractions!" "?? Well, thank you Mr. teacher, then what will we do?" "We will count our five fingers." "Ah, Mr. teacher we did that in preschool!" "Correct, in preschool we counted our fingers in ones, now we will bundle-count them in 2s and 3s and 4s using bundle-writing. In this way we will see that a total can be counted in three different ways: overload, standard and underload. Look here:

Outside, we have   I I I I I = II I I I = II II I = II II I+

Inside, we write: T =   5   = 1B3 2s = 2B1 2s = 3B-1 2s

We will call this to recount 5 with flexible bundle-numbers. Now count the five fingers in 3s and 4s in the same way. Later, we will count all ten fingers."

The following class, Peter began by rehearsing.

"Welcome class. Yesterday we saw that an outside total can be recounted in different units, and that the result inside can be bundle-written in three ways, with overload, standard and underload. Today we will begin by recounting twenty in hands, in six-packs and in weeks. Why twenty? Because counting in twenties was used by the Vikings who also gave us the words eleven and twelve, meaning one-left and two-left in Viking language."

Later, Peter introduced the recount-formula:

"Here we have 6 cubes that we will count in 2s. We do that by pushing away 2-bundles, and write the result as T = 6 = 3B 2s. We see that the inside division stroke looks like an outside broom pushing away the bundles. And asking the calculator, 6/2, and we get the answer 3 predicting it can be done 3 times. We can illustrate this prediction with a recount formula 'T =

(T/B)xB' saying that 'from the total T, T/B times, B can be pushed away'. So, from now on, 6/2 means 6 recounted in 2s; and 3x2 means 3 bundles of 2s. And since it is counted in tens, 42 is seen as 4B2 or 3B12 or 5B-8 using flexible bundle-numbers.

Now let us read 42/3 as 4bundle2 tens recounted in 3s; and let us use flexible bundle-numbers to rewrite 4B2 with an overload as 3B12. Then we have T = 42 /3 = 4B2 /3 = 3B12 / 3 = 1B4 = 14. We notice that squeezing a box from base 10 to base 3 will increase the height, here from 4.2 to 14.

And, by the way, flexible bundle-numbers also come in handy when multiplying: Here 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 bundles and 56 unbundled singles, which can be recounted to remove the overload:

T = 7 x 4B8 = 28B56 = 33B6 = 336."

The third day Peter repeated the lesson with 7 cubes counted in 3s to show that where the unbundled single was placed would decide if the total should be written using a decimal number when placed next-to as separate box of ones, T = 2B1 3s = 2.1 3s. Placed on-top means missing 2 to form a bundle, thus written as T = 3B-2 3s = 3.-2 3s. Or it means recounting 1 in 3s as 1 = (1/3)x3 = 1/3 3s, a fraction.

Later, Peter introduced per-numbers and fractions as described above, which allowed Peter to work with fractions and ratios and proportionality at the same time; and later to introduce calculus as adding fractions and per-numbers by areas.

Observing the increase of performance and the disappearance of dislike, the headmaster suggested to the headmaster of the nearby primary school that Peter be used as a facilitator for in-service teacher training. This would allow primary school children to meet fractions and negative numbers and proportionality when recounting and double-counting a total in a new bundle-unit.

08. A refugee camp curriculum

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses om proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional leaning materials?'. This motivates another question 'How much mathematics can be learned as 'finger-math' using the examples of Many coming from the body as fingers, arms, toes and legs?'

So the goal of ‘refugee camp mathematics’ is to learn core mathematics through ‘Finger-math’ disclosing how much math comes from counting the fingers.

Focus 01: Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Activity 01. With outside things (sticks, cars, dolls, animals), many ones are rearranged into one many-icon with as many things as it represents. Inside, we write the icon with as many strokes as it represents. Observe that the actual digits from 1 to 9 are icons with as many strokes as they represent if written less sloppy. A discovery glass showing nothing is an icon for zero. When counting by bundling in tens, ten become ‘1 Bundle, 0 unbundled’ or 1B0 or just 10, thus needing no icon since after nine, a double-counting takes place of bundles and unbundled.

Focus 02. Counting Ten Fingers in Various Ways

Activity 01. Double-count ten fingers in bundles of 5s and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, 0B4, 0B5 or 1B0. Then continue with saying “1B1, …, 1B5 or 2B”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less4, 1B-3, 1B-2, 1B-1, 1B or 1B0. Then continue with saying “2B-4, …, 2B or 2B0”. ● Outside, show the fingers as ten ones. ● Outside, show ten fingers as 1 5s and 5 1s; inside say “The total is 1Bundle5 5s” and write ‘T = 1B5 5s’. ● Outside, show ten fingers as 2 5s; inside say “The total is 2Bundle0 5s” and write ‘T = 2B0 5s’.

Activity 02. Double-count ten fingers in bundles of tens and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, …, 0B9, 0Bten, or 1B0”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less9, 1B-8, …, 1B-2, 1B-1, 1B or 1B0.

Activity 03. Counting ten fingers in bundles of 4s using ‘flexible bundle-numbers’.

● Outside, show the fingers as ten ones, then as one ten. ● Outside, show ten fingers as 1 4s and 6 1s; inside say “The total is 1Bundle6 4s, an overload” and write ‘T = 1B6 4s’ ● Outside, show ten fingers as 2 4s and 2 1s; inside say “The total is 2Bundle2 4s, a standard form” and write ‘T = 2B2 4s’. ● Outside, show ten fingers as 3 4s less 2; inside say “The total is 3Bundle, less2, 4s, an underload” and write ‘T = 3B-2 4s’.

Activity 04. Counting ten fingers in bundles of 3s using ‘flexible bundle-numbers’.

● Outside, show ten fingers as 1 3s and 7 1s; inside say “The total is 1Bundle7 3s, an overload” and write ‘T = 1B7 3s’. ● Outside, show ten fingers as 2 3s and 4 1s; inside say “The total is 2Bundle4 3s, an overload” and write ‘T = 2B4 3s’. ● Outside, show ten fingers as 3 3s and 1 1s; inside say “The total is 3Bundle1 3s, a standard form” and write ‘T = 3B1 3s’. ● Outside, show ten fingers as 4 3s less 2; inside say “The total is 4Bundle, less2, 3s, an underload” and write ‘T = 4B-2 3s’.

Activity 05. Counting ten fingers in bundles of 3s, now also using bundles of bundles.

● Outside, show ten fingers as 3 3s (a bundle of bundles) and 1 1s; inside say “The total is 1BundleBundle1 3s” and write ‘T = 1BB1 3s’. Now, inside say “The total is 1BundleBundle 0 Bundle 1 3s” and write ‘T = 1BB 0B 1 3s’. Now, inside say “The total is 1BundleBundle 1 Bundle, less2, 3s” and write ‘T = 1BB 1B -2 3s’.

Focus 03. Counting Ten Sticks in Various Ways

The same as Focus 02, but now with sticks instead of fingers.

Focus 04. Counting Ten Cubes in Various Ways

The same as Focus 02, but now with cubes, e.g. centi-cubes or Lego Bricks, instead of fingers. When possible, transform multiple bundles into 1 block, e.g. 2 4s = 1 2x4 block; inside say "The total is 1 2x4 block" and write 'T = 2B0 4s.'

Focus 05. Counting a Dozen Finger-parts in Various Ways

Except for the thumps, our fingers all have three parts. So, four fingers have three parts four times, i.e. a total of T = 4 3s = 1 dozen finger-parts.

Focus 05 is the same as focus 02, but now with a dozen finger-parts instead of ten fingers.

Focus 06. Counting a Dozen Sticks in Various Ways

Focus 06 is the same as focus 03, but now with a dozen sticks instead of ten.

Focus 07. Counting a Dozen Cubes in Various Ways

Focus 07 is the same as focus 04, but now with a dozen cubes instead of ten.

Focus 08. Counting Numbers with Underloads and Overloads.

Activity 01. Totals counted in tens may also be recounted in under- or overloads.

● Inside, rewrite T = 23 as T = 2B3 tens, then as 1B13 tens, then as 3B-7tens. ● Try other two-digit numbers as well. ● Inside, rewrite T = 234 as T = 2BB3B4 tens, then as T = 2BB 2B14, then as T = 2BB 4B-6. Now rewrite T = 234 as T = 23B4, then as 22B14, then as 24B-6. Now rewrite T = 234 as T = 3BB-7B4, then as 3BB-6B-6. ● Try other three3-digit numbers as well.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Activity 01. Transform the three outside counting operations (push, lift and pull) into three inside operation-icons: division, multiplication and subtraction.

● Outside, place five sticks as 5 1s. ● Outside, push away 2s with a hand or a sheet; inside say "The total 5 is counted in 2s by pushing away 2s with a broom iconized as an uphill stroke" and write 'T = 5 = 5/2 2s'. ● Outside, rearrange the 2 2s into 1 2x2 block by lifting up the bundles into a stack; inside say "The bundles are stacked into a 2x2 block by lifting up bundles iconized as a lift" and write 'T = 2 2s = 2x2'. ● Outside, pull away the 2x2 block to locate unbundled 1s; inside say "The 2x2 block is pulled away, iconized as a rope" and write 'T = 5 – 2x2 = 1'.

Five counted in 2s:

I I I I I (push away 2s) II II I (lift to stack) II/II I (pull to find unbundles ones) II/II I

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Activity 01. Use a calculator to predict a bundle-counting result by a recount-formula T = (T/B)xB, saying "from T, T/B times, B is pushed away", thus using a full number-language sentence with a subject, a verb and a predicate.

● Outside, place five cubes as 5 1s. ● Outside, push away 2s with a 'broom'; inside say "Asked '5/2', a calculator answers '2.some', meaning that 2 times we can push ways bundles of 2s. ● Outside , stack the 2s into one 2x2 stack by lifting; inside say "We lift the 2 bundles into one 2x2 stack, and we write T = 2 2s = 2x2 ● Outside, we locate the unbundled by, from 5 pulling away the 2x2 block; inside we say "Asked '5-2x2', a calculator answers '1'. We write T = 2B1 2s and say "The recount-formula predicts that 5 recounts in 2s as T = 2B1 2s, which is tested by recounting five sticks manually outside."

Activity 02. The same as activity 01, but now with 4 3s counted in 5s, 4s and 3s.

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s. ● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say "seven ones recounts as 2B1 3s" and write T = 2B1 3s. ● Outside, place the single next-to the stack. Inside say "Placed next-to the stack the single becomes a decimal-fraction '.1' so now seven recounts as 2.1 3s" and write T = 2.1 3s. ● Outside, place the single on-top of the stack. Inside say "Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s" and write T = 2 1/3 3s. Now, inside say "Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s" and write T = 3.-2 3s. Finally, inside say "With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s."

Activity 02. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 5s.

Focus 12. Recount in a New Unit to Change Units, Predicted by the Recount-Formula

Activity 01. When bundle-counting, all numbers have units that may be changed into a new unit by recounting predicted by the recount-formula.

● Outside, chose 3 4s to be recounted in 5s. ● Outside, rearrange the block in 5s to find the answer T = 3 4s = 2B2 5s. Inside use the recount-formula to predict the result, and say "three fours recounts as 2B2 5s" and write T = 3 4s = 2B2 5s = 3B-3 5s = 2 2/5 5s. Repeat with other examples as e.g. 4 5s recounted in 6s.

Focus 13. Recount from Tens to Icons

Activity 01. A total counted in tens may be recounted in icons, traditionally called division.

● Outside, chose 29 or 2B9 tens to be recounted in 8s. ● Outside, rearrange the block in 8s to find the answer T = 29 = 3B5 8s and notice that a block that decreases its base must increase its height to keep the total the same. Inside use the recount-formula to predict the result, and say "With the recount-formula, a calculator predicts that 2 bundle 9 tens recounts as 3B5 8s" and write T = 29 = 2B9 tens = 3B 5 8s = 4B-3 8s = 3 5/8 8s. Repeat with other examples as e.g. 27 recounted in 6s.

* Now, inside reformulate the outside question 'T = 29 = ? 8s' as an equation using the letter u for the unknown number, u*8 = 24, to be solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign. Use an outside ten-by-ten abacus to see that when a block decreases its base from ten to 8, it must increase its height from 2.4 to 3. Repeat with other examples as e.g. 17 = ? 3s.

Focus 14. Recount from Icons to Tens

Activity 01. Oops, without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. Use an outside ten-by-ten abacus to see that when a block increases its base from 7 to ten, it must decrease its height from 3 to 2.1.

Activity 02. Use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Focus 15. Double-Counting in Two Physical Units

Activity 01. We observe that double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$. Repeat with other examples as e.g. 4$ per 5days.

Focus 16. Double-Counting in the Same Unit Creates Fractions

Activity 01. Double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%. We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Repeat with other examples as e.g. 2$/5$.

Focus 17. Mutually Double-Counting the Sides in a Block Halved by its Diagonal

Activity 01. Recount sides in a block halved by its diagonal? Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: $a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$. Use these formulas to predict the sides in a half-block with base 6 and angle 30 degrees. Use these formulas to predict the angles and side in a half-block with base 6 and height 4.

Focus 18. Adding Next-to

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Since 3*5 is an area, adding next-to in 8s means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2B5 8s.

Focus 19. Reversed Adding Next-to

Activity 01. With T1 = 2 3s and T2 adding next-to as T = 4 7s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Asking '3 5s and how many 3s total 2B6 8s?', using sticks will give the answer 2B1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

$y' = \Delta y/t = (y2 - y1)/t$.

Focus 20. Adding On-top

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4B1 5s.

Focus 21. Reversed Adding On-top

Activity 01. With T1 = 2 3s and T2 as some 5s adding to T = 4 5s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Reversed addition is called backward calculation or solving equations.

Focus 22. Adding Tens

Activity 01. With T1 = 23 and T2 = 48, what is T1+T2 id added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Focus 23. Subtracting Tens

Activity 01. "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Focus 24. Multiplying Tens

Activity 01. "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Focus 25. Dividing Tens

Activity 01. "What is 348 recounted in 6s?" Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Focus 26. Adding Per-Numbers

Activity 01. "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

Activity 02. Two groups of voters have a different positive attitude to a proposal. How to find the total positive attitude?

● Asking "20 voters with 30% positive + 60 voters with 10% positive = 80 voters with ? positive." Here we see that the unit-numbers 20 and 40 add directly whereas the per-numbers 30% and 10% add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Focus 27. Subtracting Per-Numbers

Activity 01. "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Focus 28. Adding Differences

Activity 01. Adding many numbers is time-consuming, but not if the numbers are changes, then the sum is simply calculated as the change from the start to the end-number.

● Write down ten numbers vertically. The first number must be 3 and the last 5, the rest can be any numbers between 1 and 9. In the next column write down the individual changes 'end-start'. In the third column add up the individual changes along the way. Try to explain why the result must be 5-3 regardless of the in-between numbers.

● Draw a square with side *n*. Let *n* have a small positive change *t*. Show that the square will change with two next blocks when disregarding the small *t*x*t* square. This shows that the change in an *n***n* square is 2**n***t*, so if we want to add arears under a *y* = 2**n* curve we must add very many small areas $y*t = 2*n*t$. However, since each may be written as a change in a square, we just have to find the change of the square from the start-point to the end-point. That is how integral calculus works.

Focus 29. Finding Common Units

Activity 01. "Only add with like units, so how add $T = 4ab$^2 + $6abc$?". Here units come from factorizing: $T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b)$.

Focus 30. Finding Square Roots

Activity 01. A 7x7 square can be recounted in tens as 4.9 tens. The inverse question is how to transform a 6x7 block into a square, or in other words, to find the square root of 4.2 tens. A quick way to approach a relevant number is to first find two consecutive numbers, p and p+1, that squared are too low and too high. Then an approximate value for the square root may be calculated by using that

if p^2 < N < (p+1)^2, then $\sqrt{N} \approx \frac{N + p\text{^}2}{p*2}$.

Final Remarks

A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional block numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, the need for parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

09. Do we really need parallel CURRICULA?

Why do we need different curricula for different groups of students? Why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups, so why does the number-language?

Both languages have two levels, a language level describing the 'outside' world, and a grammar level describing the 'inside' language. In the word-language, the language level is for all students and includes many examples of real-world descriptions, both fact and fiction. Whereas grammar level details are reserved for special students. Could it be the same with the number-language, teaching the language level to all students including many examples of fact and fiction? And reserving grammar level details to special students?

Before 1970, schools taught language as an example of its grammar (Chomsky, 1965). Then a reaction emerged in the so-called 'communicative turn' in language education. In his book 'Explorations in the function of language' Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself

> has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a less correct language to be used for communication about outside things and actions.

Thus, in language teaching the communicative turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language. So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

After all, the word language seems more voluminous with its many letters, words and sentence rules. In contrast, a pocket calculator shows that the number language contains ten digits together with a minor number of operations and an equal sign. And, where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. (Tarp, 2018)

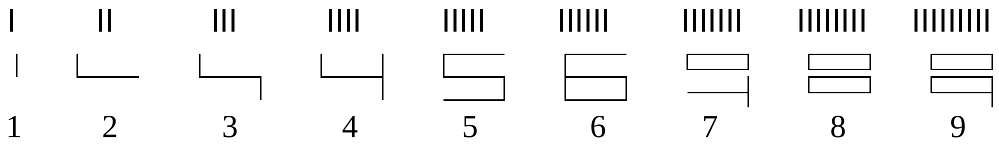

Figure 03. Digits as icons with as many sticks as they represent.

Furthermore, also the operations are icons describing how we total by counting unbundled, bundles, bundles of bundles etc. Here division iconizes pushing away bundles to be stacked, iconized by a multiplication lift, again to be pulled away, iconized by a subtraction rope, to identify unbundled singles that are placed next-to the stack iconized by an addition cross, or by a decimal point; or on-top iconized by a fraction or a negative number.

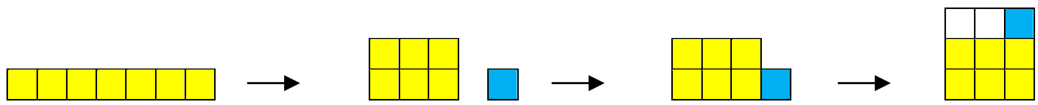

Figure 04. Seven counted as 2 3s & 1 or 2B1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

The operations allow predicting counting by a recount-sentence or formula ‘T = (T/B)*B’ saying that ‘from T, T/B times, B can be taken away’, making natural numbers as bundle- or block numbers as e.g. T = 3B2 4s or T = 3*4+2. And, using proportionality to change the unit when two blocks need the same unit to be added on-top, or next-to in a combined unit called integral calculus.

So, it seems as if early childhood education may introduce core mathematics as proportionality, solving equations, and integral calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction. Thus, there is indeed an opportunity to design a core curriculum in mathematics for all students without splitting it up in tracks. But, only if the word- and the number-language are taught and learned in the same way by describing outside things and actions in words and in numbers coming from counting and adding.

So, why not introduce a paradigm shift by teaching the number-language and the word-language in the same way through its use, and not before, thus allowing both languages being taught in the space between the inside language and the outside world.

Why keep on teaching the number-language in the space between the language and its meta-language or grammar, which makes the number-language more abstract, leaving many educational challenges unsolved despite close to half a century of mathematics education research.

Why not begin teaching children how to number, and stop teaching children about numbers and operation to be explained and learned before they can be applied to the outside world.

Why not accept and develop children's already existing 'many-sense', instead of teaching them the eight different aspects of what is called 'number-sense' described by Sayers and Andrews (2015) that after reviewing research in the Whole Number Arithmetic domain created a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

And, why not simply let children talk about counting and adding constant and changing unit-numbers and per-numbers using full sentences with a subject, a verb, and a predicate; instead of teaching them the eight different components of what is called 'mathematical competencies' (Niss, 2003), thus reducing their numbers from eight to two: count and add (Tarp, 2002)?

So maybe we should go back to the mother Humboldt university in Berlin and reflect on Karl Marx thesis 11 written on the staircase: ”Die Philosophen haben die Welt nur verschieden interpretiert; es kömmt drauf an, sie zu verändern.” (The philosophers have only interpreted the world, in various ways. The point, however, is to change it.)

10. Conclusion

Let us return to the dream of the National Council of Teachers of Mathematics, to "provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals." Consequently, "everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics." Furthermore, let us also accept what the council write about numbers: "Number pervades all areas of mathematics."

So let us look for a curriculum that allows the students to understand and use and numbers, and see how far such a curriculum can carry all students without splitting into parallel tracks.

Now, what does it mean to understand a number like 456?

Is the ability to say that the three digits obey a place-value system where, from right to left, the first digits is ones, then tens, then hundred, then thousands, then, oops no-name unless we use the Chinese name wan, then no-name, then million, then no-name, then no-name, then billions or milliards, etc. Names and lack of names that give little meaning to children where only few understand why ten has its own name but not its own icon but has two digits as 10.

On the other hand, is it the ability to understand that of course ten becomes 10 since it is short for '1 bundle and no singles'? And, that it would have been 20 had we counted in bundles of 5s instead as they do on an eastern abacus, where the two digits 10 then would be used for the bundle size 5.

So that ten is just another word for bundle B, and hundred for a bundle-bundle, BB, i.e. bundling 2 times; and thousand for bundle-bundle-bundle, BBB, i.e. bundling 3 times, etc. Here we never end in a situation with no name. As to the significance of bundles of bundles, Dienes (2002, p.1) writes:

> The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. This is why our system of numbering, introduced in the middle ages by Arabs, is called the place value system. My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. Further, in order to understand any mathematical concept, we must understand how the variables that it contains are related to each other. The concept of power has two variables: the base and the exponent. In order to get to understand the concept of power, it seems reasonable that we should have some experience of varying both variables. In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens), since for some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention. So I have been suggesting, for the past half century, that different bases be used at the start, and to facilitate understanding of what is going on, physical materials embodying the powers of various bases should be made available to children. Such a system is a set of multibase blocks, which I introduced in England, Italy and Hungary in the 1950's. Educators today use the "multibase blocks", but most of them only use the base ten, yet they call the set "multibase". These educators miss the point of the material entirely. Teachers who have used the material from the very start swear by it and would never go back to "base ten only" teaching.

Dienes talks about 'singles or tens or hundreds or higher powers' pointing out that 'The concept of power has two variables: the base and the exponent', and complaining that 'they only use the exponents zero and one'. Talking instead about bundles and bundle-bundles, the exponents 2 and 3 occur when counting the fingers in 3s: T = ten = 3B1 3s = 1BB 1 3s = 1BBB1B 2s.

Isn't it both power and beauty to transform an unorganized total into a repeated bundling with the ability that it is only the decimal point that moves if you change the number of bundlings, T = 32.1 tens = 3.21 tentens, which is not the case with romans bundling where 3 tens is 6 fives. The romans didn't stick to bundling bundles since they bundled in both fives and tens and fifties but not in 5 5s, i.e. in 25s. Power and beauty comes from bundling bundles only.

Consequently, to understand the number 456 is to see it, not as one number, but as three numberings of a total that has been bundled 0 times, 1 time, and 2 times. And to read the total as 4 bundled 2 times and 5 bundled once and 6 not bundled, or as 4 bundle-bundles and 5 bundles and 6 unbundles singles. And to write the total as T = 4BB 5B 6. And to allow the same total to be recounted with an underload as T = 4BB 6B -4, or with an overload as T = 45B 6 = 4BB 56; or as T = 45B -4 if combining overload and underload.

This understanding allows an existing unorganized total to become a number-language sentence connecting the outside subject T with an inside calculation, T = 4*B^2 + 5*B^1 + 6*B^0. Which again is an example, or specification, of an unspecified number-formula or polynomial T = a*x^2 + 5*x + 6.

The power and beauty of the number-formula is manifold. It shows four ways to unite: power, multiplication, addition and next-to block addition also called integration. By including the units, we realize that there are only four types of numbers in the world as shown in the algebra-square above, constant and changing unit-numbers and per-numbers, united by precisely these four ways: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

Furthermore, we observe that splitting a total into parts will reverse uniting parts into a total, meaning that all uniting operations have reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers. This makes root a factor-finder, and logarithm a factor-counter, and differentiation a finder of per-numbers.

And, if we use the word 'equation' for the need to split instead of unite, we observe that solving an equation means isolating the unknown by moving numbers to the opposite side with opposite calculation sign. Furthermore, using variables instead of digits we observe that the number-formula contains the different formulas for constant change as shown above.

As to a non-constant change, there are two kinds. Predictable change roots calculus as shown by the algebra-square; and unpredictable change roots statistics to instead 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for unpredictable numbers.

Thus the 'power and beauty' of mathematics resides in the number-formula, as does the ability 'to use mathematics in students' personal life, in the workplace and in further study'. So, designing a curriculum based upon the number-formula will 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

Furthermore, a number-formula based curriculum need not split into parallel curricula until after calculus, i.e. until after secondary education.

So, one number-language curriculum for all is possible, as it is for the word-language. Thus, it is possible to allow all students to learn about the four ways to unite and the five ways to split a total.

An effective way to design a curriculum for all students is to adopt the curriculum designed for the refugee camp from the beginning since it accepts and develops the number-language children bring to school. Presenting figures and operations as icons, it bridges outside existence with inside essence. All four uniting methods occur in grade one when counting and recounting in different units, and when adding totals next-to and on-top. It respects the natural order of operations by letting division precede multiplication and subtraction, thus postponing addition until after counting, recounting and double-counting have taken place. It introduces the core recounting-formula expressing proportionality when changing units from the beginning, which allows a calculator to predict inside an outside recounting result. By connecting outside blocks with inside bundle-writing, geometry and algebra are introduced as Siamese twins never to part. Using flexible bundle-numbers connects inside decimals, fractions, and negative numbers to unbundled leftovers placed next-to or on-top the outside block. It introduces solving equations when recounting from tens to icons. It introduces per-numbers and fractions when double counting in units that may be the same or different. And, it introduces trigonometry before geometry when double-counting sides in a block halved by its diagonal.

Another option is to integrate calculus in a precalculus course by presenting integral calculus before differential calculus, which makes sense since until now inverse operations are always taught after the operation: subtraction after addition etc. Consequently, differential calculus should wait until after it has been motivated by integral calculus that is motivated by adding changing per-numbers in trade and physics, or by adding percentages in statistical double-tables.

In their publication, the National Council of Teachers of Mathematics writes "High school mathematics builds om the skills and understandings developed in the lower grades. (p. 19)" If this must be like that then high school education will suffer from lack of student skills and misunderstandings; and often teachers say that precalculus is the hardest course to teach because of a poor student knowledge background.

So, we must ask: Can we design a fresh-start curriculum for high school that integrates precalculus and calculus? And indeed, it is possible to go back to the power and beauty of the number-formula as described above, and build a curriculum based upon the algebra-square. It gives an overview of the four kinds of numbers that exist in the outside world, and how to unite or split them. It shows a direct way to solve equations based upon the definitions of the reverse operations: move to opposite side with opposite calculation sign.

Furthermore, it provides 2x2 guiding questions: how to unite or split into constant per-numbers, as needed outside when facing change with a constant change-factor? And how to unite or split into changing per-numbers that are piecewise or locally constant, as needed outside when describing e.g. the motion with a changing velocity of a falling object.

As a reverse operation, differential calculus is a quick way to deliver the change-formula that solve the integration problem of adding the many area-strips coming from transforming locally constant per-numbers to unit-numbers by multiplication. Also, by providing change-formulas, differential calculus can extend the formulas for constant change coming from the number-formula. An additional extension comes from combining constant change-number and change-percent to one of the most beautiful formulas in mathematics that is too often ignored, the saving-formula, $A/a = R/r$, a formula that is highly applicable in individual and social financial decisions.

Working with constant and changing change also raises the question what to do about unpredictable change, which leads directly into statistics and probability.

So, designing and implementing a fresh-start integrated precalculus and calculus curriculum will allow the National Council of Teachers of Mathematics to have their dream come through, so that in the future high schools can provide all students "with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

As a number-language, mathematics is placed between its outside roots and its inside meta-language or grammar. So, institutionalized education must make a choice: should the number-language be learned through its grammar before being applied to outside descriptions; or should it as the word-language be learned through its use to describe the outside world? I short, shall mathematics education tach about numbers and operations and postpone applications till after this has been taught? Or shall mathematics education teach how to number and how to use operations to predict a numbering result thus teaching rooting instead of applying?

Choosing the first 'inside-inside' option means connecting mathematics to its grammar as a 'meta-matics' defining concepts 'from above' as top-down examples from abstractions instead of 'from below' as bottom-up abstractions from examples. This is illustrated by the function concept that can be defined from above as an example of a set-product relation where first component identity implies second-component identity, or from below as a common name for 'stand-by' calculations containing unspecified numbers.

Choosing the inside-inside 'mathematics-as-metamatics' option means teaching about numbers and operations before applying them. Here numbers never carry units but become names on a number-line; here numbers are added by counting on; and the other operations are presented as inside means to inside tasks: multiplication as repeated addition, power as repeated multiplication, subtraction as inverse addition, and division as inverse multiplication. Here fractions are numbers instead of operators needing numbers to become numbers. Here adding numbers and fractions without units leads to ‘mathe-matism’, true inside classrooms where 2+3 is 5 unconditionally, but seldom outside classrooms where counterexamples exist as e.g. 2weeks + 3days is 17days or 2 3/7 weeks. Here geometry and algebra occur independently and before trigonometry. Here primary and lower secondary school focus on addition, subtraction, multiplication and division with power and root present as squaring and square roots, thus leaving general roots and logarithm and trigonometry to the different tracks in upper secondary school where differential calculus is introduced before integral calculus, if at all.

Choosing the inside-outside ‘mathematics-as-manymath’ option means teaching digits as icons with as many strokes as they represent; and also teaching operations as icons, rooted in the counting process where division is a broom pushing away bundles to be stacked by a multiplication lift, again to be pulled away by a subtracting rope to identify unbundled singles. This will allow giving a final description of the total of 7 recounted in 3s as a full sentence with a subject, a verb and a predicate predicted by the recount-formula T = (T/B)xB, e.g. T = 2Bundle 1 3s = 2.1 3s = 2 1/3 3s = 3.-2 3s thus including decimal numbers and fractions and negative numbers in a natural number all de-modeled into outside existence as unbundled singles placed next-to or on-top of the bundles. Here a double description of Many as an outside block and an inside bundle-number allows outside geometry and inside algebra to be united and go hand in hand from the start.

Once counted, totals can be recounted. First in the same unit to create overloads and underloads thus introducing negative numbers. Then recounting from ten- to icon-bundles introduces solving equations by moving to opposite side with opposite calculation sign; followed by recounting from icon- to ten-bundles with multiplication tables where using overloads will show that minus times minus must be plus: T = 7 8s = 7x8 = (B-3)x(B-2) = BB -2B -3B -- 6 = 5B+6 = 56. Then double-counting in two units creates per-numbers becoming fractions with like units. Finally, recounting the sides in a block halved by its diagonal will root trigonometry before geometry, that integrated with algebra can predict intersection points.

Then follows addition and reversed addition in its two versions, on-top or next-to. On-top addition calls for recounting the totals in the same unit, thus rooting proportionality. And next-to addition means adding blocks as areas, thus rooting integral calculus. Reversed addition roots equations and differential calculus. Per-numbers are added as operators including the units, thus rooting integral calculus, later defined as adding locally constant per-numbers.

Thus, choosing the inside-outside ‘mathematics-as-manymath’ option means that the core of mathematics is learned in primary school allowing ample of time in secondary school to enjoy the number-language literature by examining existing models or producing models yourself. And it means that only one curriculum is needed for all students as in the word-language.

Furthermore, the root and use of calculus to add changing per-numbers is easily introduced at the precalculus level when adding ingredients with different per-numbers and when adding categories in statistics with different percent. And, the fact that the difficulty by adding many numbers disappears when the numbers can be written as change-numbers since adding up any number of small changes total just one change from the start- to the end-number. Which of course motivates differential calculus. Consequently, there is no need for a parallel curriculum to the traditional since everybody can learn calculus in a communicative way. Of course, one additional optional course may be given to look at all the theoretical footnotes.

To offer a completely different kind of mathematics as graph theory and game theory and voting theory risks depriving the students of the understanding that mathematics came into the world as a number-language that uses operations to predict the result of counting, recounting, and double-counting. A language that needs only four operations to unite parts into a total, and only five operations to split a total into parts.

Without calculus in the final high school curriculum, students may not understand how to add per-numbers and might add them as unit-numbers instead of as areas; and this will close many ‘doors to productive futures’ as the US National Council of Teachers of Mathematics talks about.

So why not stop teaching wrong numbers and wrong operations that need to be de-modeled to give meaning, thus generating a huge volume of pre-service and in-service teacher training and professional development as well as additional funding for a never-ending stream of new textbooks and teaching materials.

Instead, let us model how humans master many by modeling. Here different degrees of many are modeled as icons with as many sticks or strokes as they represent, and used when organizing a total as a union of unbundled singles, bundles, bundles of bundles etc.; carried out by first pushing away bundles, modeled by a division broom; then stacking bundles, modeled by a multiplication lift; then pulling away the stack to look for unbundles singles, modeled by a subtraction rope; then uniting stacks on-top or next-to modeled by a two-direction addition sign; then placing the unbundles singles next-to or on-top of the stack to be modeled as decimals, fractions or negative numbers.

This all allows the full counting process that from a total T, T/B times pulls B away to be modeled by the core formula T = (T/B)xB, expressing proportionality when shifting unites and used all over mathematics and science.

All children will acquire basic numeracy if allowed to keep the flexible bundle-numbers they create when adapting to outside quantity, to be developed by an outside-inside modeling of questions coming from counting and adding totals, and constituting the core of future curricula, easy to understand and implement for learners and teachers and parents alike.

Formulating the goals of mathematics educations as assisting children in their adaption to quantity and assisting teenagers in their adaption to double-quantity may be the radicalizing of the curriculum that is needed to allow learners formulate their own learning projects in a personalized learning environment as described in the OECD 2030 Learning Framework. And to allow them finally to keep and develop the number-language they bring to school to educate themselves and as well as schools.

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26. NORMA 24

Interplay between research and teaching practice in mathematics education

NORMA 24, The Tenth Nordic Conference on Mathematics Education will take place in Denmark from the 4th to the 7th of June 2024. The NORMA conferences offer forum for discussions and constructive interactions among researchers, teachers, teacher educators, graduate students and others interested in mathematics education research in the Nordic context.

My contributions were all rejected by the same mail:

Dear Allan Tarp, thank you very much for submitting a regular paper to NORMA 24. Unfortunately, your proposal is rejected for presentation at NORMA 24. You are of course more than welcome to participate in the conference without a contribution to the scientific program. The program committee.

From a colonized to a decolonized mathematics, from 8 to 2 competences, from non-unit to unit-numbers

Poster by Allan Tarp

Keywords: Decolonization, mathematics, competence, curriculum, numeracy.

Asking a 3year-old child “How many years next time?” shows a need to decolonize mathematics. The child reacts to 4 fingers held together 2 by 2 “That is not 4, that is 2 twos”. Adults only see inside essence, four, but the child sees outside existence, bundles of 2s in space as units when counted in time.

Comparing the claims, “1+2 = 3” and ”3x4 = 12”, we see that without units, outside examples as “1week+2days = 9days” falsify the first claim. The second claim includes the unit by predicting that 3x4 as 3 4s outside may be re-counted as 1.2 tens. So, multiplication makes mathematics a natural science that becomes decolonized once the colonization by non-unit numbers has ended.

To decolonize a colonized non-unit mathematics with 8 competences, only 2 competences are needed, count and add: counting and re-counting bring outside totals inside to be added or split depending on how they occur, as like or unlike unit or per-numbers. A counting sequence in 3s always include the units: unbundled, bundles, and bundle-of-bundles that become squares on a Bundle-Bundle Pegboard, a BBBoard, where unit-numbers are tiles. Counting before adding changes both the order and the identity of the operations. Power is in bundle-bundles, division is a broom pushing away bundles, multiplication a lift stacking them, subtraction a rope pulling away the stack to find unbundled that placed on-top of the stack becomes decimals, fractions, or negatives, e.g., 9 = 4B1 = 4 ½ = 5B-1 2s. And addition shows the two ways to add stacks, next-to by areas as integral calculus, or on-top after the units are made like by the linearity of a recount-formula showing that when re-counting 8 in 2s, 8 = 4 x 2 = (8/2) x 2, or T = (T/B) x B with T and B for the total and the bundle.

Recounting ten-bundles in digit-bundles creates equations solved by moving to opposite side with opposite sign: ux2 = 12 = (12/2) x 2, so u = 12/2. Recounting digits in tens gives the tables and early algebra on a BBBoard: 6x7 = (B-4) x (B-3) = 10B-4B-3B+4x3 (taken away twice) = 3B12 = 4B2 = 42. Recounting 6 4s as a bundle-bundle creates square roots and solves quadratics. Recounting $ in kg creates a per-number, e.g., 4$ per 5kg that bridges the units, becoming fractions with like units, and that multiplied to $ add as areas, i.e., as integral calculus. Probably all have learned this core math after primary school. Secondary school then is for the communicative turn using the number-language to write literature in its three genres, fact, fiction, and fake, as in the word-language.

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Respecting the child's innate number sense, is that Woke-math?

Working group by Allan Tarp

Keywords: Mathematics curriculum, decolonization, numeracy, arithmetic, calculus

Woke means respecting multiple identities.

Woke-math respects flexible bundle-numbers for a total instead of imposing a linear number upon it. Five fingers thus exist both as 5 1s, as 1 5s, as 1Bundle3 2s, as 2B1 2s, and as 3B-1 2s. Woke-math occurs when asked "How many years next time?", a 3year child reacts to holding fingers together 2 and 2 by saying "That is not 4, that is 2 2s." So, child sees what exists in space and time, bundles of 2s in space, and 2 of them in time when counted. Woke-math thus builds on the philosophy called existentialism holding that outside existence precedes inside constructed essence. As 'multiplication-math', Woke-math is a natural science where 3x4 = 12 states that 3 4s may be recounted in 1B2 tens. As 'addition-math', traditional math is 'mathematism' where 2+1=3 is falsified by, e.g., 2pairs+1=5.

We ask: *How to design a decolonized curriculum using 2dimensional bundle-numbers with units?*

Bundle-counting ten fingers in 3s gives power as the first operation.

0B1, 0B2, 0B3 or 1B0 or 2B-3, …, 2B2, 2B3 or 3B0 or 1BB0B0 or 1(B^2)0B0, 1(B^2)0B1 or 101.

Bundle-counting with over-loads and under-loads gives negative numbers easing calculations.

5 = 1B3 = 2B1 = 3B-1 2s. 4+58 = 4+5B8 = 5B12 = 6B2 = 62.

4x68 = 4x6B8 = 24B32 = 27B2 = 272.

Bundles in space give digits as icons, and numbers as tiles on a 10-by-10 Bundle-Bundle-Board.

There are four strokes in a 4-icon, five in a 5-icon, etc. And, 2 3s is a 2x3 tile; and a BB is a square.

Operations are icons also.

8 – 2: From 8, pull-away 2 (a rope). 8/2: From 8, push-away 2s (a broom). 4x2: 4 times lift 2s (a lift).

Recounting between icons may be predicted by a calculator.

4 5s = ? 6s. Enter '4x5/6' gives '3.more'. Enter '4x5 – 3x6' gives '2'. Answer: 4 5s = 3B2 6s.

Recounting 8 in 2s gives a 'recount-formula'.

8/2 = 4, so 8 = 4x2, or 8 = (8/2)x2, or Total T = (T/B)xB, a 'ReCountFormula' or proportionality.

Recounting 9 in 2s gives the decimal numbers, negative numbers, and fractions.

Count or calculate gives 9 = 4B1. Recount 1 in 2s gives 1 = (1/2)x2. So, 9 = 4.1 = 5B-1 = 4 ½ 2s.

Recounting from tens to icons gives equations.

How many 2s in 8? *u*x2 = 8, but 8 = (8/2)x2, so *u* = 8/2. Method: 'opposite side & opposite sign'.

Recounting from icons to tens gives tables and early algebra on a BBBoard.

Show 6x7 on a BBBoard. Pull away the top- and side-stack, add upper right corner pulled-away twice. 6 7s = ?tens. 6 7s = 6x7 = (B-4)x(B-3) = 10B –4B –3B + 4x3 = 3B + 1B2 = 4B2 = 42. And –x– = +.

Recounting to Bundle-Bundles gives squares, square roots, and quadratics.

Squaring a rectangle: Move half the excess to the top and side, and we almost have the square-root: Squaring a 6x4 rectangle, (6-4)/2 = 1. So 6x4 ≈ (6-1)x(4+1). Pull away the 1x1 corner gives 5.9 close to the calculator prediction 5.899. A (*u*+3) square has two squares, *u*^2 and 3^2, and two 3*u*-tiles, totalling *u*^2+6*u*+9. If *u*^2+6*u*+8 = 0, then (*u*+3) squared is 1 squared, which gives the solutions.

Recounting between units gives a per-number 4$ per 5kg, bridging the two units by recounting.

12$ = ? kg. 12$ = (12/4)x4$= (12/4)x5kg = 15kg. 20kg = ? $. 20kg = (20/5)x5kg = (20/5)x4$ = 16$.

With like units, per-numbers become fractions.

My share is 3$ per 5$, 3$/5$ = 3/5, how much of 200$? 200$ = (200/5)x5$ gives (200/5)x3$ = 120$.

Recounting the sides in a rectangle with height *H*, length *L*, and diagonal *D* gives trigonometry.

Height = (Height/Length) x Length = tangent (Angle) x Length. And H = (H/D) x D = sine(A) x D. A small height is almost a circle arc. So, if the length is 1, the semicircle length = $\pi \approx n$ x tan(180/*n*).

Adding next to and reversed gives primary school calculus.

Adding 2 3s and 4 5s next-to means adding areas, called integration, and differentiation if reversed.

Adding on-top gives proportionality.

Adding 2 3s and 4 5s on-top, the units first must be made the same by recounting.

Adding per-numbers and fractions gives middle school calculus.

In the bill 2kg at 3$/kg + 4kg at 5$/kg, the unit numbers add directly, but the per-numbers must be multiplied before being added. So, per-numbers add by areas, i.e., by integral calculus.

Adding like per-numbers gives power calculations.

Adding 5% means to multiply by 105%. 10 times give 105%^10 = 162.9%, or 50% + 12.9% extra.

10 times ?% gives 50%, in *u*^10=150%, the factor-finding root predicts *u*= 10√150% = 1.041 or 4.1%.

?times 5% gives 80%, in 105%^*u*=180%, the factor-counting logarithm predicts *u*=log1.05(1.80)=12.

100% split in *n* parts gives (1+1/*n*)^*n*, which for *n* large gives 2,718…, called the Euler-number, *e*.

Adding one-digit numbers as bundles:

6+9 = 2B3 6s = 2B-3 9s.

Adding letter-numbers by their common unit:

2*abcd* + 3*ace* = 2*bd***ac* + 3*e***ac* = (2*bd*+3*e*)**ac*.

Finally, we reach the 'Algebra Square',

Here, in Arabic, Algebra means to re-unite.

Unlike/like unit-numbers are united by addition/multiplication; and split by subtraction/division.
Unlike/like per-numbers are united by integration/power; and split by differentiation/root & log.

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To master or not to master math before Many, that is the question

Paper by Allan Tarp

It seems evident that the goal of mathematics education must be to learn mathematics; which once learned inside has many important outside applications that allow humans to master Many. It is only a shame that mathematics education is so difficult that it produces many 'slow learners'. Therefore, grand theory asks: With mastery of outside existence as the end goal, and with mastery of an inside institutionalized essence as a subgoal, a goal-displacement may be prevented by exchanging the two. To get an answer we observe that the mastery children develop when adapting to Many before school leads to 12 micro-curricula so completely different from the tradition that they create a Kuhnian paradigm-shift within mathematics education as radical as the change from a flat to a round earth.

Keywords: Elementary school mathematics, special education, numeracy, arithmetic, calculus

Children's 2D bundle-numbers with units replaced by 1D line-numbers without

Children talk vividly about Many before school. But then they stop doing so in school, and risk being excluded and sent to special education. An hour inside a classroom tells you why. The students no more talk about Many, instead a textbook mediated by a teacher teaches them about what they must learn first to communicate later: multidigit 1D line-numbers that obey a place value system, and that are added without units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers. The textbook thus presents a 'science', which despite 2weeks + 1day is 15days builds on the belief that 2+1 is 3 always, and thus should be called 'mathematism' true inside but seldom outside (Tarp, 2018).

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that existence precedes essence so that in a sentence, the existing subject outweighs any chosen predicate (Marino, 2004). So, 'Many' should be seen onto-logically, what it is in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget sees learning as adapting to outside existence, whereas a Vygotsky sees learning as adapting to inside institutionalized socially constructed essence.

Foucault (1995) points to 'pastoral power' that may be installed by mixing action words, verbs, and judgement words, predicates, when diagnosing humans. Not being a verb, 'math' should be replaced by 'number' in the diagnose "you don't know how to math, so we who know will now teach you". If not replaced, sociologically a school becomes a 'pris-pital-barrack' forcing constant return to the same room to be cured from the self-referring diagnose 'inability to math' under order. Also within sociology, a structure-agent debate discusses if humans should obey institutionalised essence or allow this to be constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. A Baumanian viewpoint (1990) suggests that, by monopolizing the road to mastery of Many, traditional math has created a 'goal displacement' making the institutionalized means a goal instead. So, the word 'mathematics' must leave goal descriptions to avoid a meaningless self-reference.

The tradition sees Many as an example of 1D linear cardinality always able to absorb one more thus built on the belief that '2+1 = 3 period'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, bundles of bundles, etc., e.g., T = 345 = 3*BB +

4*B + 5*1. The tradition sees mastery of math as its primary goal. A difference thus could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math on the way. And could make the tradition more meaningful by de-modelling it (Tarp, 2020). As to differences, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula (MC) to be tested with Design Research.

Counting Many with bundles, children deserve a bundle-number curriculum

Asked “How many years next time?”, a 3year old typically will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, “That’s not four, that’s two twos”, thus describing what exists in space and in time: bundles of 2s as totals in space, and two of them when counted in time as a sequence. Inside, children thus adapt to outside totals by using two-dimensional bundle-numbers with units. Also, they use full sentences as in the word-language with a subject ’that’, and a verb ‘is’, and a predicate ‘2 2s’ which abbreviated demodels a formula as a number-language sentence ‘T = 2 2s’ shown with a vertical and a horizontal rubber band on a 2D ten-by-ten pegboard, a ‘BundleBundleBoard’, or ‘BBBoard’, that now will replace the 1D number-line. So, we ask:

What math comes from a question-guided curriculum using a BBBoard and the 2D bundle-numbers with units that children bring to school to develop their number-language and mastery of Many?

MC01. Demodeling digits as icons

The tradition presents both digits and letters as symbols. A difference is letting students build digits as icons with as many sticks or strokes as they represent (Tarp, 2018) to see that inside icons link directly to outside degrees of Many; and that 5 ones differ from 1 fives; and that ten has no icon since as a bundle ten becomes 1-unit-0, or 10. 2-digit numbers thus are two numberings of bundles and of unbundled singles. A guiding question may be “There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?” Outside material may be sticks, a folding ruler, cars, dolls, a BBBoard, etc. Discussing why numbers after ten have no icon leads on to bundle-counting.

MC02. Demodeling counting sequences by always including bundles, and bundles-of-bundles

Using a place value system, the tradition never uses units in counting sequence. A difference is to practice bundle-counting in tens, fives, and threes, and always include the units. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names ‘eleven’ and ‘twelve’ are Viking names meaning ‘one left’ and ‘two left’, and that the name ‘twenty’ has stayed unchanged since the Vikings said ‘tvende ti’; The Viking tradition saying ‘three-and-twenty’ instead of ‘twenty-three’ was used in English for many years. Now it stops after 20. Now only Danes still counts in scores: 80 = 4 scores, 90 = half-5 scores. A guiding question may be “Always count what exists when bundle-counting in tens, in 5s and in 3s.” Outside material may be fingers, sticks, cubes, and a BBBoard. Including the arm as an extra finger we can count to twelve, called a dozen.

First, we count a dozen in 5s (hands): 0B1, …, 0B4, 0B5 or 1B0, 1B1, …, 1B5 or 2B0, 2B1, 2B2.

Then we count a dozen in 3s (triplets): 0B1, …, 0B3 or 1B0, …, 1B3 or 2B0, …, 2B3 or 3B0, 3B1,..

Counting cubes in 3s, 3 bundles is 1 quadratic bundle-of-bundles or 1*BB* in writing, so we repeat:

We count a dozen in 3s: 0*B*1, …, 2*B*3 or 3*B*0 or 1*BB*0*B*0, 1*BB*0*B*1, 1*BB*0*B*2, 1*BB*0*B*3 or 1*BB*1*B*0.

Counting fingers in 2s gives a total of ten as 1*BBB* 0*BB* 1*B* 0. So, power becomes the first operation.

Then we count in tens, again including the bundles: 0*B*1, …, 0*B*8, 0*B*9, 0*B*10 or 1*B*0, 1*B*1, 1*B*2. Finally, we bundle-count in tens from 0 to 111.

MC03. Demodeling multi-digit numbers with units, underloads, and overloads

Obeying place values, the tradition silences the units when writing ‘two hundred and fifty-seven’ as plain 257. A difference may be inspired by the Romans using ‘underloads’ when writing ‘four’ as ‘five less one’, IV; and by overloads when children use ‘past-counting’: ‘twenty-nine, twenty-ten, twenty-eleven’. A guiding question may be “Let us count with underloads missing for the next bundle. And with overloads saying ‘twenty-eleven’.” Outside material may be sticks, cubes, and a BBBoard.

First, we notice that five fingers can be counted in pairs in three different ways.

T = 5 = I I I I I = ~~II~~ I I I = 1*B*3, overload

T = 5 = I I I I I = ~~II~~ ~~II~~ I = 2*B*1, normal

T = 5 = I I I I I = ~~II~~ ~~II~~ ~~II~~ = 3*B*-1, underload

Using fingers and arms, first we count a dozen with underloads: 0*B*1 or 1*B*-9, 0*B*2 or 1*B*-8, …, 0*B*9 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-9, 1*B*2 or 2*B*-8.

Then in 5s (hands): 0*B*1 or 1*B*-4, 0*B*2 or 1*B*-3, …, 0*B*4 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-4, …, 2*B*2 or 3*B*-3. And in 3s (triplets): 0*B*1 or 1*B*-2, 0*B*2 or 1*B*-1, 0*B*3 or 1*B*0, 1*B*1 or 2*B*-2, …, 3*B*3 or 4*B*0 or 1*BB*1*B*0.

Cup-counting with a cup for bundles, and for bundles-of-bundles: *T* = 1]1]0 = 4]0 = 3]3 = 2]6 = 1]9.

Then we count in tens from 1 to 111, using ‘past-counting’:

… 1*B*9, 1*B*10, 1*B*11 or 2*B*1, 2*B*2, …, 2*B*11 or 3*B*1, …, 9*B*9, 9*B*10, 9*B*11 or 10*B*1 or 1*BB*0*B*1.

Counting in tens, we may also use ‘flexible bundle-numbers’ with overloads and with underloads:

T = 38 = 3*B*8 = 2*B*18 = 1*B*28 = 4*B*-2 = 5*B*-12.

Traditional carrying and borrowing become useless when using demodeling with units instead:

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92, and T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B* 12 9 *B* 2	4 *B* -2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8

92	38	336	48

Figure 1: Doing math using flexible bundle-numbers with units

MC04. Demodeling solving equations to reversing formulas

Reducing the outside fact 'the Total is 3 4s' to the inside statement '3*4 = 12' the tradition silences both the subject and the verb. And forces the total to accept the identity '1.2 tens', even leaving out the unit and the decimal point. A difference is to use full sentences with an existing outside subject, a verb, and an inside chosen predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence "$T = 5*6 = 30$" thus inside predicts that an outside total 5 6s existing on a BBBoard may be re-counted as 3 tens on a number-line thus losing its identity.

A guiding question may be. "Let us talk math with full sentences about the totals we count and how."

Pulling-away 2 from a total T will leave 'from T pull-away 2' iconized by a rope called subtraction: Before we had T. After we have T-2 and 2, so $T = (T – 2) + 2$, or $T = (T – B) + B$ in general.

We call this formula a 're-stack formula' since, with the total as a stack, we may pull away the bundle from the top and place it next-to as its own stack.

Outside asking "Adding 2 to what gives 5?", inside becomes "? + 2 = 5" in writing. Using the letter u for the unknown number, this becomes an equation '$u + 2 = 5$', easily solved outside by pulling away the 2 that was added, described inside by restacking the 5: $u+2 = 5 = (5 – 2) + 2$, so $u = 5 – 2$.

So inside, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the definition of the number '5-2':

"5 minus 2 is the number u that with 2 added gives 5".

We count and re-count in bundles. Re-counting 8 1s in 2s, we use '/' to iconize a broom pushing away 2s. So '8/2 = 4' is an inside prediction for the outside action 'From 8, push-away 2s, 4 times.'

Having been pushed away, the 2-bundles are stacked. This is iconized by an 'x' for lifting the bundles, so '4x2 = 8' is an inside prediction for the outside action '4 times stacking 2s gives 8 1s.'

Re-counting 8 in 2s gives a 'recount formula' 8 = (8/2) x 2, outside showing a stack with the height 8/2 and the width 2, and with the total 8 as the area. So, with outside totals as inside areas, totals add by areas, called integral calculus.

With unspecified numbers it says: $T = (T/B) \text{ x } B$, or $T = (T/B) * B$, simply stating that when recounting a total T in Bs, T contains T/B Bs.

Outside asking "How many 2s in 8", inside is the equation '? * 2 = 8', or '$u * 2 = 8$' easily solved outside by pushing away 2s, described inside by recounting 8 in 2s:

$u * 2 = 8 = (8/2) * 2$, so $u = 8/ 2$.

Again, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the formal definition of '8/2': "8 divided by 2 is the number u that multiplied with 2 gives 8.

MC05. Demodeling decimals, fractions, and negatives as names for the unbundled singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question may be “How to see the unbundled singles?”. Outside materials may be cubes and a BBBoard.

Before recounting 9 in 2s outside, inside we let a calculator predict the result: Entering 9/2 gives ‘4.more’ predicting that ‘9 contains 4 2s, and more’ that are found by outside pulling away the 4 2s, predicted inside by entering ‘9 – 4 * 2’ giving 1. So, inside, the calculator predicts that 9 recounts as 4*B*1 2s, which is also observed outside.

Recounting 9 cubes in 2s, the unbundled can outside be placed on-top of the stack. Inside it may be described by a decimal point separating the bundles from the unbundled: *T* = 4*B*1 2s = 4.1 2s. Likewise, when counting in tens: *T* = 4*B*2 tens = 4.2 tens = 4.2 * 10 = 42.

Seen Outside as a part of a bundle, inside we can count it in bundles as a ‘fraction’, 1 = (1/2) * 2 = 1/2 2s; or we can count what is missing in a full bundle, 1 = 1*B*-1 2s.

Again, we see the flexibility of bundle-numbers: *T* = 4*B*1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Likewise, when counting in tens: *T* = 4*B*2 tens = 4 2/10 tens = 4.2 tens = 5.-8 tens.

MC06. Demodeling multiplication tables and equations as changing number units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon-unit to another, from icons to tens, or from tens to icons, or into a square.

A guiding question may be “How to change number units?”. Outside materials may be a BBBoard.

Asking ‘3 4s = ? 5s’, we inside predict the result by entering on a calculator the 3 4s as 3*4, to be counted in 5s by dividing by 5. The answer ‘2.more’ predicts that 3 4s contains 2*B* &more 5s. To find the unbundled singles, outside we pull away the 2 fives from the 3 4s; inside we predict this by entering ‘3*4 – 2*5’. The answer ‘2’ then predicts that 3 4s may be recounted as 2 5s & 2, or 2*B*2 5s.

Asking “40 = ? 5s”, we predict by solving the equation “u * 5 = 40” by recounting 40 in 5s: u * 5 = 40 = (40/5) * 5, so u = 40/5.Changing units also change the form of the height and width of the stack.

Asking “6 8s = ? tens”, or “6 * 8 = ?”, we inside predict the result by looking at a ten-by-ten square with 6 and 8 as *B*-4 and *B*-2 on the sides. We then see that the 6*8 stack is left when from the *B***B* stack we pull away a *B**2 and a 4**B* stack, and then add the 4*2 stack that was pulled away twice.

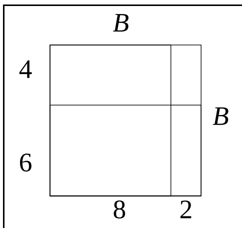	T = 6 * 8 = (*B*-4) * (*B*-2) = *BB* – 2*B* – 4*B* + 4*2 = 4*B*8 = 48	$T = \begin{pmatrix} 1B & -4 \\ 1B & -2 \end{pmatrix}$ = 1*BB* – 2*B* – 4*B* + 4*2 = 10*B* – 6*B* + 8 = 4*B*8 = 48	$T = \begin{pmatrix} 6B & +4 \\ 8B & +2 \end{pmatrix}$ = 48*BB* + 12*B* + 32*B* + 8 = 48*BB* + 44*B* + 8 = 52*BB* 4*B* 8 = 5248

Figure 2: Multiplying 6*8 and 64*82 as binomials

Here, 6*8 exists as 6 8s outside. Inside recounting in tens shows that clearly minus times minus is plus. This view is an alternative to the multiplication controversy created by the ‘YouCubed’ website.

Inside, multiplying two ‘less-numbers’ horizontally creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles.

MC07. Demodeling proportionality as per-numbers changing physical units

The tradition sees shifting physical units as an application of proportionality. Typically, finding the unit cost will answer questions as “with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?” A difference is to use ‘per-numbers’ (Tarp, 2018) coming from double-counting the same total in two units, e.g., T = 3$ = (3$/2kg) * 2kg = p * 2kg, with the per-number p = 3$/2kg, or 3/2 $/kg. A guiding question may be “How to change physical units?”. Outside materials may be coloured cubes.

Recounting in the per-number allows shifting units:

T = 6 kg = (6/2)*2 kg = (6/2)*3 $ = 9$; and T = 15$ = (15/3)*3$ = (15/3)*2kg = 10 kg.

Alternatively, we recount the units: $ = ($/kg)*kg = (3/2)*6 = 9; and kg = (kg/$)*$ = (2/3)*15 = 10.

With like units, per-numbers become fractions: 1$/4$ = ¼. The tradition teaches fractions as division: 1/4 of 12 = 12/4. A difference is to see a fraction as a part of a bundle counted in bundles, 1 = (1/4)*4, so 1/4 = 1 part per 4. Finding 3/4 of 12 thus means finding 3parts per 4 of 12 that recounts in 4s as: T = 12 = (12/4) * 4 = (12/4) * 3parts = 9parts, so 3 per 4 is the same as 9 per 12, or 3/4 = 9/12. Likewise, 3/4 of 100 means finding 3 parts per 4 of 100 = (100/4)*4, giving 75parts per 100 or 75%.

MC08. Demodeling square roots as recounting stacks as Bundle-Bundle squares

The tradition postpones squares, square roots and quadratics to upper secondary school. A difference is to see a square as a BB, a Bundle-Bundle. This allows stacks with overloads to be squared. Materials may be books, windows, doors, and a BBBoard, which also may be used to solve quadratics.

Squaring a 6-by-4 stack, its side is called √24, with lines to iconize the square. To find √24, half of the top surplus, ½*(6-4) = 1, is place it next to the side, which gives a 5-by-5 square less the upper right 1-by-1 square, so the side must be reduced by u, where 5*u = ½*1, giving u = 0.1. So √24 = 4.9 almost since we miss a small upper right square. Inside, a calculator predicts that √24 = 4.899.

To solve the quadratic equation u^2 + 6u +8 = 0, we look at a (u+3) x (u+3) square divided in four sections, u^2, and 3u twice, and 9 = 8+1. Since u^2 + 6u +8 = 0, (u+3) squared is 1 = 1 squared. Hence u+3 = +1 or -1, so there are two solutions, u = -3+1 = -2, and u = -3-1 = -4.

MC09. Demodeling trigonometry as recounting the sides in a stack halved by its diagonal

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, recounting the sides in a stack halved by its diagonal. A guiding question may be “How to recount the sides in a stack halved by its diagonal?”. Outside materials may be tiles, cards, a BBBoard, and books.

Recounting the height in the base, height = (height/base) * base = tangent A * base, shortened to

h = (h/b) * b = tan A * b = tan A bs, thus the formula: tangent A = height / base, or tan A = h/b.

This gives a formula for the length of a unit circle containing many right triangles: A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 100 1.8s. With A as 1.8

degrees, the circle and the tangent, *h*, are almost identical. So, the length of a half circle is ½*C* = 100 * *h* = 100 * tan 1.8 = 100 * tan (180/100) = 3.1426 ≈ π, where π is tan (180/n)*n, for n large enough.

MC10. Once counted and recounted, stacks may be added on-top or next-to

The tradition sees numbers as 1D line-numbers with addition defined as counting on. A difference is to accept children's 2D bundle-numbers that add next-to and on-top. A guiding question may be "How to add 2 3s and 4 5s on-top and next-to?". Materials may be cubes and a BBBoard.

Adding 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing or pulling, inside by recounting to change units. The recount formula predicts the result when entering (2*3+4*5)/*B*, where *B* may be 3 or 5 or 8. With like units, digits add on-top: 8+5 = 2*B*(8-5) 5s = 2*B*(5-8) 8s.

Adding stacks next-to by areas is called integral calculus; and differential calculus if reversed asking "4 5s plus how many 3s gives 5 8s?". Outside, we pull-away 4 5s from the total *T* before recounting in 3s, which is predicted inside by a calculator: (5*8 – 4*5)/3, or $\Delta T/3$. Using a difference to calculate the change in the total, ΔT, before using division to recount in 3s, this is called differential calculus.

Adding BB-squares as a square, its side is the 'lower diagonal' in the major square placed first.

MC11. Adding unspecified letter-numbers

The tradition sees adding letter-numbers as an application of a distributive law. A difference is to find the common unit. In $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, *T* may be seen as 3 *ab*s, or as (3*a*) *b*s, or as (3*b*) *a*s. To avoid confusion the 's' will be omitted, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$. Since totals need a common unit to add, this must be first found as $T = 3ab + 4ac = 3b *a + 4c *a = (3b+4c) *a = 3b+4c\ a$s.

MC12. Demodeling integral calculus as adding per-numbers or fractions

Adding numbers without units may be called 'mathematism', true inside but seldom outside where, e.g., 2m + 3cm = 203cm. A difference respects that the recount-formula shows that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding. A guiding question may be "What is 2kg at 3$/kg plus 4kg at 5$/kg?" Outside materials may be a BBBoard with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

Inside we see that unit-numbers add directly. Whereas per-numbers first must be multiplied to become unit-numbers. And since multiplication creates areas, per-numbers add by their areas, i.e., as the area under the per-number curve. And again, adding areas is called integral calculus. Again, the opposite is called differential calculus asking "2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg?" The two connect by the fact that when adding differences, the middle terms disappear leaving only the difference between the end and initial numbers: $(b – a) + (c – b) + (d– c) = d – a$.

We now know all 4 ways to unite parts into a total, and to split a total in parts, the 'Algebra-square' (Tarp, 2018) that respects the Arabic meaning of the word algebra, to re-unite. And that shows how an outside total, *T*, is brought inside by a predicate describing how the total occurs as united or split into the four existing number types, unlike and like unit-numbers, and unlike and like per-numbers.

Operations unite/ split Totals in	Unlike	Like
Unit-numbers m, s, kg, $	$T = a + n$ $T – n = a$	$T = a*n$ $T / n = a$
Per-numbers m/s, $/kg, $/100$ = %	$T = \int a*dn$ $dT/dn = a$	$T = a\text{^}n$ $\sqrt[n]{T} = a \quad n = \log_a T$

Figure 3: The 4 ways to unite parts into a total, and the 5 ways to split a total into parts

Conclusion and recommendation

We asked what mathematics comes from replacing an essence-based with an existence-based curriculum answering the question 'how many?' by numbering outside totals inside by (re)counting. Digits occur as icons with as many strokes as they represent, thus becoming units when numbering totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads. Bundling bundles also leads to squares and square roots; and to power as the first of the operations; which are icons also, but with different meanings and opposite order. Division now means counting iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Thus, both proportionality and trigonometry occur in year one, as well as formulas and functions as full number-language sentences allowing calculators to predict numbering results. So 8/2 is 8 counted in 2s, and 6*8 is 6 8s, a stack that recounted in tens changes both width and height, and that introduces early algebra when written with underloads: $6*8 = (B\text{-}4)*(B\text{-}2) = (10\text{-}4\text{-}2)*B + 4*2 = 4B8 = 48$. Once counted and recounted, totals may add on-top, or next-to by areas as in integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own flexible bundle-number with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as a self-organized pre- and in-service teacher training asking the subject instead of the instructor as exemplified on the MATHeCADEMY.net website.

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' We could ask if this is possible if an educational tradition rejects the child's own 2D flexible bundle-numbers with units, and replaces them with inflexible 1D line-numbers without units? So maybe the time has come where mathematics education should stop teaching 'mathematism' to children and instead begin to learn from them how to master Many with their flexible bundle-numbers with units.

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A proposal for a symposium

Celebrating Niss and Skovsmose and Tarp, all close to 80 years old.

The symposia title is “School mathematics, past & present & future, 1974 & 2024 & 2074.”

The objectives of the session are to ask representatives for three Danish research directions grounded before 1980: “School mathematics, how was it in 1974? What has your direction been working with the past 50 years? What does your direction expect to happen in the coming 50 years?”

In 1981 the Danish educational journal ‘Pædagogik (3)’ asked three young scholars born in 1944, 1944 and 1945 to give a status on school mathematics in Denmark. Mogens Niss called his article ‘Mathematics Education in a Society, for the society or for the population?’ Ole Skovsmose called his article ‘Critical mathematics’. And Allan Tarp called his article ‘Mathematics, a quantitative description of the real world.’. The articles may be found at the Royal Danish Library, www.kb.dk, and serves as symposium papers. The symposium also celebrates the three scholars about to turn 80.

Overview and structure of the presentation. There will be three sessions. In the first session each presenter has ten minutes. In the next 30 minutes there will be three two-persons dialogs. In the last session the floor may send in short written contributions to be handled by a moderator.

Scholarly or scientific significance. The presenters represent three aspects on school mathematics. The Niss-direction sees university mathematics as an educational task to quote Freudenthal. So, it holds that the goal of mathematics education is to learn mathematics as defined by the university.

The Skovsmose-direction accepts this but stresses a critical attitude to the real-world examples and applications that are included. So, it holds that the goal of mathematics education is to use mathematics as a tool to unmask injustice, and to work for equality and equity.

The Tarp-direction holds that existence precedes essence, so mastery of Many should precede mastery of math. Children thus are allowed to develop their innate number language to enable them to unite and split like and unlike unit- and per-numbers to allow a communicative turn in school math. These directions constitute stress fields allowing school math to develop in different directions.

The Niss- and Skovsmose-directions will discuss what should be stressed in school math, its close-ness to university mathematics to allow students to become math graduates, or its real-world applications allowing the students to become critical citizens.

The Niss- and Tarp-directions will discuss if the goal of math education is to master inside math before it can be applied to master outside Many, or if mastery of Many automatically implies mastery of core math.

The Skovsmose- and Tarp-directions will discuss which grand theories to use in math education, Marxism critical to applications or Existentialism sceptical to core predicates by deconstructing them.

27. Curriculum Proposal at a South African teacher college

Curriculum (postmodern, nominalistic, praxis-based)

Content list

0. Quantifying reality
Qualities and quantities, word language and number language
Quantifying: Naming degrees of many, number systems
Numbers as Totaling narratives: 647.3: T = 6(100) + 4(10) + 7(1) + 3(1/10)
Operations as Totaling techniques:
Changing/constant unit-numbers totaled by addition/multiplication
Changing/constant per-numbers totaled by integration/power

I. Totaling constant per-numbers, linearity

1.1 Constant per-number, proportionality	T = a*x
1.2 Constant units per day	T = b+a*x
1.3 Constant percentage per day	T = b*a^x
1.4 Constant percent per percent	T = b*x^a
1.5 Linear programming	

II. Totaling variable predictable per-numbers, local linearity

2.1 Variation:	Change Δ, rate r
2.2 Polynomials: Totaling bundles	T = ao + a1*x + a2*x^2 + a3*x^3 +…
2.4 Local variations, differentials	dT = T'dx
2.5 Integrals	$\int$ f dx = $\int$ dT = ΔT = T2-T1
2.6 Differential equations	T' = f

III. Totaling stochastic numbers

3.1 Stochastic variation	X = Xm ± 2*Xvar
3.2 Probability	$X = n*p \pm 2*\sqrt{n\cdot p\cdot(1-p)}$

IV. Generalizing to several variables

4.1 Calculus	$dT = (\partial T/\partial x)dx + (\partial T/\partial y)dy$
4.2 Vector spaces	T = ar+bs
4.3 Vector analysis	div, rot, grad

V. Modeling reality
5.1 Modeling reality: Factual and fictional numbers and equations
Modeling the big 4: Points, Particles, Populations, Pecunia
5.2 Geometry and trigonometry
5.3 Physics and Chemistry
5.4 Biology and Politics
5.5 Economics

A History, Philosophy and Didactics of Mathematics

A History of Mathematics (and trade)

Gather/hunters:
 Situated number names
Agriculture (Egypt, Babylon, etc.):
 Number systems
Trade
 Greeks: Geometry
 Arabs: Decimals, algebra and trigonometry
 Italians: Percents, power/root/log
 England: Calculus
Industry
 England: Vector analysis
Information technology
 USA: Operational Research

A Philosophy of Mathematics, Externalizing Mathematics

Mathematics: Top-Down or Bottom-Up - Platonic or nominalistic - crystalized from above or constructed from below - made by nature or by culture - discovered or invented – revealed or named - a mirror of nature or a clientification of quantification?

A Didactics of Mathematics, Internalizing Mathematics

Presenting mathematics
Mathematics before applications or problems before solutions.
Expressions before equations or number sentences before calculations.
y=a*x, an example of a linear function f(x)= a*x , being an example of a relation between sets or an example of totaling a constant per-number T= a*n, being an abstraction of trade, motion etc.

Learning mathematics
Platonism: Concepts are implanted, knowledge is imported and exported as echoes, learning by heart, copying
Constructivism: Concepts are individually constructed on the basis of operations/communication (radical/social constructivism, Piaget/Vygotsky, Bruner), the cognitive revolution
Nominalism: Social concepts are names emerged from social practices, knowledge is partly narratives, partly tacit, learning is situated. The social praxises creating the narratives of geometry and algebra are "earth measuring" and "reuniting".

Curriculum Architecture

The social need for mathematics: A cultural tool to be applied or a number language for quantifying and predicting - a set of universal concepts or a social praxis of uniting and splitting totals.
The social control of mathematics: Abstraction based praxis or praxis based abstractions - Exams, assessments.

The curriculum space
Totaling narratives between abstract algebra and cultural practices:

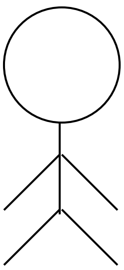
SET
PRODUCT
RELATION
FUNCTION

T=a*sin(mx)+ b*cos(nx)
T=Sfdx
T=ao+a1*x+a2*x^2+…
T=a^n
T=a*x+b*y
T= a*n
T=a+n

Energy, Particles, Molecules, Ions, Bacteria, Plants, Reptiles, Mammals, Humans, Food, Agriculture, Trade, Industry, Politics, Economy, Ecology, (IT) Information Technology

NATURE CULTURE

The two Language Houses



WORD-language	NUMBER-language	
Grammar	Mathematics	META-LANGUAGE
WORDS	NUMBERS (count./calc.)	LANGUAGE
sentences	equations	

PRAXISES REAL-WORLD

The Teacher as a Researcher

The three modern research paradigms looking for convincing consequence
Quantifying the field
Qualifying the field Top-Down: Theorized Ground
Qualifying the field Button-Up: Grounded Theory
The postmodern research paradigm looking for inspiring contingency

	Platonic Mathematics	**Nominalistic Mathematics**
Numbers	Never units	Always units
Subject rationale	Learn Math to apply Math	Practicing quantifying and calculating. (calculating is quicker than counting)
Whole numbers	1, 2, 3 3 + 4 = 2 + 5 = 7 different number names 467: Place values	T = 9 = 4(2) + 1 = 3(3) = 1(2) + 2(3) + 1 = etc. T = T/b(b), rearranging, changing T = 4(100) + 6(10) + 7(1)
Addition	1 + 1 = 2 a universal Truth	T = 1cm + 1m = 101cm, T = 1R + 1$ = 8R T = 1 + 1 = ? Local truths Change before adding: T = a(x) + b(y)
Multiplication	All numbers can be added and multiplied **Never**: divide by 0	All numbers can be multiplied **Never**: add unlike units
Fractions	$\frac{2}{5}$ An element of the set of rational numb. $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$ $\frac{2}{5} + \frac{1}{4} = \frac{2\cdot4}{5\cdot4} + \frac{1\cdot5}{4\cdot5} = \frac{8}{20} + \frac{5}{20} = \frac{13}{20}$ **Never**: $\frac{2}{5} + \frac{1}{4} = \frac{2+1}{5+4} = \frac{3}{9}$ **Never**: 2% + 1% = 3%	T = $\frac{2}{5}$(6) = 2(6/5): Fractions change units T = $\frac{2}{5}$(10) + $\frac{1}{5}$(15) = 7 = $\frac{7}{25}$(25) or T = $\frac{2}{5}$(15) + $\frac{1}{5}$(10) = 8 = $\frac{8}{25}$(25) or … T = $\frac{2}{5}$(5) + $\frac{1}{4}$(4) = 3 = $\frac{3}{9}$(9) or … T = 2%(10) + 1%(40) = 0.6 = $\frac{0.6}{50}$(50) = 1.2%(50) or …
Equations	4 + 3*x = 10 (4 + 3*x)-4 = 10-4 4 has the inverse -4 (3*x + 4)-4 = 6 + is commutative 3*x + (4-4) = 6 + is associative 3*x + 0 = 6 0 is neutral under + 3*x = 6 (3*x)* $\frac{1}{3}$ = 6* $\frac{1}{3}$ 3 has the inverse $\frac{1}{3}$ (x*3)* $\frac{1}{3}$ = 2 * is commutative x*(3* $\frac{1}{3}$) = 2 * is associative x* 1 = 2 1 is neutral under * x = 2 **L = {x∈R|4 + 3*x = 10} = {2}**	4 + 3*x = 10 4 + (3*x) = 10 3*x = 10-4 x = $\frac{(10-4)}{3}$ x = 2 ------------------------------ *Forward and backward calculation:* 4 + (3*x) = 10 + 4 ↑↓ -4 3*x = 10-4 *3 ↑↓ /3 x = $\frac{(10-4)}{3}$
Fundamental functions:	f(x) = b + a*x linear function f(x) = b*a^x exponential function f(x) = b*x^a potential function f(x) = ao + a1*x + a2*x^2 + …a polynomial	Totaling constant T = b + a*x units per day T = b*a^x percentages per day T = b*x^a percentages per percentages T = ao + a1*x + a2*x^2 + … totaling bundles
Integrals	$\int_a^b f(x)dx = \lim \Sigma f(x) * \Delta x$	$\Delta T = \int_a^b f(x)dx$ Totaling variable units/day
Differentials	f(x) is differentiable in xo if f(x) is continuous in xo and lim Δf/Δx exist	F is differentiable where f is locally linear Locally linear calculations: (1 + t)^n = 1 + n*t, sin(n + t) = sin(n) + cos(n)*t
Differential equations	$\int_a^b f(x)dx = F(b)-F(a)$, $F(x) = \int f(x)dx$	$\int_a^b f(x)dx = \int_a^b dF = F(b)-F(a)$, $dF = f(x)dx$

Worksheets in Bottom-up Understandings

Equations

Teacher/students = Explainer/understander

How can “Equations” be explained/understood?

Metaphysics	Example or application of an equation, i.e. a statement that can be true or false
TopDown	Platonism
Word	4+3*x = 10
BottomUp	Nominalism
Practice	Used for telling about a calculation of a total.

A Nominalistic Bottom-Up understanding of 4+3*x = 10

A practical problem: 3 kg @ ? R/kg total 10 R including a fee of 4 R

T = 4+3*x = 10 is a totaling narrative telling about a total and how it was calculated:

	*3		+4	
x	------>	3*x	------>	4+3*x
6/3	<------	10-4	<------	10
	/3		-4	

An equation is solved by turning the calculation around.

A Platonic Top-Down up understanding of 4+3*x = 10

4, 3, x and 10 are examples of numbers. + and * are examples of binary operations. = is an example of an equivalence relation. 4+3*x = 10 is an example of en equation, i.e. a statement that can be true or false.

A statement can be transformed by identical operations on both sides of the equation to keep the equivalence and to make it easier to identify the truth set of the equation.

4+3*x	= 10	
(4+3*x)-4	= 10-4	*4 has the inverse -4*
(3*x+4)-4	= 6	*+ is commutative*
3*x+(4-4)	= 6	*+ is associative*
3*x+0	= 6	*0 is identity under +*
3*x	= 6	
(3*x)* $\frac{1}{3}$	= 6* $\frac{1}{3}$	3 *has the inverse* $\frac{1}{3}$
(x*3)* $\frac{1}{3}$	= 2	** is commutative*
x*(3* $\frac{1}{3}$)	= 2	** is associative*
x* 1	= 2	*1 is identity under **
x	= 2	
L={x∈R|4+3*x =10}={2}		

4+(3*x)	=	10	
3*x	=	10-4	= 6
x	=	$\frac{(10-4)}{3}$	= 2

Forward and backward calculation:

4+(3*x) = 10

+4 ↑↓ *-4*

3*x = 10-4 = 6

**3* ↑↓ */3*

x = $\frac{(10-4)}{3}$ = 2

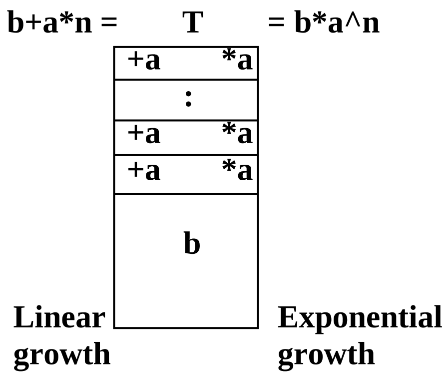
Linear and Exponential Functions

	T	a	b	n	T = a+b*n	T = a–b*n	$T = a + \frac{b}{n}$	$T = a - \frac{b}{n}$	$T = a*b^n$	$T = a+b^n$
17		2	1.75	4	9	–5	2.438	1.563	18.758	11.379
18		2.5	1.65	5	10.75	–5.75	2.83	2.17	30.575	14.730
19		3	1.55	–3	–1.65	7.65	2.483	3.517	0.806	3.269
20		3.5	1.45	–2	0.6	6.4	2.775	4.225	1.665	3.976
21	13		1.35	4.5	6.925	19.075	12.7	13.3	3.369	9.141
22	14		1.25	5.5	7.125	20.875	13.773	14.227	4.103	10.588
23	15		1.15	–2.5	17.875	12.125	15.46	14.54	21.273	14.295
24	16		1.05	–1.5	17.575	14.425	16.7	15.3	17.215	15.071
25	17	6		5	2.2	–2.2	55	–55	1.232	1.615
26	18	6.5		6	1.917	–1.917	69	–69	1.185	1.502
27	19	7		–2	–6	6	–24	24	0.607	0.289
28	20	7.5		–1	–12.5	12.5	–12.5	12.5	0.375	0.080
29	23	8	0.55		27.273	–27.273	0.037	–0.037	–1.766	–4.530
30	14	8.5	0.45		12.222	–12.222	0.082	–0.082	–0.625	–2.135
31	15	9	0.35		17.143	–17.143	0.058	–0.058	–0.487	–1.707
32	20	9.5	0.25		42	–42	0.024	–0.024	–0.537	–1.696

Invisible parenthesis reduces multiple calculations to single calculations

T = ?	**T = b+a*n**	T = ?	**T = b*a^n**
b = 100 a = 4 n = 5	T = b+(a*n) T = 100+(4*5) T = 120	b = 100 a = 1.04 n = 5	T = b*(a^n) T = 100*(1.04^5) T = 121.665
	exc. 17-20		
b = ?	T = b+a*n	b = ?	T = b*a^n
T = 100 a = 4 n = 5	T = b+(a*n) T–(a*n) = b 100–(4*5) = b 80 = b	T = 100 a = 1.04 n = 5	T = b*(a ^n) $\frac{T}{(a\ \hat{}n)} = b$ $\frac{100}{(1.04\ \hat{}5)} = b$ 82.193 = b
	exc. 21-24		
a = ?	**T = b+a*n**	a = ?	**T = b*a^n**
T = 100 b = 60 n = 5	T = b+(a*n) T–b = a*n $\frac{(T–b)}{n} = a$ $\frac{(100–60)}{5} = a$ 8 = a	T = 100 b = 40 n = 12	T = b*(a ^n) T/b = a ^n $\sqrt[n]{(T/b)} = a$ $\sqrt[12]{(100/40)} = a$ 1.079 = a
	exc. 25-28		
n = ?	**T = b–a*n**	n = ?	**T = b*a^n**
T = 100 b = 130 a = 2 n =	T = b–(a*n) T+(a*n) = b a*n = b–T $n = \frac{(b–T)}{a}$ $\frac{(130–100)}{2} = 15$	T = 100 b = 80 a = 1.04	T = b*(a ^n) T/b = a ^n $\frac{\log (T/b)}{\log a} = n$ $\frac{\log (100/80)}{\log 1.04} = n$ 5.689 = n

exc. 29-32



b: Initial number
T: Terminal number

n: Growth times

Increment per time — a — Growth per time

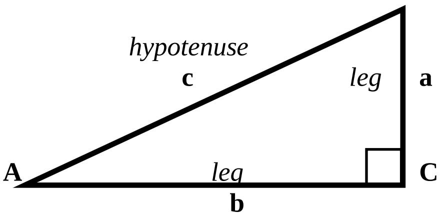
Linear growth has a constant increment: +5£, +5£, +5£. Linear growth is also called growth by adding or gradual growth.

Exponential growth has a constant growth percentage: +5%, +5%, +5%. Adding 5% makes the terminal number T 105% of the initial number b, i.e. T=b*105%=b*1.05 Exponential growth is also called growth by multiplying or interest growth

Trigonometry

A triangle has three known pieces (angles, sides) and three unknown pieces to be calculated.

The Greeks:

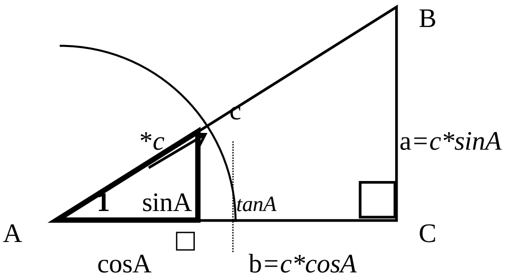


Equations

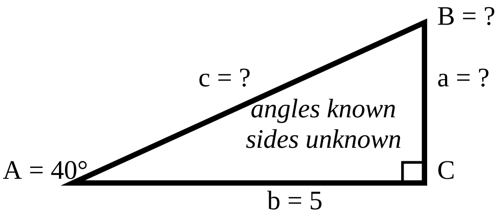
A+B = 90	A+B+C=180
$a^2+b^2 = c^2$	Pythagoras

Problem: One of the equations will contain two unknowns

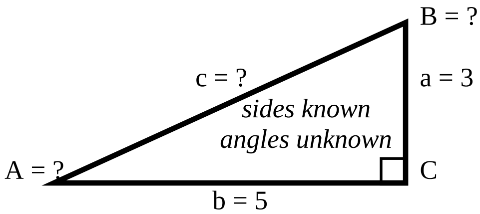
The Arabs: Inside a big triangle is a small standard triangle, which is named and described in tables.
SinA is **seen** from A. CosA is **at** A



sin A	$= \frac{\mathbf{a}}{\mathbf{c}} = \frac{\text{seen}}{\text{hyp}}$	= cos B	
cos A	$= \frac{\mathbf{b}}{\mathbf{c}} = \frac{\text{at}}{\text{hyp}}$	= sin B	
tan A	$= \frac{\mathbf{a}}{\mathbf{b}} = \frac{\text{seen}}{\text{at}}$	$= \frac{\text{sinA}}{\text{cosA}}$	$= \frac{a/c}{b/c}$



a = ?	$\text{tanA} = \frac{a}{b}$	**c = ?**	$\text{cosA} = \frac{b}{c}$
A=40 b=5	B*tanA = a 5*tan40=a **4.195=a**	A=40 b=5	C*cosA = b $c = \frac{b}{\text{cosA}}$ **c=6.527**



A = ?	$\text{tanA} = \frac{a}{b}$	**c = ?**	$a^2 + b^2 = c^2$
a=3 b=5	$\text{A=INVtan}\ \frac{a}{b}$ $\text{A=INVtan}\ \frac{3}{5}$ **A=30.96**	a=3 b=5	$\sqrt{(a^2 + b^2)} = c$ $\sqrt{(3^2 + 5^2)} = c$ $\sqrt{34} = c$ **5.831 = c**

Problems / **Answers**

	a	**b**	**c**	**A**	**B**	**C**		**a**	**b**	**c**	**A**	**B**
1			3,917	33,3		90		2,149	3,275			56,7
2			6,520	42,5		90		4,401	4,810			47,5
3			8,423	62,5		90		7,471	3,889			27,5
4	8,597			51,0		90			6,959	11,061		39,0
5	9,620			65,9		90			4,298	10,537		24,1
6	3,787			21,5		90			9,620	10,339		68,5
7	2,661		4,628			90			3,787		35,1	54,9
8	3,889		6,266			90			4,913		38,4	51,6
9	2,763		7,015			90			6,448		23,2	66,8
10	3,480	6,243				90				7,147	29,1	60,9
11	2,866	8,597				90				9,062	18,4	71,6
12	8,597	8,085				90				11,802	46,8	43,2
13	7,471	6,959			43,0	90				10,210	47,0	
14		5,527			32,7	90		8,597		10,220	57,3	
15			9,560		52,4	90		5,834	7,574		37,6	

Differential Calculus

Linear curves have constant slopes (gradients, per-numbers) $\frac{\Delta y}{\Delta x} = a$

*At linear curves the **total change** is Δy = a*Δx*

Curved curves are locally linear with locally constant slopes (gradients, per-numbers) $\frac{dy}{dx} = y'$

At locally linear curves the **local change** is dy = a*dx, and the **total change** is $\Delta y = \int dy = \int y'dx$

Locally linear functions: $(1 + dx)^n = 1 + n*dx$
$e^{dx} = 1 + dx$
$\sin(n + dx) = \sin(n) + \cos(n)dx$

$y = x^n$
x→ x + dx
y→ $(x + dx)^n = (x(1 + dx/x))^n = x^n(1 + dx/n)^n = x^n(1 + ndx/x) = x^n + n*x^n/x*dx = y + n*x^{n-1}*dx$
hence $dy = n*x^{n-1}*dx$
thus $dy/dx = n*x^{n-1}$

$y = e^x$
x→ x + dx
y→ $e^{(x + dx)} = e^x e^{dx} = e^x*(1 + dx) = e^x + e^x*dx = y + e^x dx$
hence $dy = e^x *dx$
thus $dy/dx = e^x$

If A = f*g then ΔA = Δf*g + f*Δg + Δf*Δg ~ Δf*g + f*Δg

and $\frac{\Delta A}{A} = \frac{\Delta(f*g)}{f*g} \sim \frac{\Delta f*g}{f*g} + \frac{f*\Delta g}{f*g} = \frac{\Delta f}{f} + \frac{\Delta g}{g}$, thus $\frac{\Delta(f*g)}{f*g} \sim \frac{\Delta f}{f} + \frac{\Delta g}{g}$

For small changes $\frac{d(f*g)}{f*g} = \frac{df}{f} + \frac{dg}{g}$, and $\frac{d(f*g)}{f*g}/dx = \frac{df}{f}/dx + \frac{dg}{g}/dx$, or $\frac{(f*g)'}{f*g} = \frac{f'}{f} + \frac{g'}{g}$

And $(f*g)' = f'\frac{f*g}{f} + \frac{f*g}{g}g' = f'g + fg'$, thus $(f*g)' = f'g + fg'$

If **$y = x^n$** then $dy/y = d(x^n)/x^n = n* (dx/x)$
and $dy = n*y/x\ dx = n*x^n/x\ dx = n*x^{n-1}\ dx$
thus **$dy/dx = n*x^{n-1}$**

The Pascall Triangle

e	T0	T1	T2	T3	T4	T5
	1	1	1	1	1	1
1	1	2	3	4	5	6
2	1	3	6	10	15	21
4	1	4	10	20	35	56
8	1	5	15	35	70	126
16	1	6	21	56	126	252
32	1	7	28	84	210	462

Notice:

T2 = Σ T1
Δ T2 = T1

etc.

Tn is growth of degree n
e is exponential growth

Limits and the ε-δ Definition

Linear change means constant rate of change (gradient):
If y = a*x, then y/x = a If meter = 3* second, then m/s = 3
If Δy = a*Δx, then Δy/Δx = a

Formal definitions of constancy:

1. Constant globally

f is constant c globally if for all arbitrary small numbers ε, $|\Delta f| = |f(x)-c| < \varepsilon$ for all x.

$$\forall\, \varepsilon>0 : \quad |\Delta f| = |f(x)-c| < \varepsilon \quad \text{for all } x$$

Is f(x) = 5 is globally constant?
Let ε = 0.001
$\|\Delta f\| = \|f-5\| = \|5-5\| = 0$, and 0 < ε for all x. Hence Is f(x) = 5 is globally constant

2. Constant in an interval

f is constant c in an interval if there exist a number δ>0 such that f is globally constant in the interval]xo-δ;xo + δ[, i.e. in a neighbourhood around xo.

$$\exists\,\delta > 0 \;\; \forall\, \varepsilon > 0 : \quad |\Delta f| = |f(x)-c| < \varepsilon \quad \text{when } |x-xo| < \delta$$

Is f(x) = 5 for x<3 and 7 for x≥3 constant in an interval?
Let ε = 0.001
$\|\Delta f\| = \|f-5\| = 0$ for x < 3 and 2 for x ≥ 3,
$\|\Delta f\| = \|f-7\| = 2$ for x < 3 and 0 for x ≥ 3, hence
f is constant 5 in all intervals to the left of x = 3.
f is constant 7 in all intervals to the right of x = 3.
f is not constant in any intervals containing x = 3.

3. Constant locally, practically constant, continuous

f is locally or practically constant or continuous in a point xo if for all practical values ε of 0 we can find a neighbourhood around xo where Δf is practically 0.

$$\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0: \quad |\Delta f| = |f(x)-c| < \varepsilon \quad \text{when } |x-xo| < \delta$$

Is f(x) = 5x locally constant in x = 3?
Let ε = 0.001
$\|\Delta f\| = \|5*x - 5*3\| = \|5*(x-3)\| < 0.0005 < \varepsilon$ for $\|x-3\| < 0.0001$
Hence f is locally constant 15 in the point x = 3

4. Linear locally, differentiable

f is locally linear or differentiable in a point x if the gradient f' = Δf /Δx is locally constant c in x, i.e. if for all practical values ε of 0 we can find a neighbourhood around x where $|\Delta f'| = |(\Delta f/\Delta x) - c|$ is practically 0.

Is f(x) = (1 + x)^n locally linear in x = 0?
f(x) can be expanded as a polynomial: f(x) = (1 + x)^n = 1 + n*x + n* (n-1)/2*x^2 + ..
$\|\Delta f'\| = \|(\Delta f/\Delta x) - c\| = \|(f(x)-f(0))/(x-0) - c\| = \|((1 + n*x + n*(n-1)/2*x^2)-1)/(x-0) - c\|$
$= \|(n*x + n*(n-1)/2*x^2)/x - c\| = \|(n + n*(n-1)/2*x) - n\| = \|n*(n-1)/2*x\| \to 0$ for $x \to 0$.
Hence f is locally linear in the point x = 0 and f' = n.

28. Celebrating the Luther year 1517 with some Theses on Mathematics and Education

To celebrate the 500year anniversary of the 95 Luther theses I decided to write two feature articles to a Danish newspaper. The first asks why mathematics, which was created as a straightforward natural science about the physical fact Many has to be presented as a metaphysical self-referring science that transforms many potential users to losers. The second asks If Europe really need Compulsory School Classes, arguing that the North American self-chosen half-year blocks might be a better way to support adolescents in their complicated identity work after puberty.

Mathematics, Banality or Evilness

Mathematics is steeped in evilness right from the first to the last class in the 12-year school, which we leave our children and young people to in the belief that the school will prepare them to master their environment and its two languages, the word language, and the number language called 'math' by the school. Strange, for we master our world through actions, by reading and writing and by counting and adding, so why is it necessary to learn to ' math'?

Thus the evilness of mathematics begins with its name; and by claiming that counting and adding are mere applications of mathematics, which, as such, of course, must first be learned before it can be applied; and which, unfortunately, is so difficult to learn, that it requires an extra effort leading still more to fail.

Also mathematics hides its origin. The ancient Greek Pythagoreans used the word as a common name for their four knowledge areas, music and stars and shapes and numbers, that constitutes ancient and medieval basic training, quadrivium, as recommended by the Greek philosopher Plato.

With music and astronomy out, today mathematics is just a common name for the two remaining areas, geometry, which in Greek means earth-measuring; and algebra, which in Arabic means to reunite numbers, and again hidden by the school, claiming instead that algebra means to search for patterns.

Algebra followed when the Renaissance replaced Roman numbers as CCXXXIV with the Arabic number 234 = 2 ten-tens and 3 tens and 4 ones = 2*10*10 + 3*10 + 4*1 showing algebra's four ways to unite numbers. Addition unites unlike numbers such as 3 + 4. Multiplication unites like plus-numbers such as 3 + 3 + 3 + 3 = 3*4; power unites like multipliers such as 3*3*3*3 = 3^4; and the three number-blocks 200, 30 and 4 are united by next-to addition, also called times-plus calculation, or integration, the Latin word for uniting.

And blocks is exactly what children bring to school. Asking a three-year child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, it's not four, that is two twos!"

So children come to school with two-dimensional block-numbers all carrying a unit, corresponding to Lego-blocks that stack as 1, 2, 3 or more 4ere. By combining geometry and algebra in their shapes and buds, blocks are highly suitable as a basis for connecting the starting point, children's block-numbers, with the final goal: algebra's uniting block-numbers illustrated by geometrical shapes.

However, the school is ignoring this and instead it teaches one-dimensional line-numbers located on a number line with each their name; and where the system will only be visible in the late twenties, where many children count over by saying 'ten-and-twenty' instead of 'thirty'. This then allows the school to pass a dyscalculia-diagnose and to institutionalize a corresponding dyscalculia-treatment supported by a growing dyscalculia-research with an associated dyscalculia-industry.

Evilness occurs when the school itself installs dyscalculia in the child by teaching line-numbers instead of block-numbers, thus teaching today's two-dimensional Arabic numbers, used by communities and kids, as if they were one-dimensional ancient Roman numbers.

Both number systems count by bundling.

Roman numbers use linear bundling: in a row of sticks, 5 1s are bundled to a V, 2 V'er to an X, 5 X's to a L, 2 Ls to a C, and so on. So a Roman number remains a one-dimensional string of letters as I, V, X, L, C etc.

Arabic numbers use rectangular bundling: in a row of sticks, twelve 1s are bundled to 1 ten-bundle and 2 unbundled, written as 12. Bundles then stack to a block of e.g. 4 10s, until ten bundles of 10s create a new block with the unit ten-ten or hundred, which then again stack in a block until ten of them create the unit ten-ten-ten or one thousand, etc.

So, where Roman numbers never have units, Arabic numbers always have, just as in children's own number system.

Nevertheless, the school teaches only in numbers without units. Likewise, the school does not distinguish between 2 * 3 = 6 and 2+3 = 5. The former is always true since 2 3s can be recounted to 6 ones. The latter is true only if the omitted units are the same: 2 days + 3 days is 5 days, but 2 weeks + 3 days is 17 days, and 2 days + 3 weeks is 23 days. Mathema-tics without units should be called 'mathema-tism', something that is true inside, but seldom outside a classroom. This would allow seeing if its diagnoses are created by teaching mathematics as mathematism.

Its evilness begins when mathematics neglects children's own Arabic number system and impose on them a Roman number system. It continues by forcing children to add before counting; and by forcing upon children the four operations in the order addition, subttraction, multiplication and division, where the last is presented so difficult that it triggers new dyscalculia diagnoses.

It is in fact the opposite order that is the natural. We count by bundling, so 7 sticks are counted in 3s by removing 3s many times, which is division predicted by a calculator as '7/3 = 2. something'. Then the 2 3s are stacked, which is multiplication. Removing the stack to look for unbundled is subtraction, predicted by a calculator as '7 – 2*3 = 1'. So, the calculator prediction holds true: 7 = 2.1 3s. Which shows that a natural number is a decimal number with a unit where the decimal point separates bundles form the unbundled. In contrast to the school that writes 5.6 tens as 56, i.e. without a unit and with a misplaced decimal point, and even calls such a number a natural number. An effective way to create even more diagnoses.

So counting includes the three operations division, multiplication and subtraction, and in that order.

After counting, it is natural to learn re-counting, back-counting and double-counting to change unit, or to create or remove an overload occurring when removing or adding. Thus, 7 can be recounted in the same unit 3s with or without an overload as 1.4 3s or 2.1 3s.

Recounting in a new unit means asking e.g. ‘how many 4s is 2 3s?'. We get the answer by a manual recounting, or by asking the calculator for a prediction: 2*3/4 = 1. something and 2*3 – 1*4 = 2, so 2 3s = 1.2 4ere.

Recounting the tens is done by pure multiplication: 3 8ere = 3*8 = 24 = 2.4 tens.

Back-counting from tens leads to solving equations. The question '5 tens is how many 4ere?' becomes the equation 50 = 4*x. The solution is obtained by recounting 50 in 4s, x = 50/4. So an equation is just another word for a back-counting, which means using the opposite operation, i.e. moving a number to the opposite side with the opposite sign. A natural approach easy to understand.

But, again silenced by the school, instead postponing equations to later grade levels. Here equations are presented as examples of open statements expressing equivalence between two numbers-names, and which teachers learn to solve using an abstract neutralization method.

Double-counting in different colors leads directly to the most important numbers, 'per-numbers', used to change units: If 3 red corresponds to 4 blue then 5 red correspond to how many blue? Or later: If 3 kg cost 4 $ then what is the cost of 5 kg? To answer we use the per-number 4$/3kg to recount the kilo-number 5 in 3s, 5/3, so many times we must pay 4$.

Changing unit is one of the two core areas of mathematics. However, the school does not recognize words as re-counting, back-counting, double- counting, or per-numbers. Instead, it uses the word 'proportionality', and again postpones it to later grades and makes it so difficult that new diagnoses are issued.

Why must children not learn the different ways of counting already in pre-school, where they count by themselves, time after time? Why does the school hide the great advantages in counting before adding? After all, totals must be counted before they can be added?

In addition, addition is not well defined: Should two blocks be added on-top or next-to each other, also called integration, the Latin word for uniting?

On-top addition means recounting to a common unit. But the school insists on using a so-called carry-method, which creates new diagnoses.

At the same time, the school only works with totals counted in tens. It is therefore unnecessary to change unit and to do next-to addition, the second main area of mathematics, and therefore more important than on-top addition; and that can be learned as early as pre-school by posing Lego-blocks next to each other and ask '3 2s plus 5 4s total how many 6s?' Nevertheless, school postpones it to the last school year with the claim that only the very best can learn next-to addition.

Reversing next-to addition is called differentiation. It asks e.g. '3 2s plus how many 4s gives 7 6s?'. Here we first remove the 3 2s with a minus before we recount the rest in 4s by division. So in reversed next-to addition subtraction comes before division. Of course, for in next-to addition, multiplication comes before addition.

But, the school does not recognize the words next-to addition or reversed next-to addition, nor does it recognize the word times/plus calculation or minus/division calculation. Instead, it introduces the Latin words integral and differential calculus and postpone both to the upper-secondary level where they are presented in reversed order, i.e. reversed next-to addition before next-to addition. Which makes both hard to understand with a high failure rate as a consequence.

A sly way to sabotage any high school reform. The parliament would like everyone to learn forward and reversed next-to addition, but both teachers and their teachers, the university professors, protest loudly: It cannot be done!

Of course it can, you just need to teach what is in the world, blocks to be united or split, and in that order, i.e. integration before differentiation. It is that simple to make calculus accessible to all.

So if the school allowed children and young people to meet its root Many as it naturally occurs in the world, i.e. as block-numbers that are counted, re-counted, back-counted, and double-counted, to be added on-top or next-to and forward or reversed, then everyone would learn everything in mathematics.

However, then no longer can mathematics be used for exclusion, which is precisely the school's main task, according to the sociologist Bourdieu. We think we got rid of the nobility with its privileges, but instead of a blood-nobility we got a knowledge-nobility protecting its monopoly on today's most important capital form, knowledge capital, by using the school to exercise what he calls symbolic violence.

The word-language cannot be used for exclusion since it is learnt before school. In contrast to the number-language that school can make so hard that is will be accessible to the nobility's own children alone. In other words, the same technique as the mandarin class used when they made the Chinese alphabet so difficult that only their children could pass the state's official exams.

But why do teachers accept to teach evil mathematics? Because of the banality of evil as described by Arendt in her book about Eichmann in Jerusalem. Here Arendt points to the lurking evil stored in blindly following orders in institutions originally created to ensure that good thing happens.

To keep your job, you must obey orders, 'conform or die'. Institutions do not compete as does the private labor market where 'compete or die' ensures control by the users' needs.

Together with skeptical post-modern thinking, also Arendt finds inspiration in the last century's great philosopher, Heidegger, who points out that to realize your existential potential you must have an authentic relationship with the surrounding things. To ensure this, we continually must ask if a thing's true existence is shown or hidden by institutionalized essence claims.

So, as an institution, mathematics education should continually ask whether it mediates an authentic image of its subject, the physical fact Many. Or, whether the institution is caught in what the sociologist Baumann calls a 'goal displacement', where the initial goal is transformed into a subordinate instrument to a new target: the institution's self-preservation.

Mathematics education could be a framework for children's and young people's authentic meetings with its physical root, Many. Instead, it has become an attempt to cure self-created diagnoses.

To deal with Many is simple and banal, so why drown the banality of mathematics in evilness?

Sensory perception, experience and common sense are the worst enemies of evil mathematics. So practice existence before essence, also in mathematics education. Which instead should comply with the international PISA-intention: To equip a population with knowledge and skills for the realization of their individual potentials.

Consequently, please drop the evil mathematics. Allow the child to develop its existing number language through guided learning meetings with its root, Many. Remove the evil textbooks on line-numbers and addition before counting. Use blocks and playing cards to illustrate block-numbers and activities such as counting, re-counting, back-counting and double-counting followed by forward and reversed on-top and next-to addition; and swap differential and integral calculus in high school, so all young people learn next-to addition both forth and back.

Again, Luther is right: Contact can be established individually without an institutionalized intermediary.

Does Europe really need Compulsory School Classes?

Compulsory classes force children and young people to follow the year group and its schedule. Compulsory classes made sense when created in Prussia about 200 years ago in an agricultural society; and also in industrial society with its permanent life jobs. In an IT-society, compulsory classes make sense in primary school: with both mother and father in changing self-realizing jobs, the first 3-4 school-years children need a warm and loving nanny with only one class, quickly getting a gaze of each child's characteristics and needs.

On the other hand, compulsory classes mean disaster in secondary school with young people who have left childhood and started an extensive identity work to uncover and develop their personal potential and talent. Here a compulsory class is the last thing they need, which is evident when observing the seven sins of compulsory classes.

Noise. Having an activity imposed that you do not master or find interesting, you quickly switch to other activities, surfing the Web or chatting with others in the same situation. The result is noise, which can be so violent that the rest of the class must wear hearing protectors.

Absence. Once you have given up on learning you feel a desire for absence, perhaps even to drop out. But that will hurt the school's economy, so you will not be allowed to leave the class regardless of your extent of absence.

Bullying. When you finally meet up again after an absence, it is tempting to bully those who meet every day.

Drinking. Especially if they do not want to participate in the extended weekend drinking starting in lower secondary school and coming to full expression in das Gymnasium, where many are sent to the hospital at the annual welcome parties or get hurt under excessive drinking on study tours.

Substitute teachers. Once you have conquered the territory, it is natural to bully also the various teachers who come to visit. Some can take it, others cannot and take a long-term sick leave. Skilled substitute teachers are expensive, so often a recent high school graduate is selected instead, or cleaning personnel.

Bottom marks. The extent of mental absence is shown by the written marks. Thus, in Denmark with 5 passing marks, the three lowest are given when answering correctly 16%, 33% or 50% at the final exam in mathematics at the end of lower secondary school. And here the international passing level at 70% gives the second-highest mark. The low level of learning can, however, be hidden by replacing written tests with oral, which is much more effective to increase the marks with floods of leading questions. Denmark is virtually the only country in the world maintaining an oral exam. Its credibility is illustrated by the joke, which is often exchanged over coffee table during an exam: With a friendly external examiner, a good teacher can examine a chair to a passing mark, provided the chair stays quiet.

War against boys. In a compulsory class, girls and boys are forced to go along, although the girls are two years ahead in development. It provides both with a skewed impression of the opposite sex, and school dislike makes boys leave school before upper secondary school, where there are two girls for every boy. In short, compulsory classes pump boys out of school to remain in the outskirts, while girls are pumped into the juggernaut universities in Copenhagen; and in Aarhus, where they then move to Copenhagen after graduation, since that is where the jobs are. With the absence of boys, girls find another girl and a sperm bank so that together they can get a single child.

Which creates the compulsory class' most fatal consequence, a birth-rate in Europe at 1½ child per family. A quick calculation shows that with 0.75 child per woman, Europe's population will halve twice over the course of 100 years. A population decline unprecedented in history.

Unlike in the North American republics. Here young people do not have multi-year compulsory classes. Instead, they are welcomed to high school with recognition: "Inside, you carry a talent that it is our mutual job to uncover and develop through daily lessons in self-chosen half-year academical or practical blocks together with a teacher who only teaches one subject. If successful we say 'good job, you have talent, try out more blocks'. If not we say 'good try, you have courage to try out the unknown, now let's find another block for you to try out. And at the last year you can try out college blocks."

Thus, the absence of multi-year educational defeats allows you to enter a local block-organized college at 18 and get a two-year practical diploma degree or continue at a regional college and get a four-year job-directed bachelor's degree.

Without compulsory classes, Europe could do the same, so that every other boy could be an engineer at the age of 22; and at the age of 25 have a well-paid job, a family, and three children ensuring state survival: one for mother, one for father, and one for the state.

As demonstrated in North America, compulsory classes are not a biological necessity.

As mammals, we are equipped with two brains, one for routines and one for feelings. When we raised up on our hind legs, we developed a third brain to keep balance; and to hold concepts since we could now use the front legs to grab the food and eat it or share it with others. In this way, gripping could provide the holes in our head with our two basic needs, food for the body and information to the brain. For by assigning sounds to what we grasp, we develop language to transfer information between brains.

In fact, we have two languages, a word language and a number language. At home children learn to talk and to count. Then as an institution, the school takes over and teaches children to read and to write and to calculate, and to live together with others in a democracy.

The ancient Greek sophists saw enlightenment as a prerequisite for democracy: knowing the difference between nature and choice, we can avoid hidden patronization in the form of choice presented as nature. The philosophers had the opposite view: Choice does not exist, since all physical things are but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Consequently, people should give up democracy and accept the patronage of the philosophers.

The Christian Church eagerly took over the idea of metaphysical patronage and converted the academies into monasteries, until the Reformation recovered the academies. Likewise, nor emperors nor kings had anything against being inserted by the Lord's grace.

Metaphysical patronage ended with Newton's three times no. "No, the moon does not move among the stars, it falls to the ground like an apple. No, moons and the apples do not follow a metaphysical unpredictable will; instead they follow their own will, which is predictable because it follows a formula. And no, a will does not maintain order, it changes it."

Once Newton discovered the existence of a non-metaphysical changing will, this created the foundation for the 1700 Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. The result was two republics, one in the United States and one in France. The United States still has its first Republic, France its fifth, since Prussia tried to overthrow the French Republic again and again.

France first got upper hand by mobilizing the population with enlightenment and democracy. As a counter measure, Prussia created a strong central administration with an associated 'Bildung' education with three goals: The population must be kept unenlightened so it will not demand democracy. Instead, Bildung must install nationalism transforming the population into a 'people', Germans, obeying the almighty Spirit by fighting other 'people', especially the French with their democracy. Finally, from the population, its elite must be sorted out to form a new central administration; and receive classical Bildung to become a new knowledge-nobility to replace the old blood-nobility, which was unable to strangle French enlightenment and democracy.

The rest of Europe eagerly took over the Prussian Bildung education. One might expect that when Europe became republics, its school form would follow. Here is only to say that still it is not too late. But it requires a comprehensive school reform, for the two school forms are very different.

In continental Europe, compulsory classes are replaced by a mess of competing compulsory lines in upper secondary school and with a confusion of more or less coordinated lines at the tertiary level leading to a 3year bachelor degree, usable only if supplemented with a 2year university directed master degree.

In the North American republics, compulsory classes stop after primary school. With self-chosen half-year blocks, learners can try something new each half year and continue if the trial was successful; and, as important, get out if it turns out to be an area outside your personal talent.

At the same time, the mark reflects the personal effort. Thus, at a half-year math block you can collect 700 points. The daily assignments give 100 points based on neatness, completeness and correctness. Late delivery does not count. The final test counts 200 points; and 400 points come from five tests, of which the lowest is neglected.

The 700 points corresponds to 100%, and the characters A, B and C correspond to 90%, 80% and 70% of the points. A score below 70% means that the block must be retaken or be replaced by another block.

At 18 you can continue at a regional four-year college, or a local two-year community college, which is divided into quarters so it's easy to take blocks while you work or during summer holidays.

Likewise, the block system makes it easier to change job in case of unemployment or a desire for new challenges.

But why don't Europe do the same? Because Europe is so over-institutionalized, that it cannot imagine a society without institutions. And once you have chosen institutions, the school is used to create public servants through compulsory classes in primary school and in a myriad of compulsory lines at the secondary and university level.

And compulsory classes mean disappearance of the freedom to develop your personal potential. Instead school struggle with its well documented seven sins. Sins, Europe believes it can eliminate through its political system. If it has not died out before.

29. Invitation to a Dialogue on Mathematics Education and its Research

March 2014. Inspired by the Chomsky-Foucault debate on Human Nature, www.youtube.com/watch?v=3wfNl2L0Gf8

Bo: Welcome to the MATHeCADEMY.net channel. My name is Bo. Today we discuss Mathematics education and its research. Humans communicate in languages, a word-language and a number-language. In the family, we learn to speak the word language, and we are taught to read and write in institutionalized education, also taking care of the number-language under the name Mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Today governments control education, guided by a growing research community. Still international tests show that the learning of the number language is deteriorating in many countries. This raises the question: If research cannot improve Mathematics education, then what can? I hope our two guests will provide some answers. I hope you will give both a statement and a comment to the other's statement.

Welcome to John. John has …

John: Thank you, Bo.

Bo: And welcome to Allan. Allan has been working as an ethnographer in different parts of education from secondary school to teacher education. Allan has created the web based MATHeCADEMY.net teaching teachers to teach Mathematics as a natural science about Many. In addition, Allan has written a series of papers for the ICME congresses collected in an ICME-trilogy.

Allan: Thank you, Bo.

01. Mathematics Itself

Bo: We begin with Mathematics. The ancient Greeks Pythagoreans used this word as a common label for what we know, which at that time was Arithmetic, Geometry, Astronomy and Music. Later Astronomy and Music left, and Algebra and Statistics came in. So today, Mathematics is a common label for Arithmetic, Algebra, Geometry and Statistics, or is it? And what about the so-called 'New Math' appearing in the 1960s, is it still around, or has it been replaced by a post New-Math, that might be the same as pre New-Math? In other words, has pre-modern Math replaced modern Math as post-modern Math? So, I would like to ask: 'What is Mathematics, and how is it connected to our number-language?'

John: Sentence. Sentence. Sentence. …

Allan: To me, it is the need to communicate about the natural fact 'Many' that created the number-language. In space, we constantly see many examples of Many; and in time Many is present as repetition. So, if Mathematics means what we know, we might want to add about Many, and use the word 'Manyology' as a parallel word for Mathematics.

To deal with Many we perform two actions, we count, and we add to answer the basic question 'how many'. This resonates with the action-words algebra and geometry meaning to reunite numbers in Arabic and to measure land in Greek. We count a given total in singles, bundles, bundles of bundles, etc. as shown by a number as five hundred and forty-three, consisting of 3 singles, 4 ten-bundles and 5 ten-bundles of ten-bundles. We see that all numbers carry units as ones, tens, ten-tens etc. Having the same unit, the 4 ten-bundles are added on-top of each other; and having different units, the 5 tens-tens and the 4 tens are added next-to each other as areas, also called integration, where shifting unit is called linearity. So, a three-digit number shows the core of Mathematics, which is linearity and integration. The number also shows the four different ways to unite numbers: by multiplication as in 4 tens, by power as in ten-tens, by vertical on-top addition as in 3 ones, and by horizontal next-to addition as in the juxtaposition of the three blocks with different units. Showing its bundle-size ten when written as 54.3 tens, the total also shows that singles can be written as decimals or as fractions where the 3 singles become 0.3 tens or 3 counted in tens, 3/10.

With unspecified bundle-number, a three-digit number becomes a formula, where the bundle-number can be found by reversing addition, also called solving equations.

So, Mathematics is very easy; and also, very easy to make hard. You just replace Mathematics with ‘Metamatism’, a mixture of ‘Meta-matics’ and ‘Mathema-tism’.

Mathematism is true in a library but not in a laboratory. Thus, statements as ‘2 + 3 is 5’ are found in any textbook even if it is falsified by countless outside examples, as e.g. 2 weeks and 3 days total 17 days.

Metamatics defines its concepts as examples of abstractions instead of as abstractions from examples, i.e. top-down and from above instead of bottom-up and from below. Thus, Metamatics defines a formula as an example of a set-product where first-component identity implies second-component identity, instead of, as Euler did, as a name for a calculation containing both numbers and letters. Defining concepts as examples of the ultimate abstraction, a set, makes Metamatics self-referring, and thus meaningless according to Russell’s set-paradox saying that the set of sets not belonging to itself will belong to itself if it does not belong, and vice versa. To avoid this paradox, Russell proposed a type-theory to distinguish between examples and abstractions, meaning e.g. that a fraction is not a number. Unwilling to accept this, modern set theory removes the difference between an element and a set, i.e. between an example and an abstraction, which still makes Metamatics meaningless since you can survive on examples of food but not on the label food; they enter different holes in the head.

Summing up, Mathematics can be a grounded natural science about the natural fact Many, thus becoming a number-language showing how numbers are built by using four different ways to unite: multiplication, power, on-top and next-to addition, that can all be reversed. However, Mathematics can also be an ungrounded self-referring Metamatism with set-derived definition and with statements that are claimed to be true even when confronted by counterexamples. In other words, Mathematics can be easy and accessible to all, or it can be made hard and accessible to an elite only.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

02. Education in General

Bo: Thank you, John, and Allan. Now let us talk about education in general. On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. To survive, plants need minerals, pumped in water from the ground through their leaves by the sun. Animals instead use their heart to pump the blood around and use the holes in the head to supply the stomach with food and the brain with information. Adapted through genes, reptiles reproduce in high numbers to survive. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a few children per year. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared. Education takes place in the family and in the workplace, and in institutions with primary, secondary, and tertiary education for children, for teenagers and for the workplace. Continental Europe uses words for education that do not exist in the English language such as Bildung, unterricht, erziehung, didactics, etc. Likewise, Europe still holds on to the line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics have block-organized talent developing education from secondary school. As to testing, some countries use centralized test where others use local testing. And some use written tests and others oral tests. So, my next question is ‘what is education?’

John: Sentence. Sentence. Sentence.

Allan: We adapt to the outside world through experience and advice, i.e. we are educated by the outside world and by other human beings. Children like to feel the outside world; teenagers like to gossip about it and about themselves; and adults must exchange actions with money to support a family. Thus, it makes sense to institute both primary, secondary, and tertiary education to serve the needs of children, teenagers, and adults. As an institution, education contains an element of force. Our language came from naming what we can grasp or point to, i.e. through a from-the-hand-to-the-head principle, called greifen-begreifen in German. So guiding children with concrete material to grasp, and teenagers with gossip to listen to makes education successful as described in Psychology by Piaget and Ausubel. On the other hand, forcing abstractions upon children and teenagers before introducing concrete materials or gossip excludes many children and teenagers from learning, thus creating a monopoly of knowledge as described in Sociology by e.g. Foucault and Bourdieu.

As to the space-and-time structure of education, primary education for children should be line-organized with yearly age-group-nannies as guides bringing the outside world to the classroom to develop concepts about nature described by a number-language, and concepts about society described by a word-language. In late primary school, this double nanny becomes two different nannies. Daily, the children also express themselves through music, art, or motion. The priority of to-do-subjects over to-be-subjects changes from primary to secondary school.

Transformed from children to teenagers able to have children of their own, the curiosity changes from the outside to the inside world, from things to persons. Being biologically programmed to remember gossip is useful if information about nature and society takes the form of gossip, i.e. statements with known subjects. Experimenting now is with what is inside oneself, e.g. as to talents. Consequently, secondary school should offer daily lessons in self-chosen half-year blocks to allow the individual teenager to test personal talents. If successful, the school says ‘good job, you need more of this’. If not, the school says ‘good try, you need to try something else’ to express admiration for the courage it takes to try out something new. This is how the North American republics organize a bottom-up secondary and tertiary education.

Being highly institutionalized, Europe hangs on to its line-organized school system preparing for public, created by Humboldt in Berlin shortly after 1800. Furthermore, the word ‘education’ is replaced by words as ‘unterricht’ and ‘erziehung’ and ‘Bildung’. Unterricht means handing down to those below you, and erziehung means dragging them up. These top-down words come from the Platonic patronizing view that the goal of education is to transmit and exemplify abstract knowledge.

The success of the French Enlightenment republic came from enlightening its population. To protect autocracy, the Prussian king asked Humboldt to create a school that could replace the blood-nobility unable to stop the French with a knowledge-nobility to occupy a strong public administration and to receive Bildung so it could go to court. This Bildung school should have two more goals: to prevent democracy, the population must not be enlightened; instead, the population must be transformed into a people proud of its history and willing to protect it against other people, especially the people from the French republic. To hide its anti-enlightening agenda, teacher education is based upon a special subject called didactics, confusing the teachers by claiming to determine the content of Bildung.

So to sum up, education can be bottom-up enlightenment allowing children to experiment with the outside world brought to the classroom, and allowing teenager to experiment with their inside talents through daily lessons in self-chosen half-year blocks that inform about the outside world in the form of gossip. Or education can be top-down Bildung trying to make the students accept patronization by abstract knowledge created at a distant university, where the best of them might be accepted later.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

03. Mathematics Education

Bo: Thank you, John, and Allan. Now let us talk about education in Mathematics, seen as one of the core subjects in schools together with reading and writing. However, there seems to be a difference here. If we deal with the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. However, you cannot Math, you can reckon. At the European continent reckoning, called 'Rechnung' in German, was an independent subject until the arrival of the so-called new Mathematics around 1960. When opened up, Mathematics still contains subjects as fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc. Today, Europe only offers classes in Mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. Therefore, I ask, 'what is Mathematics education?'

John: Sentence. Sentence. Sentence. …

Allan: The outside world contains many examples of Many: many persons, many houses, many days, etc. So, to adapt to the outside world, humans need to be deal with the natural fact Many, and this should be the goal of Mathematics education since the main contents of Mathematics was created as precisely that: statistics to count Many, algebra to reunite Many and geometry to count spatial forms. To deal with Many, we count and add. Counting takes place in the family and therefore integrates into preschool in a natural way. Since primary school only allows counting in tens, preschool can profit from the golden learning opportunities coming from icon-counting in numbers less than ten. Here first-order counting allows five ones to be bundled as one fives, transformed into one five-icon containing five strokes if written in a less sloppy way. Now second-order counting can count in icons so that seven sticks can be recounted in 1 five-bundle and two unbundled singles, written as 1 and 2 5s, or as 1.2 5s using the decimal point to separate bundles and unbundled. Which again can be recounted as 2.1 3s where changing units later is called proportionality and linearity. Once counted, totals can be added. To add on-top the units must be the same, so one of the totals must be recounted in the other's unit. Added next-to each other, the totals are added as areas which is called integration. And reversing addition means creating opposite operations to predict the result. Here the operations occur in their natural order, which is the opposite of what the school presents: to count in 5s we take away 5s many times, which is division. Then the bundles are stacked, which is multiplication. We might want to recount a stack by taking away one bundle to change it into singles, which is subtraction. Finally, stacks can be added on-top or next-to. By meeting concrete examples of Many, children learn to count and recount by bundling and stacking; and to add on-top and next-to. Later physical units introduce children to per-numbers when double-counting in two different units as e.g. 5 $ per 3 kg, or 5/3 $/kg.

Telling Mathematics as gossip makes learning easy for teenagers, biologically programmed to remember statements about known subjects. The formula for a number as 543, i.e. 5 tens-tens and 4 tens and 3 ones show the four ways to unite numbers: Multiplication, power, on-top addition and next-to addition, also called integration. With an unknown bundle-number, the number-formula becomes a polynomial containing basic relations between variable numbers as proportional, linear, exponential, power and quadratic formulas that tabled and graphed show the different forms of constant changing unit-numbers in pre-calculus. As to calculus, per-numbers can be constant in three different ways: globally, piecewise, and locally also called continuous; all added to totals by the area under the per-number graph i.e. by combining multiplication and addition. Reversed, the combination of subtraction and division, called differentiation, allows the per-number to be determined from the area. Many teenagers enjoy the beauty of uniting geometry and algebra in coordinate-geometry allowing a geometrical prediction of algebraic solutions and vice versa; as well as the fascinating post-diction by statistics of unpredictable numbers in probability.

To sum up: Mathematics education can be easy if grounded in the roots of Mathematics, the natural fact Many, to be dealt with by counting and adding making a natural number a decimal number with a unit. Counting and recounting in icons before counting in tens brings the core of Mathematics, linearity, and integration, to preschool; and allows solving equations and fractions to be introduced in the beginning of primary school as reversed addition and recounting in different physical units. Or Mathematics can be hard by allowing only counting in tens, by presenting a natural number without a decimal point and a unit, and by transforming Mathematics to Mathematism by adding numbers without units, claiming e.g. that 2 plus 3 is 5 despite many counterexamples, and by postponing proportionality and integration to the beginning and end of secondary school.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

04. The Learner

Bo: Thank you, John, and Allan. Now let us talk about at the humans involved in Mathematics education: Governments choose curricula, build schools, buy textbooks, and hire teachers to help learners learn. We begin with the learners. The tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning takes place through internal construction. Therefore, I ask 'what is a learner?'

John: Sentence. Sentence. Sentence. …

Allan: Again, let us assume that we adapt to the outside world through actions, physical and verbal. So, learning means acquiring proper actions, some verbalized and some tacit. Repetition makes learning effective. Repetition takes place in the family and in the workplace and can take place in school through daily lessons both for children and for teenagers. Also, allowing learners to carry out most of the homework at school will minimize the effect of the learners' different social backgrounds.

Again, we must distinguish between a child, a teenager, and an adult. Its biology programs a child to learn by grasping as described by Piaget, and a teenager to learn by gossip as described by Ausubel stressing the importance of connecting new knowledge to what the learner already knows. An adult is motivated to learn something from its use in the workplace. Piaget describes individual learning as creating schemata that can assimilate new examples or be accommodated to assimilate divergent examples. In contrast, Vygotsky describes learning as being able to connect the learner's individual knowledge zone with the abstract concepts of the actual knowledge regime.

The four answers to the question: "Where do concepts come from? From above or from below? Form the outside or from the inside?" create four learning rooms. The two traditional rooms, the transmitter room, and the constructivist room, say "above and outside" and "above and inside". The two hidden alternatives, the "fairy-tale room" and the apprentice room, say "below and outside" and "below and inside". The traditional rooms take Mathematics for granted and see the world as applying Mathematics. The hidden rooms have the opposite view seeing Many as granted and as a creator of Mathematics through the principle 'grasping by grasping'. The transmission room and the fairy-tale room facilitate learning through sentences with abstract and concrete subjects. The constructivist room and apprentice room facilitate learning through sentence-free meetings with abstract or concrete subjects.

A block-organized education allows the learners to change classes twice a year with a "good job" greeting if successful and a "good try" greeting if less successful aiming at keeping alive the curiosity of the teenager as to which talent is hidden inside. In Europe, its line-organized education forces the learner to stay in the class even if being less successful, or to be removed from class to special education, of to be to leave education and find a job as an unskilled worker.

To summarize: As to children, learning can be concept-building through daily contact with concrete materials. Or, learning can prevent concept-building by excluding concrete materials and by sporadic lessons. As to teenagers, learning can be expanding their personal narrative with authorized gossip enforced by daily lessons in self-chosen half-year blocks. Or learning can be preventing their narratives from growing by teaching unknown fact about unknown subjects, again enforced by sporadic lessons. Finally, to adults learning can be grounded in workplace examples, or learning can be ungrounded encapsulated knowledge claimed to become maybe useful later.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

05. The Teacher

Bo: Thank you, John, and Allan. Now let us talk about the teacher. It seems straightforward to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a 'Lehrer' thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate 'unterrichtung and erziehung and to develop qualifications and competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. And until lately, educating lehrers took place outside the university in special lehrer-schools. Thus, being a teacher does not seem to be that well-defined. Therefore, my next question is 'what is a teacher?'

John: Sentence. Sentence. Sentence. …

Allan: As with learning, we must differentiate between teaching children in primary school, teaching teenagers in secondary school and teaching adults in tertiary schools. A parent is an adult helping the child to supply its stomach with food and its brain with information, based upon a relationship of trust. Removed from the home in an institution, a child will look for a substitute parent, a nanny. To prevent them from becoming competing parents, a nanny only teaches one year-group and has only one class. The first year of primary school the nanny slowly splits up the outside world in things that we count and humans that we communicate with or about, thus laying the foundation to the two basic knowledge areas: nature with a number-language and society with a word language. At the end of primary school, a class has two nannies specialized in each of the two basic knowledge areas.

In secondary school, the teacher role changes from a nanny to an expert with special training in one or two subjects. Now teachers have their own classroom where they teach the different daily half-year groups in their subject in the form of gossip. Half-year classes allow the teachers and the learners to maintain a good relationship, since at the end of the half year all learners leave the class thanked with a "good job" if successful and a "good try" if less successful.

In tertiary education, the degree of specialization is higher demanding a master's degree in a theoretical subject or a license in a trade or in a craft.

At a block-organized university taking additional blocks allows a teacher to change career from primary to secondary or tertiary education, or to business, engineering, or other crafts, and vice versa. And the final choice between teaching preschool or primary or secondary school can be postponed to later in teacher education. In contrast, Europe's line-organized education forces a choice between the different level to be made before tertiary school, and forces teachers to stay in their public office for the rest of their working life.

To summarize, a teacher has different roles at block- and line-organized schools. At the former, a teacher for children is a nanny splitting up the world in two subject areas: nature with a number-language and society with a word-language. And for teenagers, teachers are experts telling about their specific knowledge area in the form of gossip. Both are educated at a university and able to change career by taking additional blocks. In line-organized education, a teacher specializes in several subjects, have several classes each day, and follows a class for several years. And once a teacher, always a teacher since line-organized universities typically force students to start all over if wanting to change form one line to another.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

06. The Political System

Bo: Thank you, John, and Allan. Now let us talk about governments. Humans live together in societies with different degrees of patronization. In the debate on patronization, the ancient Greek sophists argued that humans must be enlightened about the difference between nature and choice to prevent patronization by choices presented as nature. In contrast, the philosophers saw choice as an illusion since physical phenomena are but examples of metaphysical forms only visible to philosophers educated at Plato's Academy who consequently should be accepted as patronizors. Still today, democracies come in two forms with a low and high degree of institutionalized patronization using block-organized education for individual talent developing or using line-organized education for office preparation. As to exams, some governments prefer them centralized and some prefer them decentralized. As to curricula, the arrival of new Mathematics in the 1960s integrated its subfields under the common label Mathematics. Likewise, constructivism meant a change from lists of concepts to lists of competences. However, these changes came from Mathematics and education itself. So, my question is: 'Should governments interfere in Mathematics education?'

John: Sentence. Sentence. Sentence. …

Allan: A government must create an educational institution forcing children and teenagers to spend so much of their life in it that some Greenland teenagers even talk about being condemned to school. Thus, a government must decide how much force it will allow the educational institution to exercise. Likewise, a government should know the root and agenda of their present educational institution as well as alternatives practiced elsewhere in the world.

As to curricula, a government must decide if schools present concepts as exemplified from above or abstracted from below. As to structure, a government must choose between the block-organized enlightenment education of the North American Democracies aiming at developing individual talents; and the line-organized Bildung education in Europe created in Berlin around 1800 to prevent democracy from spreading from France and aiming at preparing for public offices.

Besides politicians, a government also includes public servants, called mandarins in the ancient Chinese empire. In Europe the French sociologist Bourdieu has pointed out that the mandarin class forms a new knowledge-nobility using the educational system to exercise symbolic violence so that their children inherit the parents' lucrative public offices; and that Mathematics is especially well suited for this purpose. Some countries, as e.g. Denmark, even hold on to oral exams, thus giving additional advantages to mandarin children.

In Europe, spreading out economical capital by creating a welfare state made socialist parties strong. However, they seem to neglect to spread out knowledge capital as well. After all, where economical capital is split up in a 'what I win, you lose' game, knowledge capital can be enjoyed by all in an all-win game. To me this paradox shows the strength of the mandarin class in Europe.

So to sum up. Yes, governments must create educational institutions, but should minimize its force as much as possible. Consequently, education should be block-organized from secondary school, and school subjects should be teaching grounded categories and knowledge. That is, Mathematics education must meet the human need to deal with the natural fact Many by counting and adding, i.e. by recounting in different units to root proportionality, by adding also next-to to root integration, and to reverse addition to root solving equations. And no, Europe should not hold on to its Humboldt line organized Bildung preparing the mandarin children to inherit their parents' public offices, created 200 years ago by the German nobility to induce nationalism into the population to keep democracy from spreading from France.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

07. Research

Bo: Thank you, John, and Allan. Now let us talk about research. Tradition often sees research as a search for laws built upon reliable data and validated by unfalsified predictions. The ancient Greek Pythagoreans found three metaphysical laws obeyed by physical examples. In a triangle, two angles and two sides can vary freely, but the third ones must obey a law. In addition, shortening a string must obey a simple ratio-law to create musical harmony. Their findings inspired Plato to create an academy where knowledge meant explaining physical phenomena as examples of metaphysical forms only visible to philosophers educated at his academy by scholasticism as ‘late opponents’ defending their comments on an already defended comment against three opponents. However, this method discovered no new metaphysical laws before Newton by discovering the gravitational law brought the priority back to the physical level, thus reinventing natural science using a laboratory to create reliable data and test library predictions. This natural science inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research outside the natural sciences still uses Plato scholastics. Except for the two Enlightenment republics where American Pragmatism used natural science as an inspiration for its Grounded Theory, and where French post-structuralism has revived the ancient Greek sophist skepticism towards hidden patronization in categories, correctness and institutions that are ungrounded. Using classrooms to gather data and test predictions, Mathematics education research could be a natural science, but it seems to prefer scholastics by researching, not Math education, but the research on Math education instead. To discuss this paradox I therefore ask, ‘what is research in general, and within Mathematics education specifically?’

John: Sentence. Sentence. Sentence. …

Allan: A ‘pencil-paradox’ illustrates the trust-problem in research. Placed between a ruler and a dictionary, a pencil can itself falsify a number by pointing to a different number, but it cannot falsify a word by pointing to a different word, so where number-statements may express natural correctness, word-statements express a political correctness valid inside a ruling truth regime. In other words, using numbers, natural science produces universal truth, and using words, human and social sciences produce local and temporary truths always threatened by competing truth regimes or paradigms as Kuhn called them. Psychology has a paradigm war between behaviorists and constructivists, and within constructivism between Vygotsky and Piaget disagreeing as to whether the learner shall adapt to the ruling paradigm or the other way around. Sociology has a paradigm war called the actor-structure controversy, where the North American republics see social life as created by the symbolic interaction between independent actors, while the institutionalized Europe traditionally sees social life as determined by structures like the gravitational laws of natural science. But accepting word-statements as being not nature but choice has created a research genre studying the social construction of different word-paradigms. The two Enlightenment republics have found ways around the pencil-paradox. North American reaction against traditional philosophy has created American Pragmatism and its symbolic interactionism insisting that categories and theory be grounded in observations. Thus, you must not enter a field with preconceived categories, and generated categories must accommodate to field resistance, thus paralleling the generation of collective and individual knowledge as described by Piaget both accepting the priority of observations as in natural science. Here counter-examples do not reject a category but splits it into sub-categories. In other words, both the courtroom and Grounded Theory base their categories upon action-statements and reject is-statements as prejudice, reserved for the judge and the researcher.

In the second Enlightenment republic, the French, patronization hidden in ungrounded words, sentences and institutions has developed the post-structural thinking of Derrida, Lyotard and Foucault. Derrida recommends deconstructing patronizing categories. Lyotard recommends challenging political correctness by inventing paralogy as dissension to the ruling consensus. Foucault recommends using concept archeology to uncover the pastoral power of the so-called human sciences, instead being disciplines disciplining themselves and their subject, thus silencing competing disciplines and forcing ungrounded identities upon humans as diagnoses to be cured by normalizing institutions applying these human sciences.

Inspired by this French skeptical thinking, postmodern contingency research has found another solution to the pencil paradox. Often postmodern thinking is seen as meaningless since its skepticism also must apply to itself. However, postmodern skepticism is a meta-statement about statements about the world and therefore not one of the statements about the world, against which it directs its skepticism. Of course, the liar paradox saying, ‘this sentence is false’ and being false if true and vice versa makes self-reference problematic, but postmodern thinking avoids self-reference by its meta-statement ‘Everything can be different, except the fact that everything can be different’. Thus, the ancient sophist warning against mixing up nature and choice makes it possible for postmodern contingency research to discover false nature by finding hidden alternatives to choices presented as nature. Within Mathematics education research, contingency research has successfully pointed out hidden alternatives to unquestioned traditions within numbers, operations, equations, teacher education, etc. as seen on the MATHeCADEMY.net website.

To sum up, research can be a bottom-up activity using outside world observations to generate categories and theories to test predictions, especially successful with the number-statements of natural sciences. Or research can be a top-down activity forcing the outside world to assimilate to operationalized categories from the ruling paradigm and using scholasticism to produce new researchers as late opponents defending comments on already defended comments against three opponents.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

08. Conflicting Theories

Bo: Thank you, John, and Allan. Of course, Mathematics education research builds upon and finds inspiration in external theories. However, some theories are conflicting. Within Psychology, constructivism has a controversy between Vygotsky and Piaget. Vygotsky sees education as building ladders from the present theory regime to the learners’ learning zones. Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing categories to grow out of concrete experiences and observations. Within Sociology, disagreement about the nature of knowledge began in ancient Greece where the sophists wanted it spread out as enlightenment to enable humans to practice democracy instead of allowing patronizing philosophers to monopolize it. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. As counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe’s line-organized office preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the post-structuralism of the French Enlightenment republic. Likewise, the two extreme examples of forced institutionalization in 20th century Europe, both terminated by the low institutionalized American Enlightenment republics, made thinkers as Baumann and Arendt point out that what made termination camps work was the authorized routines of modernity and the banality of evil. Reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job forces you to carry out both good and evil orders. As an example of a forced institution, this also becomes an issue in Mathematics Education. So I ask: What role do conflicting theories play in Mathematics education and its research?

John: Sentence. Sentence. Sentence. …

Allan: To me, Sociology is the basic theory when discussing Mathematics education and its research. Sociology asks the basic question: in the social space, do we need patronization or can we find mutual solutions using the threefold information-debate-choice method of a democracy? As pointed out, the debate on patronization began in ancient Greece between the philosophers and the sophists; and the debate is still with us today between socialist top-down critical theorists and skeptical bottom-up postmodern theorists. As a social institution, education contains an element of force, that can be patronizing or emancipating providing what is called 'Mündigkeit' in German. Europe maximizes the force-component by using line-organized office preparing education to force humans to stay in the line as long as possible, and to accept that their difficulties are caused by their inferiority to the children of the public office holders helping their children inherit their offices created to patronize the population. Whereas North America from secondary school minimizes the force-component by using daily lessons in self-chosen half-year blocks to uncover and develop the individual talent of the learner.

Likewise, Mathematics can serve both purposes. Presented from above as top-down falsified Metamatism, it becomes so hard to learn that it forces many learners to stop learning it. This is a minor problem with half-year blocks since leaving Mathematics does not force you to leave school, but it is a big problem at line-organized schools where leaving the line means leaving school for good. Presented bottom-up from below grounded in the natural fact Many, Mathematics becomes easy to learn; and the learner can keep on choosing more blocks until the interest may disappear, or in Europe the ordinary learner can stay longer on the line to the dislike of the public office holders, the mandarins.

Likewise, the controversy within Psychology between Vygotsky and Piaget as to how learning takes place also serves both sociological purposes. Presented top-down from above, concepts become hard to learn and force many learners to stop learning the concepts and to accept patronization by those who succeed learning them. In contrast, bottom-up concepts grounded from below in the outside world are easy to learn for children through the concrete material that roots the concepts; and for teenagers since knowing the subject of the sentence gives a Grounded Theory the form of gossip.

The need to keep their job forces teacher to follow the orders of their specific institution. When trained, teachers should as potential change agents be informed about the many choices of an educational institution and within Mathematics, so the individual teacher knows the difference between choice and nature, i.e. what can be changed and what cannot, in order to prevent being a victim of the banality of evil.

To sum up, a civilized teacher education should inform about the many examples of conflicting theories in Mathematics, in education and in research and should put more emphasis on the sociological consequences of unnecessary force in these three institutions.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

09. Me and Mathematics Education and Research

Bo: Thank you, John, and Allan. Now let us talk about your own experiences with Mathematics education and its research. In addition, I would like to ask you who are the most important theorists in Math education research in your opinion.

John: Sentence. Sentence. Sentence. …

Allan: I met Mathematics before the arrival of the so-called new Math. In elementary school we had reckoning, and in middle school we had written and oral reckoning together with arithmetic and geometry, and finally about 5% of us went on to the European high school called a 'gymnasium' where we met the word Mathematics for the first time; finally, at the university, Mathematics was to new Math from day one. Repetition and its roots to the outside world made reckoning easy to learn, likewise with geometry where we learned to construct different figures and met formal definitions and proofs. Introduced as letter-reckoning made arithmetic strange and difficult, especially when reducing letter fractions came along. At the gymnasium, the epsilon-delta definition of real numbers from day one killed the interest of most students; and likewise, during the first year at the university when geometry was replaced by n-dimensional linear algebra. Here Mathematics changed to Metamatics with top-down set-derived definitions and general proofs without examples to sort out the elite for graduate studies. Most students dropped out or failed the exam. I passed, but to get a meaningful job I decided to shift to architecture. However, at a Belgian library I met American textbooks presenting algebraic topology bottom-up as abstractions from examples instead of the other way around and I decided to become a math teacher teaching bottom-up meaningful Mathematics instead of the top-down meaningless Metamatics, that made the textbooks so hard to access for the students in the gymnasium.

As a teacher I learned, that using words derived from its roots made concepts much more understandable. Thus, most students had problems with the traditional textbook definitions and theorems of exponential functions introduced after the set-derived definition of a function. In contrast, saying that when growing by a constant multiplier, the end value y is the initial value b multiplied with the multiplier c x times, written as y equal b multiplied with c to the power of x made one student remark: 'Hey Mr. Teacher, this we already know, when do you teach us something we don't know?' So, I began to look for root-based names for the Mathematical concepts and was surprised to find the root of calculus as adding variable per-numbers, and to find that when epsilon and delta changes places we define a piecewise instead of a locally constant formula. Likewise, introducing integral calculus before differential calculus took the hardness out of calculus.

The discovery that hidden alternatives can change Mathematics from hard to easy brought me to Mathematics education research. Here the beauty and simplicity of the ancient Greek sophist warning against false nature by saying that unenlightened about the difference between nature and choice we risk being patronized by choices presented as nature made me develop contingency research aiming at discovering hidden alternatives to choices presented as nature. Likewise, I admired the beauty and simplicity of American Sociology where Berne talks about the three states of communication, parent, child and adult. These three states create two effective ways of communicating, child-parent where both accept the presence of authority, and adult-adult where both accept its absence; and several ineffective ways not agreeing upon the role of authority. In addition, I was fascinated about the resemblance between Piaget in Psychology and American Grounded Theory both inspired by natural science and describing how individual and collective learning means adapting knowledge to the outside world by assimilation and accommodation. And finally, I was caught by postmodern or post-structural skeptical thinking developed in the threatened French Enlightenment republic warning against patronization in our most basic institutions: our words, beliefs, cures and schools. Here I saw the patronizing techniques of the school: hiding understandable alternatives forces children and teenagers to accept the ruling choices as nature.

Searching for contingency, I found hidden words as icon-counting, next-to addition, reversed addition, and per-numbers. In addition, I found that Mathematics was created as a natural science about the natural fact Many. By teaching in the US, I found that teenagers can be allowed to develop their personal talent if Europe's line-organized office preparing education with forced classes are replaced with North American block-organized talent developing education with daily lessons in self-chosen half-year blocks. Furthermore, I found that Bourdieu might be right when warning against a knowledge nobility that use their public offices to protect the line-organized education to ensure that their children inherit their offices. And finally, Baumann's and Arendt's work on the extreme institutionalization in 20th century Europe made me realize that the problems in Mathematics education and its research might be caused by an exaggerated institutionalization that by forcing teachers to follow authorized routines makes them subjects to the banality of evil without knowing it and without wanting to be so.

10. How to Improve Mathematics Education

Bo: Thank you, John, and Allan. Let us finish by looking at what this is all about, Mathematics education. The first International Congress on Mathematics Education, ICME 1, took place in 1968, so we can say that Mathematics education research has about the same age as the new Mathematics emerging in the 1960s. With half a century of research, we should expect the problems in Mathematics education to have disappeared or at least decreased considerably. However, the decreasing results of international tests indicates that the opposite is the case. The paradox that researching Mathematics education seems to create more problems than solutions motivate my last question 'how can Mathematics education be improved?'

John: Sentence. Sentence. Sentence. …

Allan: Indeed, we have a paradox when the problems in Math education increase with its research. To solve it we can ask how well-defined Mathematics and education and research is? Or, as in the fairy tale Cinderella we can look for hidden alternatives that might please the prince and make the paradox disappear? The ruling tradition presents Mathematics as ungrounded Metamatism with meaningless self-referring concepts, and with statements falsified by the outside world. The hidden alternative presents Mathematics as grounded science about the natural fact Many. These two alternatives entail two different forms of teaching. One presents concepts as created from above as examples from abstractions as shown in the textbooks; the other show how concepts are created from below as abstractions from examples, facilitated by concrete material for children and relevant gossip for teenagers.

Theorists also come in two forms. One uses the Platonic tradition to present physical phenomena as examples of metaphysical forms discovered by and investigated by philosophers. The other sees theory as grounded in and adapting to its underlying reality that generates the theory's concepts and validates its statements.

Research also comes in two forms. One is self-referring scholasticism commenting on comments already defended against three opponents. The other is Grounded Theory seeing individual and collective knowledge creation as parallel processes, creating schemata that adapt to the outside world. Finally, education also comes in two forms, as line-organized office-preparation or as block-organized talent-developing.

So, to me, the choice within four factors determines the success of Mathematics education. Problems occur if Mathematics presents itself as Metamatism, if only top-down theorists are used, if research is scholastic, or if education uses force by choosing line-organized office preparation. When chosen simultaneously as in Europe, Mathematics education is in deep trouble, which of course suits the knowledge nobility well. To be successful, Mathematics must grow from its roots in the natural fact Many, only grounded bottom-up theorists must be used, research must be a natural science using the classrooms to generate categories and test predictions; and education must minimize its force by choosing block-organized talent development from secondary school. Having implemented the three latter, the North American republics only need to change Metamatism to grounded Mathematics to make their Mathematics education successful.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

Bo: Thank you, John, and Allan. I began by expressing the hope that you could provide some answers to the question 'If research cannot improve Mathematics education, then what can?' I now see that this debate has resulted in a several suggestions that I am sure practitioners and politicians will be eager to work with and be inspired by. Thank you, John, and Allan, for your time and for sharing your views with us.

John: You are welcome, Bo. I enjoyed very much to take part in this debate. Allan: So did I, Bo.

30. MrAlTarp YouTube videos

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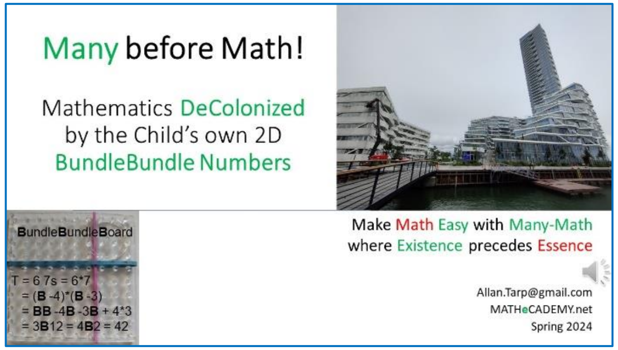
Many before Math, Math decolonized by the child’s own BundleBundle-Numbers with units.



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“Welcome children, I teach math, which is about the numbers on this number line, and that is built upon the fact that one plus one is two as you can see here. So …”

Then a child stopped me, holding four fingers together two by two: “Mister teacher, here is one, and here is one. You say it is two, but we can all see, that it is four”. The child separated the fingers, and then held three fingers together on both hands before separating them. “And, here we see that two times three gives six. So, multiplication can be trusted, but not addition. Therefore, please take away your number line addition with its flat-earth-math. Instead, please help us with the numbers we bring to school, multiplication bundle-numbers as 2 3s and 4 5s that we can see on this ten-by-ten peg board, a Bundle-Bundle board, or a BBBoard. And that we would like to add next-to as eights, or on-top as 3s or 5s. If we add them next-to, we add areas, which my uncle calls integral calculus, and if we add them on-top the units must be changed to the same unit, which my uncle calls proportionality. He says it is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school instead of having them colonized by your line numbers without units. 2404.

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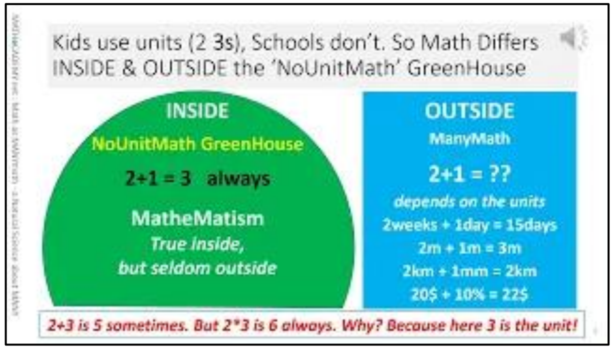
Online math opens for a communicative turn in number language education.



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“Of course, the goal of mathematics education is to master mathematics before it can be applied to master Many later.” Seeing this as a ‘goal displacement’ making a means a goal, sociology and difference research suggests a communicative turn as in foreign language education in the 1970s: Mastering Many maybe a more accessible way to master mathematics later. Likewise, existentialism holds that existence should precede essence. Preschoolers seeing 4 fingers 2 by 2 as ‘2 2s’ shows they master Many with 2D bundle-numbers with units. In this ‘BundleNumber-math’ adding is preceded by counting, which de-models division and multiplication as icons for a broom and a lift to push-away from the total T the unit-bundles B to be lifted into a stack. They combine in a ‘recount-formula’, T = (T/B)xB, predicting that T contains T/B Bs. Subtraction iconizes a rope to pull-away the stack to find unbundled that are placed on-top as decimals, fractions, or negatives. Addition iconizes adding on-top or next-to. De-modeling allows both school and education students to be guided by the concrete subject on their desktop instead of by an instructor on a screen, as exemplified by the MATHeCADEMY.net. 2306.

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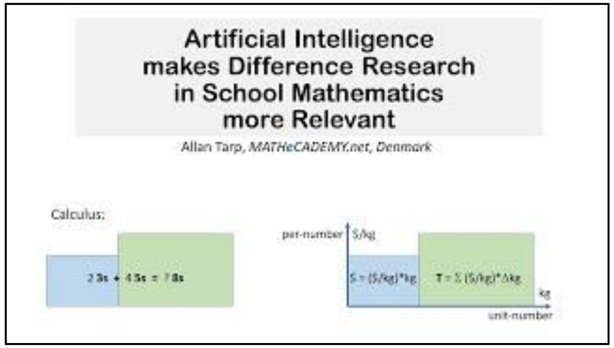
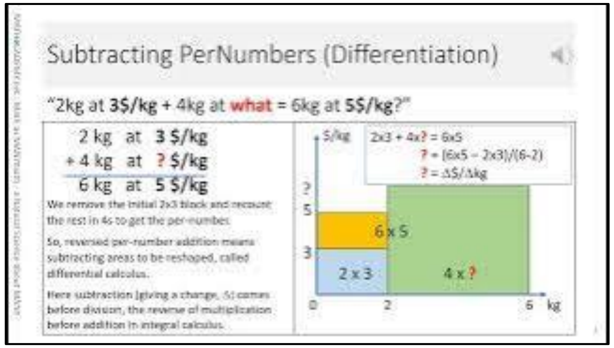
AI and Difference Research in Math Education



https://youtu.be/4EPqjz8evd4.

Research typically is seen as an example of a top-down or bottom-up lab-lib cooperation where laboratory observations are deduced from or are inducing library concepts. In the top-down version, library theory generates a hypothesis that, validated or falsified in the laboratory, leads to a stronger or adapted theory. In the bottom-up version, laboratory observations lead to categories, that additional observations may split into more nuanced subcategories. Artificial Intelligence has access to the library, but laboratory data must be added. Top-down research thus may be generated very quickly with a quality depending on the reliability of the data input, which may be difficult to check. In contrast, AI is of less help to bottom-up research typically generating new categories not yet present in the library. Also Difference Research ‘searching for differences making a difference’ (Tarp, 2018) may now be more relevant since although AI may locate existing differences, it cannot invent new differences. Nor can it examine the difference they make. Examples of difference research are bundle-numbers with units, operations as icons for counting, re-counting to change unit and to solve equations, per-numbers coming from recounting in two units, integration as addition of locally constant per-numbers, trigonometry before geometry, mathematism adding numbers without units, mastering Many before math, etc. Links Flexible Bundle Numbers Develop the Childs Innate Mastery of Many (https://youtu.be/z_FM3Mm5RmE). 2306.

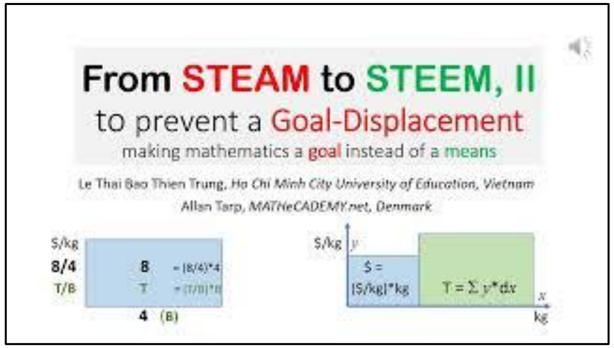
Continuous means locally constant



https://youtu.be/Cncg_2VEypY.

Deconstruction replaces strange words with natural words. Continuous thus means locally constant. Differentiable means locally linear. Calculus means reuniting per-numbers. Also watch the MrAlTarp video "Deconstructing Calculus" https://youtu.be/yNrLk2nYfaY. 2302.

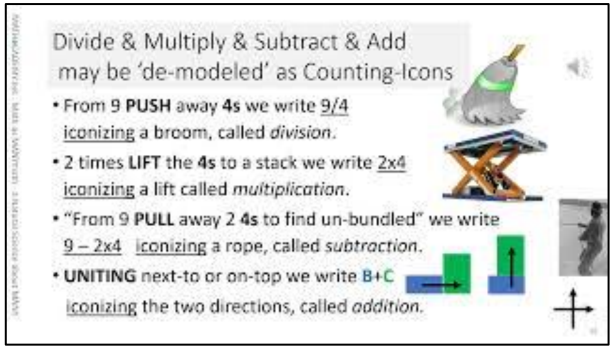
From STEAM to STEEM part II



https://youtu.be/pg9GeT9hG8M.

With only 4 Types of Numbers, Unit-Math is Easy. Algebra, in Arabic, means to reunite. With only 4 types of numbers, there are only 4 ways to unite, and 5 ways to split: Addition and multiplication unite changing and constant unit-numbers. Subtraction and division split into changing and constant unit-numbers. Integration and power unite changing and constant per-numbers. Differentiation and root/log split into changing and constant per-numbers. Here root means factor-finding, and logarithm means factor-counting. A Final Question concerning the 4th of the UN Sustainable Goals, Quality Education: Should Ethical Quality Education force children inside a 'no-unit-math' greenhouse that slowly strangles their innate number-language by using line-numbers to learn no-unit addition that folds outside? When children's innate mastery of Many just waits to be developed by flexible bundle-numbers, available at their fingertips. 2206.

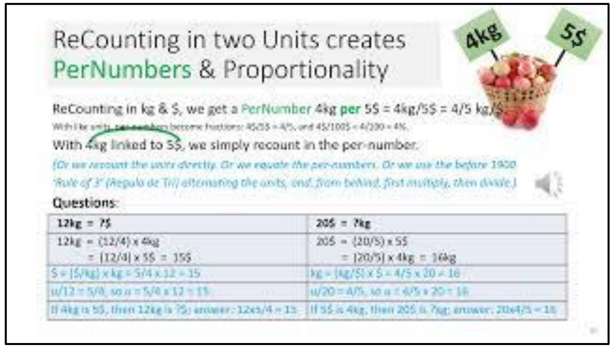
Flexible Bundle Numbers Develop the Childs Innate Mastery of Many



https://youtu.be/z_FM3Mm5RmE.

Apparently, we have 2 Mathematics Paradigms, one without Units, and one With Units • an inside 'no-unit-math' paradigm, where 1 plus 2 is 3 always, and • an outside 'unit-math' paradigm, where 1 plus 2 depends on the units The 'unit-math' paradigm builds on the philosophy, EXISTENTIALISM, where EXISTENCE precedes ESSENCE. So, unit-math describes real existence, and neglects institutionalized essence. The outside, 'unit-math' paradigm, provides the same mathematics, as the inside, 'no-unit-math' paradigm, only in a different order. And, the 'unit-math' paradigm, avoids the inside paradigm's 'mathema-tism', with its falsifiable, addition-claims. So, to become a full science, mathematics should leave, its 1 plus 2 is 3, 'no-unit-math' greenhouse, and accept that, of course, numbers cannot add, without units. It should teach the outside 'counting-before-adding', 'unit-math', paradigm, where Numbers and operations are icons, linked directly to existing things, and actions. 2112.

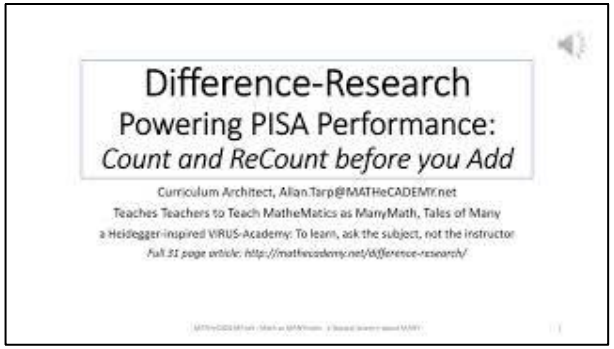
Children's innate Mastery of Many developed by flexible bundle-numbers



https://youtu.be/1v1PKi_rAOM.

Outside, addition folds but multiplication holds, since factors are units while addition needs like units. This creates two paradigms in mathematics, an outside 'unit' paradigm, and an inside 'no-unit' paradigm making mathematics a semi-greenhouse. To make mathematics a true science with valid knowledge, we ask what mathematics can grow from bundle-numbers with units, being areas instead of points on a number line. Concretely constructed, digits become number-icons with as many sticks as they represent, and operations become counting-icons for pushing, lifting and pulling away bundles to be added next-to or on-top. Recounting 8 in 2s creates a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. By changing units, it occurs as proportionality formulas in science; it solves equations; and it shows that per-numbers and fractions, T/B, are not numbers, but operators needing numbers to become numbers. Fractions, decimals, and negative numbers describe how to see the unbundled. Recounting sides in a box halved by its diagonal allows trigonometry to precede plane and coordinate geometry. Once counted, total may add on-top after recounting makes the units the same; or next-to addition by adding areas as in integral calculus, which also occurs when adding per-numbers. So, mathematics created outside the 'no-units' greenhouse is the same as inside, only the order is different, and all is linked directly to outside things and actions making it easier to be applied. And, with multiplication preceding it, addition only occurs as integral calculus, unless inside brackets with like units. 2109.

To master Many Recount before Adding



https://youtu.be/eEYoU-bvPrI.

Outside, addition folds but multiplication holds, since factors are units while addition presupposes like units. This creates two paradigms in mathematics, an outside 'unit' paradigm, and an inside 'no-unit' paradigm making mathematics a semi-greenhouse. To make mathematics a true science with valid knowledge, we ask what mathematics can grow from bundle-numbers with units, being areas instead of points on a number line. Concretely constructed, digits become number-icons with as many sticks as they represent, and operations become counting-icons for pushing, lifting and pulling away bundles to be added next-to or on-top. Recounting 8 in 2s creates a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. By changing units, it occurs as proportionality formulas in science; it solves equations; and it shows that per-numbers and fractions, T/B, are not numbers, but operators needing numbers to become numbers. 2107.

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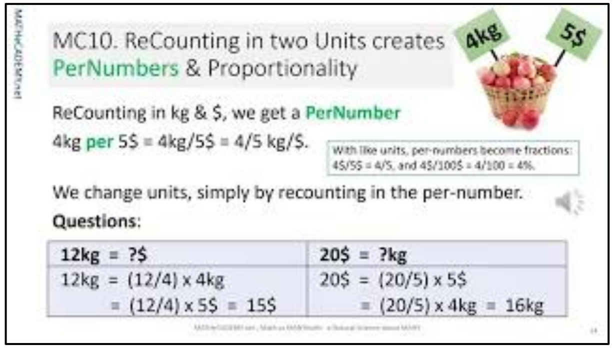
Bring Back Brains from Special Education in Mathematics



https://youtu.be/MZOAZK49omg.
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From loser to user, from special to general education, learning inside mathematics through outside actions. Paper presented at the ICME 14, 2021, Topic Study Group 36, Research on classroom practice at primary level. Abstract: Although eager to begin school, some children soon fall behind and are sent to special education teaching the same at a slower pace. Wanting mathematics education to be for all leads to the question: Is this so by nature or by choice? Can it be otherwise? Observing how children communicate about Many before school, this paper asks what kind of mathematics can be learned if accepting the bundle-numbers as 2 3s that children bring to school. Using Difference Research, it turns out that accepting numbers with units means that counting, recounting, and solving equations come before adding on-top or next-to introduce integral and differential calculus as well as proportionality in early childhood education. So, it is possible to institute an ethical mathematics education that transforms losers to users returning to general education as stars teaching fellow students and teachers how to master Many. 2107.
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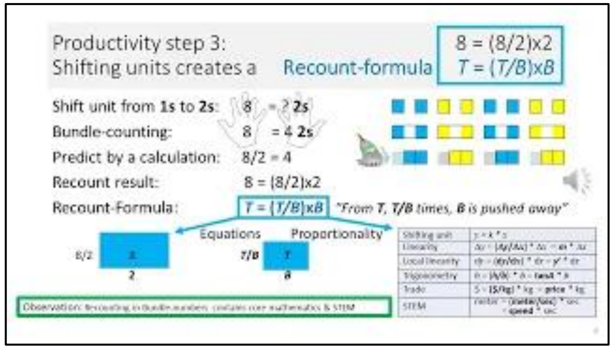
From STEAM to STEEM



https://youtu.be/t7Cf0qgBcWE.
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Yes, core mathematics may be learned through its historic root, economics, describing how humans share what they produce Asking “How many did I produce?” roots counting, predicted by division iconizing a broom pushing away bundles, to be stacked by a multiplication-lift, to be pulled away by a subtraction-rope to look for unbundled singles, to be added on-top or next-to, thus rooting decimal and negative numbers Recounting in a new unit creates a recount formula, used to solve equations, and to change units as in most STEM formulas Uniting stacks on-top or next-to roots proportionality or calculus So why make mathematics hard when it may also be easy & meaningful? 2107.
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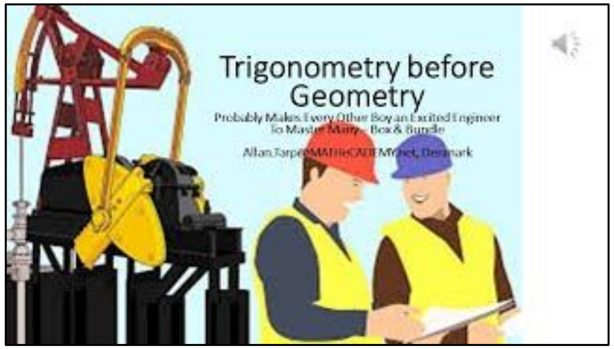
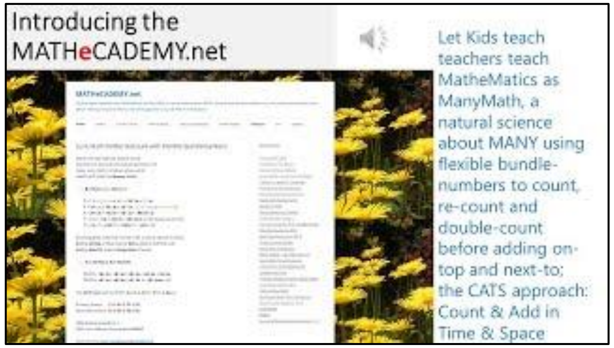
Trigonometry Before Geometry Probably Makes Every Other Boy an Excited Engineer



Before 1970 both foreign language and mathematics were hard to learn because the two taught grammar before language. Then a turn took place in foreign language education allowing students to learn it through communication. Mathematics education never had a similar turn, so it is still hard for many. Therefore, this video asks if it is possible to learn mathematics as communication. Being inspired by the fact that children communicate about the physical fact Many with two-dimensional box- and bundle-numbers with units, a curriculum is designed where trigonometry is rooted in a mutual recounting of the three sides in a box halved by its diagonal. So, the answer is: Yes, core mathematics can be learned as communication about boxes since it is directly connected to counting and recounting Many in boxes and bundles. Details at http://mathecademy.net/imcic-spring-2021/. 2103.

https://youtu.be/yEG27dNgCzE.

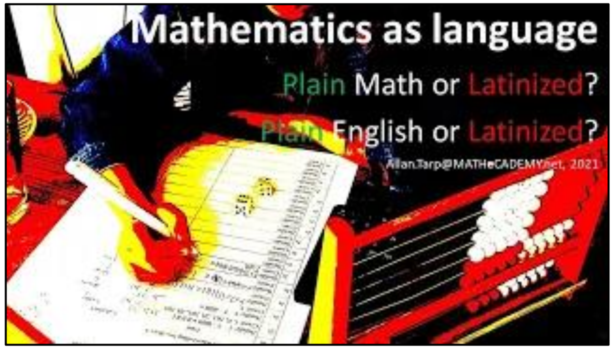
Introducing the MATHeCADEMY dot net



https://youtu.be/CbRDG64onKA.

The MATHeCADEMY.net teaches teachers teach mathematics as ‘many-math’, a natural science about Many. It is a virus academy saying: To learn mathematics, don’t ask the instructor, ask Many. To deal with Many, we Count and Add in Time and Space. The material is question-based. Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects? Secondary School. COUNT: How can we count possibilities? How can we predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? QUANTITATIVE LITERATURE, what is that? Does it also have the 3 different genres: fact, fiction and fiddle? PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn. The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher’s routine-assignment. Each pair is the opponent on the essay of another pair. Details at http://mathecademy.net/. 2103.

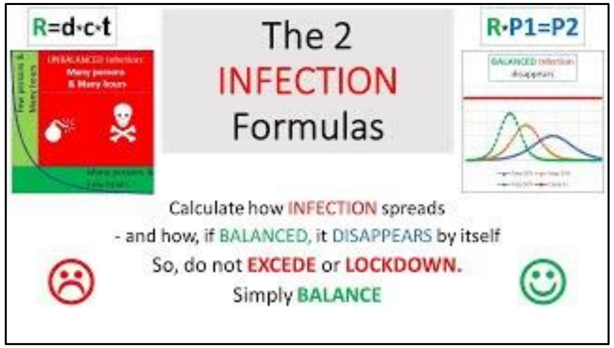
Mathematics language or grammar



Before 1970 both foreign language and mathematics were hard to learn because the two taught grammar before language. Then a turn took place in foreign language education allowing students to learn it through communication. Mathematics education never had a similar turn, so it is still hard for many. Therefore, this video asks if it is possible to learn mathematics as communication. Being inspired by the fact that children communicate about the physical fact Many with two-dimensional box- and bundle-numbers with units, a curriculum is designed respecting this. So, the answer is: Yes, core mathematics can be learned as communication about boxes since it is directly connected to counting and recounting Many in boxes and bundles. Details at http://mathecademy.net/imcic-spring-2021/. 2103.

https://youtu.be/h-0RmgX4E0w.

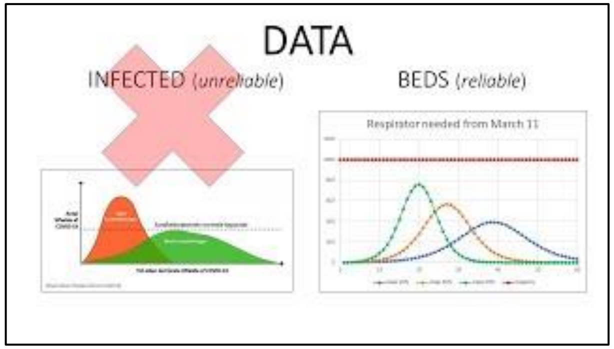
The two infection formulas, part 1



Part one is about the two infection formulas. Part two will be about the Danish lockdown. Infection follows 2 simple formulas that predict its behavior. The first infection formula is a 3-factor formula. It tells us, how the infection spreads. For example, the infection may explode in unbalanced meetings, where many persons are together, for many hours. Therefore, balance meetings in space, and in time. Few persons in many hours, or many persons in a few hours. The second infection formula is a hill-formula. It tells us, that if balanced, the infection will disappear by itself. Therefore, the message of the 2 infection formulas is: do not exceed. Do not lockdown. Simply balance. Details may be found at http://mathecademy.net/corona-infection-model/. 2006.

https://youtu.be/nUsnQa6gi0U.

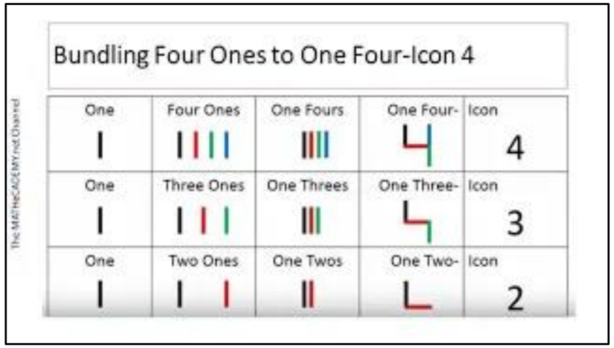
The two infection formulas, part 2



Part one is about the two infection formulas. Part two is about the Danish lockdown that was based on five things. It used unreliable data from infected, not reliable data from hospitalized. It neglected the 2 infection formulas saying: balance, do not lockdown. It scared the population with the Italian after-ski virus-greenhouses, even if they could never occur in Denmark, according to the 3-factor formula. It scared the parliament to pass a state of emergency. It silenced time as an infection factor. As a consequence, there never was a Danish corona crisis. Instead, there was a corona scandal, that created a financial crisis. Details may be found at http://mathecademy.net/corona-infection-model/. 2006.

https://youtu.be/EKPpu7LWbKc.

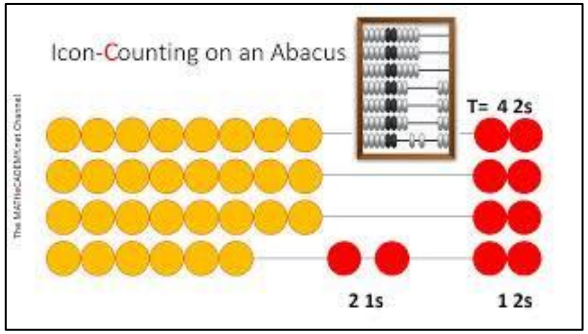
CupCount and ReCount before you Add



Math dislike disappears when children count and recount before they add and use cups to count in icons before counting in tens. Made with ezvid, free download at http://ezvid.com. 1605.

https://youtu.be/IE5nk2YEQIA.

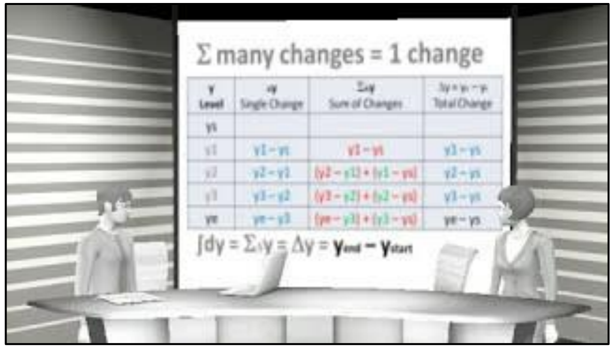
Preschoolers learn Linearity & Integration by Icon-Counting & NextTo-Addition



In PreSchool, allowing IconCounting and NextTo-Addition leads to Linearity & Integration. Thus adding 2 3s and 4 5s on-top, both must be recounted in a common unit, which is linearity. And added next-to each other as 8s, their areas are added, which is integration. Made with ezvid, free download at http://ezvid.com. 1402.

https://youtu.be/R2PQJG3WSQY.

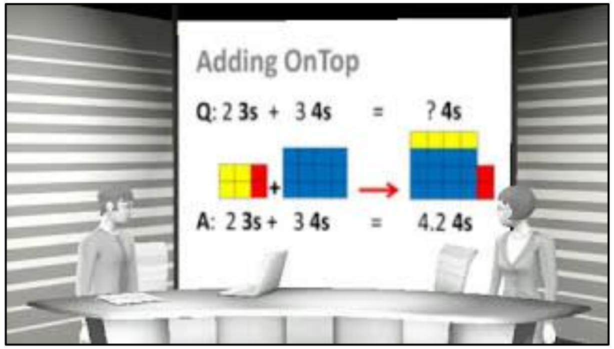
Deconstructing Calculus



Calculus adds per-numbers by areas. Adding 3 kg at 4 $/kg and 5 kg at 6 $/kg, the unit-numbers 3 and 5 are added as 3+5, and the per-numbers 4 and 6 are added as 3*4+5*6, i.e. as the area under the per-number graph, i.e. as integration combining * and +. As reversed integration, differentiation produces per-numbers by combining - and /. 1305.

https://youtu.be/yNrLk2nYfaY.

Deconstructing PreSchool Mathematics



This video on early childhood mathematics shows that children can learn linearity, integration and solving equations when allowed to count in icons less than ten, e.g. in 3s or 5s. To be added on-top, a unit must be changed, later called proportionality or linearity. Adding next-to as 8s is later called integration. And reversed addition is solving equations. Script and screens can be found on MATHeCADEMY.net/videos. 1305.

https://youtu.be/qgCwVZnALXA.

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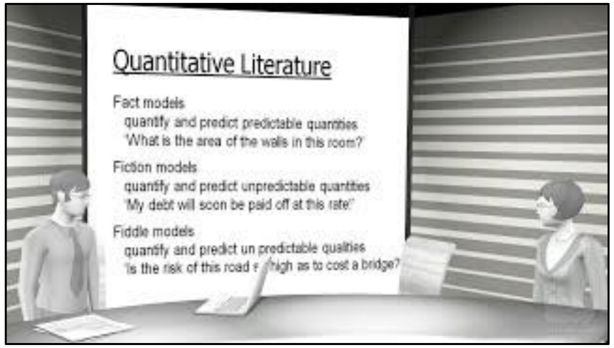
<td>Deconstructing PreCalculus Mathematics

</td>
<td>This video on deconstruction in mathematics education shows how precalculus can be made easy by three stories about formulas: how formulas predict, how formulas are transformed into functions with two variables or equation with one variable - both to be handled by a graphical calculator, and how formulas can be used to real world modeling. 1208.

https://youtu.be/3C39Pzos9DQ.</td>
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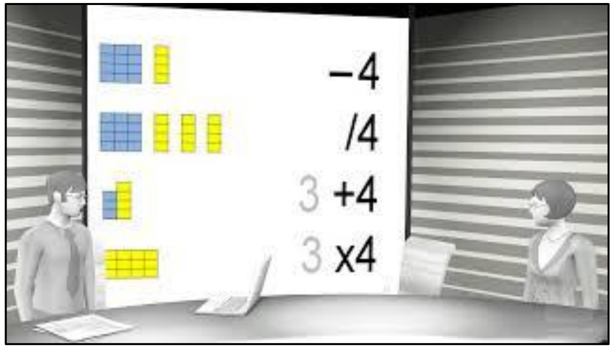
<td>Deconstructing Fractions

</td>
<td>This video on deconstruction in mathematics education connects fractions to its root, the leftovers when performing icon-counting. To deal with Many, we total by bundling in icon-numbers less than ten, or in tens needing no icon as the standard bundle. When bundling in 5s, 3 leftovers become 0.3 5s or 3/5 5s, thus leftovers root both decimal fractions and ordinary fractions. 1207.

https://youtu.be/PtRuk0EWmaQ.</td>
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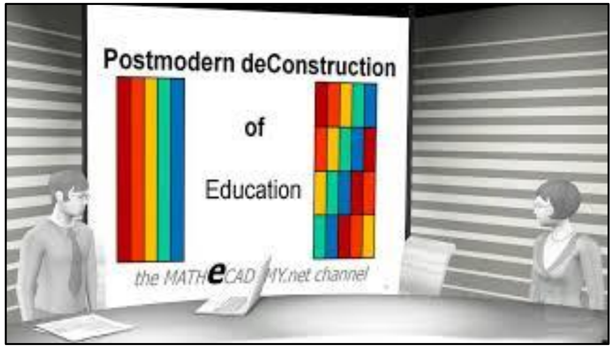
A Postmodern Deconstruction of World History



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This video on postmodern deconstruction describes world history as the history of trade. First eastern lowland pepper and silk was traded with western highland silver, then eastern cotton was moved west and traded with northern industrial products, and finally electrons replaced the silver and cotton economy with an information economy. Script and screens can be found on MATHeCADEMY.net/videos. 1206.

https://youtu.be/xQAdrI_CvyY.

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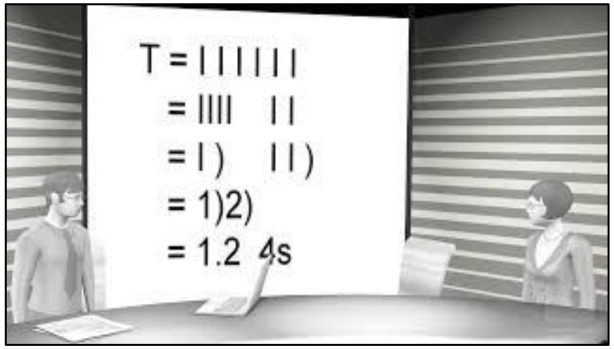
8 Missing Links of Mandarin Math I



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This video on deconstruction in mathematics education describes how natural mathematics is made difficult by removing eight links to its roots, the natural fact Many. The missing links make mathematics a privilege to a mandarin class wanting to monopolize public offices. Reopening the eight missing links will make mathematics easy and accessible to all. This session is addressing primary mathematics. The second session is addressing secondary mathematics. 1206.

https://youtu.be/sTJiQEOTpAM.

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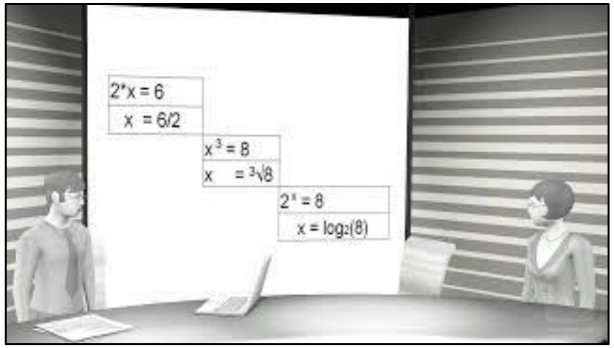
8 Missing Links of Mandarin Math II  	This video on deconstruction in mathematics education describes how natural mathematics is made difficult by removing eight links to its roots, the natural fact Many. The missing links make mathematics a privilege to a mandarin class wanting to monopolize public offices. Reopening the eight missing links will make mathematics easy and accessible to all. The first session is addressing primary mathematics. This session is addressing secondary mathematics. 1206. https://youtu.be/MItYFL-3JnU.
A Postmodern Mathematics Education 	Paul and Allan are asked to discuss eight questions on postmodern mathematics education: What is meant by postmodern? What is meant by modern? What is the root of postmodern thinking? Who is the most important postmodern thinker? What is mathematics? What is postmodern mathematics? What is postmodern research? 1204. https://youtu.be/ArKY2y_ve_U.

Two-level Table of Contents

Contents