Math Ed & Research 2020

No Power Point Presentations

The same Mathematics Curriculum for Different Students
Addition-free STEM-based Math for Migrants
Math Dislike Cured with Inside-Outside Deconstruction
Developing the Child's Own Mastery of Many
What is Math - and Why Learn it?
Flexible BundleNumbers

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Preface

To be edited

01. CONCEPTUAL CHANGE AND COMPULSION IN PRIMARY MATHEMATICS EDUCATION

STATEMENT OF THE THEORETICAL PROBLEM

The OECD 2015-report ‘Improving Schools in Sweden’ describes how “one out of four students” do not “demonstrate competencies to actively participate in life (p. 3)”. So, we ask: Is it so by nature, or could it be different? Difference research (Tarp, 2018) may give an answer.

AN ACCOUNT OF THE THEORETICAL PROPOSAL BEING MADE

Adapting to outside quantity, children develop two-dimensional double-numbers as 2 3s. Using sticks, children see digits as icons with as many sticks as they represent if written less sloppy. And, recounting 8 in 2s, children see bundle-counting as pushing away bundles, iconised as a broom called division; stacked as 4x2, iconized as a lift called multiplication; to be pulled away, iconized as a rope called subtraction in order to add the unbundled singles next-to or on-top, iconized as a cross showing the two directions.

Writing 8 = (8/2)x2, unspecified numbers gives a recount-formula T = (T/B)xB, saying “from a total T, T/B times, bundles of Bs may be taken away and stacked.” By changing units, this formula expresses proportionality, thus becoming the typical STEM-formula.

Added next-to or on-top, unbundled become decimals, fractions or negatives thus introducing ‘flexible bundle-numbers’: T = 7 = 2B1 3s = 2.1 3s = 2 1/3 3s = 3.-2 3s = 1.4 3s.

Rephrasing “recount 8 in 2s” to “how many 2s are there in 8?” creates the equation ux2 = 8 solved be recounting 8 in 2s, ux2 = 8 = (8/2)x2, where the solution u = 8/2 comes from moving numbers to opposite side with opposite sign.

Recounting between icons and tens leads to tables and equations when asking T = 6 7s = ? tens, and T = 78 = 7.8 tens = ? 9s.

Double-counting in different physical units creates per-numbers as 3$/4kg that bridge the units when recounting in the per-number; and become fractions when recounted in the same unit; and add by areas as integral calculus since multiplying before adding transform per-numbers into areas.

Recounting the sides in a block halved by its diagonal creates trigonometry.

REVIEW OF THE RELEVANT LITERATURE

Mathematics education sees its goal as mastering university mathematics, seen as a pure self-supporting science theorizing one-dimensional number sets organised by different operations: addition, multiplication and power; all with inverse operations: subtraction, division, root and logarithm; and together creating additional number sets when introducing letters for unspecified numbers.

As an alternative Kuhnian paradigm, ManyMath sees mathematics as a natural science about the outside fact Many as shown by geometry and algebra meaning earth-measuring and reuniting in Greek and Arabic; and developing children’s already existing mastery of Many with two-dimensional double-numbers, which makes the existing literature on one-dimensional single-numbers little relevant. Instead theoretical guidance comes from seeing mathematics education as an institutionalized goal-directed treatment of human brains, thus being theorized by sociology, philosophy, and psychology. Here however, internal controversies necessitate choices to be made. In sociology, this project chooses agency over structure by using Bauman, Habermas and Foucault; in philosophy it chooses empiricism over rationalism by using existentialism; and in psychology it chooses nature over culture by choosing Piaget over Vygotsky.

Clarifying the Novel Contribution of this Particular Project

ManyMath education has the goal to outside master Many, where traditional mathematics education has the goal to inside master mathematics so other subjects later may apply it outside.

Accepting and developing the double-numbers children create when adapting to Many, ManyMath teach, not numbers, but numbering, using functions from grade one as number-language sentences that, as in the word-language, contains a subject, a verb, and a predicate; and where numbers always carry units.

By counting and recounting before adding, double-numbers with units let children meet the core of mathematics in grade one including equations and calculus, allowing the following years to study footnotes and expand what is already known.

Finally, the concept per-number is a new concept that unite e.g. proportionality, fractions and calculus. The recount-formula shows that per-numbers and fractions are not numbers, but operators needing numbers to become numbers, so adding without units creates 'mathematism', true inside but seldom outside classrooms where 2weeks+3days is 17days.

Empirical Research that Could Test the Validity of the Theoretical Proposal

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of ManyMath difficult in a traditional education, except for where it is stuck, e.g. division and fractions. But ManyMath may be tested outside the main track: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

Implication of this Work for Further Theory Development

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Mastering Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as e.g. in Vietnam.

Implications for Practice

Brains help animals adapt to the outside world by making proper behavioural choices to satisfy their basic needs as expressed by the holes in their heads: food for the stomach, and information for the brain. Additional brain capacity allows humans to share information by developing a word- and a number-language describing outside qualities and quantities, thus needing only few children in a lifetime to reproduce if systematizing adaption through institutionalized education teaching children about their outside world, and teenagers about their inside talents and potentials, using and developing the students' native languages, and refraining from colonizing students' brains by forcing foreign languages upon them. Thus, to develop the students' mastery of Many, outside abstracted ManyMath must replace today's inside derived university mathematics.

References

OECD. (2015). *Improving schools in Sweden*. www.oecd.org/education/school/improving-schools-in-sweden-an-oecd-perspective.htm.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

02. CALCULUS CONCEPTUALLY CHANGED: FROM DEIFIED TO REIFIED

STATEMENT OF THE THEORETICAL PROBLEM

Created to add per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier and to all students. Difference research (Tarp, 2018) may give an answer.

AN ACCOUNT OF THE THEORETICAL PROPOSAL BEING MADE

Reified, integral calculus occurs in grade one when performing next-to addition of bundle-numbers as e.g. T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 3kg at 2$/kg + 5kg at 4$/kg = 8kg at ? $/kg, again with differential calculus coming from the reverse question: 3kg at 2$/kg + 5kg at ? $/kg = 8kg at 4$/kg. Here the per-number graph is piecewise constant c, i.e. there exists a delta-interval so that for all positive epsilons, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus also added by their areas, and never without units.

In high school adding per-numbers occurs when the distance travelled with a varying per-number P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P is found by slicing the area thinly. If writing the area strips as differences, addition will make all middle terms disappear and leave only the first and last terms. Alternatively, the last strip represents the change of the area, dA, and may be written as dA = P*dx, thus motivating developing differential calculus to find a formula A with the property that A' = dA/dx = P. Here, looking at the shadow of an f*g book directly gives the formula (f*g)'/(f*g) = f'/f + g'/g, exemplified e.g. by (x^2)' = 2x.

REVIEW OF THE RELEVANT LITERATURE

Mathematics education sees its goal as mastering university mathematics, seen as a pure self-supporting science theorizing one-dimensional number sets organised by different operations. And later introducing differentiation and integration as operations on functions seen as set-relations or subsets of set-products where first-component identity implies second-component identity.

As an alternative Kuhnian paradigm, ManyMath sees mathematics as a natural science about the outside fact Many as shown by geometry and algebra meaning earth-measuring and reuniting in Greek and Arabic; and developing children's already existing mastery of Many with two-dimensional double-numbers that when added next-to by their areas presents integral calculus before differential, which makes irrelevant the existing calculus literature doing the opposite and neglecting double-numbers.

Instead theoretical guidance comes from seeing mathematics education as an institutionalized goal-directed treatment of human brains, thus being theorized by sociology, philosophy, and psychology. Here however, internal controversies necessitate choices to be made. In sociology, this project chooses agency over structure by using Bauman, Habermas and Foucault; in philosophy it chooses empiricism over rationalism by using existentialism; and in psychology it chooses nature over culture by choosing Piaget over Vygotsky.

Clarifying the Novel Contribution of this Particular Project

ManyMath education has the goal to outside master Many, where traditional mathematics education has the goal to inside master mathematics so other subjects later may apply it outside.

Accepting and developing the double-numbers children create when adapting to Many, ManyMath teach, not numbers, but numbering, using functions from grade one as number-language sentences that, as in the word-language, contain a subject, a verb, and a predicate. The tradition insists on teaching single-numbers, and postpones functions to high school.

In grade one, on-top addition of double-numbers leads to proportionality making the units the same, and next-to addition leads to integral calculus by adding areas, and to differential calculus when reversed. In middle school both reappear when adding or subtracting piecewise constant per-numbers by their areas in mixture problems, to be followed in high school by locally constant per-numbers instead, again with integration preceding differentiation. The tradition only teaches on-top addition of single numbers, and teaches fractions as mathematism being added without units. Mixture problems are not illustrated geometrically to show the relationship with calculus, and the concepts of piecewise and local constancy are absent. Finally, differential calculus is taught before integral calculus, and local constancy and linearity is called continuous and differentiable.

Empirical Research that Could Test the Validity of the Theoretical Proposal

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of Reifying Calculus difficult inside a traditional education, except for where it is stuck, e.g. adding fractions without units. But it may be tested outside: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

Implication of this Work for Further Theory Development

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Reifying Calculus to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as e.g. in Vietnam.

Implications for Practice

Institutionalized education systematizes adaption by teaching children about their outside world, and teenagers about their inside talents and potentials. Thus, to develop the students' mastery of Many, outside abstracted ManyMath must replace todays inside derived university mathematics by presenting calculus at all three school levels.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

03. THE POWER OF BUNDLE- & PER-NUMBERS UNLEASHED IN PRIMARY SCHOOL: CALCULUS IN GRADE ONE – WHAT ELSE?

In middle school, fraction, percentage, ratio, rate, and proportion create problems to many students. So, why not teach it in primary school instead where they all may be examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting, and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept, and learn from the mastery of Many that children possess until mathematics takes it away?

MATHEMATICS IS HARD, OR IS IT?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries, and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Also among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as e.g. The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always 'to learn mathematics'. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

RESEARCH METHOD

Difference research searching for differences has uncovered hidden differences (Tarp, 2018c). To see if the differences make a difference, phenomenology (Tarp, 2018a), experiential learning (Kolb, 1984), and design research (Bakker, 2018) may create cycles of observations, reflections, and designs of micro curricula to be tested in order to create a new cycle for testing the next generation of curricula.

OBSERVATIONS AND REFLECTIONS 01

Asked "How old next time?" a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers '$T = (T/B)\text{x}B$' saying "from T, T/B times, B may be pushed away and stacked". Consequently, search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a generation-1 curriculum.

GUIDING QUESTIONS

The following guiding questions were used: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g. if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "What to do if a bundle is not full?", "How to bundle-count seconds, minutes, hours, and days?", "How to double-count lengths in centimeters and inches?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s in 8 may be written as $u*2 = 8$, how can this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when recounted stacks are placed on a squared paper?", "What happens when turning or stacking stacks?"

OBSERVATIONS AND REFLECTIONS 02

Data and ideas allowed designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC 01: Digits as Icons

With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. In this way, all digits may be iconized, and used as units for bundle-counting (Tarp, 2018b).

MC 02: Bundle-counting Ten Fingers

A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with 'underloads' counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s, 2s.

In 5s with bundles: 0*B*1, …, 0*B*4, 0*B*5 no 1*B*0, 1*B*1, …, 1*B*4, 1*B*5 no 2*B*0.

In 5s with bundles and underloads: 1*B*-4, 1*B*-3, …, 1*B*0, 2*B*-4, … , 2*B*0.

MC 03: Counting Sequences Using Tens and Hundreds

In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus preparing the ground for later writing out a multi-digit number fully as a polynomial, $T = 345 = 3BB4B5 = 3*B\text{^}2 + 4*B + 5*1$.

Count 10, 20, 30, …, 90, 100 etc. Then 1*B*, 2*B*, …, 9*B*, ten*B* no 1*BB*.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1*BB*, 2*BB*, …, 9*BB*, ten*BB* no 1*BBB*.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1*BB*0*B*, 1*BB*1*B*, …, 1*BB*9*B*, 1*BB*ten*B* no 2*BB*0*B*.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1*B*1, 2*B*1, 3*B*1, 3*B*2, …; 3*B*4.

MC 04: Cup-counting and Bundle-bundles

When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. *T* = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write *T* = 3*B*1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup, and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 1*BB*0*B*1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. *T* = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, *T* = 5*B*0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 2*BB*1*B*0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1*BBB*, so with bundle-bundle-bundles we write *T* = 1*BBB*0*BB*1*B*0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC 05: Recounting in the Same Unit Creates Underloads and Overloads

Recounting 8 1s in 2s gives *T* = 4*B*0 2s. We may create an underload by borrowing 2 to get 5 2s. Then *T* = 5*B*-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then *T* = 3*B*2 2s = 2*B*4 2s = 1*B*6 2s. Later, such ‘flexible bundle-numbers’ will ease calculations.

MC 06: Recounting in Tens

With ten fingers, we typically use ten as the counting unit thus becoming 1*B*0 needing no icon.

Q. *T* = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write *T* = 3 4s = 1*B*2 tens = 1]2 tens = 1.2 tens, or *T* = 2*B*-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. *T* = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write *T* = 6 7s = 4*B*2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write *T* = 6 7s = 5*B*-8 tens = 5]-8 tens = 5.-8 tens = 3*B*12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. *T* = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. *T* = 6 7s = 6*7 = (*B*-4) * (*B*-3) = *BB*-3*B*-4*B*+4*3 = 10*B*-3*B*-4*B*+1*B*2 = 4*B*2 since the 4 3s must be added after being subtracted twice.

MC 07: Recounting Iconizes Operations and Creates a Recount-formula for Prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a ‘recount-formula’ 8 = (8/2)*2 = 8/2 2s, or *T* = (*T*/*B*)**B* = *T*/*B* *B*s, saying “From *T*, *T*/*B* times, we push away *B* to be stacked”. Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering ‘7/2’ gives the answer ‘3.some’ predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering ‘7-3*2’ gives the result ‘1’, thus predicting that 7 recounts in 2s as 7 = 3*B*1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as *T* = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as *T* = 8 = 2 2/3 3s thus introducing fractions. Or, as *T* = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. *T* = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC 08: Recounting in Time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as *T* = 60days = (60/7)*7days = 8*B*4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC 09: Double-counting in Space Creates Per-Numbers or Rates

Counting in space has seen many units. Today centimeter and inches are common. ‘Double-counting’ a length in inches and centimeters approximately gives a ‘per-number’ or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. *T* = 12in = ?cm; and *T* = 20cm = ?in

A1. *T* = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and A2. *T* = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC 10: Per-numbers Become Fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s: *T* = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s: *T* = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s: *T* = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18. Here we enlarge both numbers in the fraction by 12/2 = 6.

MC 11: Per-numbers Become Ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: *T* = 12 = 2*B*2 5s = 3*B*-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle. Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6, and B gets the fraction 4/6 of the total.

MC 12: Prime Units and Foldable Units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s. Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime units and foldable units up to two dozen.

MC 13: Recounting Changes Units and Solves Equations

Rephrasing the question "Recount 8 1s in 2s" to "How many 2s are there in 8?" creates the equation 'u*2 = 8' that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution u = 8/2 to u*2 = 8 thus comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation: u*2 = 4*2 = 8, OK.

Recounting from tens to icons gives equations: "42 is how many 7s" becomes u*7 = 42 = (42/7)*7.

MC 14: Next-to Addition of Bundle-Numbers Involves Integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. T = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or A2. T = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC 15: On-top Addition of Bundle-Numbers Involves Proportionality

To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. T = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. T = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or T = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC 16: Reversed Addition of Bundle-Numbers Involves Differentiation

Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + u*5 = 3 8s is solved by moving to opposite side with opposite calculation sign

u*5 = 3 8s – 2 3s = 3*8 – 2*3, so u = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC 17: Adding and Subtracting Tens

Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: T = 4B6 tens = 4.6 tens = 46. Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: T = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4BB5B6 + 2BB6B9 = 6BB11B15 = 7BB12B5 = 7BB2B5 = 725.

Subtraction: 456 – 269 = 4BB5B6 – 2BB6B9 = 2BB-1B-3 = 2BB-2B7 = 1BB8B7 = 187

Multiplication: 2 * 456 = 2 * 4BB5B6 = 8BB10B12 = 8BB11B2 = 9B1B2 = 912

Division: 154 /2 = 15B4 /2 = 14B12 /2 = 7B6 = 76

MC 18: Next-to Addition & Subtraction of Per-Numbers and Fractions is Calculus

Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square.

When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square. Thus per-numbers add by areas, i.e. by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2\text{-}T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC 19: Having Fun with Bundle-Bundle Squares

On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

$n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

Two $s*s$ bundle-bundles form two squares that halved by their diagonal d gives four half-squares called right triangles. Rearranged, they form a diagonal-square $d*d$. Consequently, $d*d = 2*s*s$

Four $c*b$ playing cards with diagonal d are placed after each other to form a $(b+c)*(b+c)$ bundle-bundle square. Below to the left is a $c*c$ square, and to the right a $b*b$ square. On-top are 2 playing cards. Inside there is a $d*d$ square and 4 half-cards. Since 4 half-cards is the same as 2 cards, we have the formula $c*c + b*b = d*d$ making it easy the add squares, you just square the diagonal.

MC 20: Having Fun with Halving Stacks by its Diagonal to Create Trigonometry

Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 0.5 approximately. So, the tan-number comes from recounting the height in the base.

MC 21: Having Fun with a Squared Paper

A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying "3up_per_1out, 2up_per_1out, …, 3down_per_1out" allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC 22: Having Fun with Turning and Combining Stacks

Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus 2 3s may be stacked 4 times to the box T = 4*(2*3) that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

DISCUSSION AND RECOMMENDATION

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many.

To find an answer, phenomenology, experiential, and design research is used to create a cycle of observations, reflections, and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions on mastering Many. The first observation is that children use two-dimensional bundle-numbers with units instead of the one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting, and double-counting core activities leading to proportionality by combining division and multiplication, thus reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking e.g. 'how many 5s are 3 tens?' And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see that flexible bundle-numbers ease traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education, all teaching single numbers without units, will deport testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in e.g. division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus directly gives the proportionality recount-formula $8 = (8/2)*2$ or $T = (T/B)*B$ used in STEM, and to solve equations. Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not lean from children instead?

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04. LEARN CORE MATHEMATICS THROUGH YOUR KID'S TILE-MATH: RECOUNTING BUNDLE-NUMBERS AND EARLY TRIGONOMETRY

Fifty years of mathematics education research has failed to create a mathematics for all. This raises the Cinderella question: are there hidden unnoticed alternatives that may make the prince dance? There are. Education may be different, and also math may be different from today's 'meta-matism'. Adapting to Many, children develop bundle-numbers with units as 2 3s having 1 1s as the unit, i.e. a tile also occurring as bundle-of-bundles, e.g. 3 3s. Recounting a total in bundles gives a recount-formula to solve equations, to change units when adding on-top or when adding areas next-to as in calculus, also occurring when adding per-numbers or fractions coming from double-counting in two units. Double-counting sides in a tile halved by its diagonal leads to trigonometry. Asking 'What kind of mathematics may grow from tiles?' this paper uncovers 25 micro-curricula for outdoor STEM education.

POOR PISA PERFORMANCE, A PERMANENT PANDEMIA?

When evaluating the effect of mathematics education, poor PISA performance occurs all over the world, despite 50 years of increasing research and funding. Thus, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life" (p. 3).

Thus, it seems that it is now demonstrated beyond any doubt that mathematics is indeed difficult to learn so research should continue to understand why this is the case. But then again, can we be sure that what is called mathematics education is not something else.

Education can be different as seen when comparing continental Europe using multi-year lines to prepare teenagers for offices in the public or private sector, with North America using self-chosen half-year blocks to support the identity work of their teenagers.

History shows that also mathematics can be different. The Pythagoreans used it as a label for their knowledge about Many by itself, in space, in time and in space and time, also called arithmetic, geometry, music and astronomy. Where North America still uses specific names, Europe teaches 'metamatism' combining set-derived 'meta-matics', defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples, with 'mathematism' true inside but seldom outside by adding without units (Tarp, 2018).

Finally, math education may suffer from a goal displacement (Bauman, 1990) by changing mathematics from being an inside means to master the outside goal, Many, to being a goal in itself, thus unable to develop the mastery of Many, children bring to school through adaption to Many. Seeing mastery of Many as the end goal, phenomenology has found that Many presents itself to children as bundle-numbers, e.g. 2 3s. (Tarp, 2018). This creates a basis for cycles of experiential learning (Kolb, 1984) and design research to design and test micro-curricula (MC) leading to new cycles. So, respecting children's own two-dimensional bundle-numbers we now ask: What kind of mathematics may grow from tiles?

MC01. Counting by Bundling and Iconizing

Chopsticks placed on tiles show that digits are icons with as many sticks as they represent if written less sloppy. Counting fingers in 3s, we may include the word bundle in the counting sequence by saying '0*B*1, 0*B*2, 0*B*3 no 1*B*0; 1*B*1, 1*B*2, 1*B*3 no 2*B*0; 2*B*1, 2*B*2, 2*B*3 no 3*B*0, 3*B*1'. Counting and stacking tiles and using a folding ruler or a rope to show a bundle, we see that 3 bundles is 1 bundle-of-bundles or a 3x3 square of tiles, so we should instead say '2*B*3 no 3*B*0 no 1*BB*0*B*0'.

To include time in bundle-counting in 3s, we place a cube on each of ten neighboring tiles e.g. on a chess board. A cube moves to the next tile, and both move on to the next tile where they unite to 1 bundle that moves to the tile above, from where it moves to the next tile. This is repeated until tile 9 where 3 bundles unite to 1 bundle-of-bundles that move to the tile above, from where it moves to the last tile, thus showing that ten recounts as 1*BB*0*B*1 3s. Now the same is repeated with bundle-counting in 4s, then in 2s.

MC02. Formulas Predict

Eight persons are bundle-counted in 2s by asking 2s to go to neighboring tiles 4 times. Observing the total 8 splits into 4 2s, we write T = 8 = 4*2. Using an uphill stroke to iconize a broom pushing away bundles, the action 'from 8, push away 2s' may be entered on a calculator as '8/2', thus predicting 4 before carrying out the action. This allows rewriting 8 = 4*2 to 8 = (8/2)*2, or $T = (T/B)*B$ using unspecified numbers, saying 'From *T*, *T*/*B* times, *B* can be pulled away'. This 'recount-formula' predicts changing units. And, rephrasing recounting to "how many 2s in 8?" allows formulating recounting as an equation $u*2 = 8$ solved by $u = 8/2$, i.e. by moving a number to opposite side with opposite calculation sign.

MC03. Unbundled Become Decimals, Fractions or Negative Numbers

Can we predict the result of rearranging in 4s persons placed on tiles as 2 7s? Entering 2*7/4, a calculator says '3.some'. Using a horizonal stroke to iconize a rope pulling away stacks, entering 2*7-3*4 gives the answer '2' thus predicting that 2 7s recount as 3 4s and 2. The unbundled 2 may be placed next-to the stack reported as a decimal number, T = 3*B*2 4s = 3.2 4s, or on-top counted as bundles, 2 = (2/4)*4 = 2/4 4s, reported as a fraction, T = 3 2/4 *B* 4s; or, if counting what is needed for an extra bundle, reported by a negative number, T = 4 *B*-2 4s = 4.-2 4s. The prediction is then tested with persons or cubes placed on tiles.

MC04. Double-counting Creates Per-numbers

Traveling through a row of tiles is rewarded with 3 cubes per 2 tiles thus creating the 'per-number' 3/2 cubes/tiles. Travelling 12 tiles thus gives 3 cubes 6 times, which can be precited by recounting in the per-number: *T* = 12 tiles = (12/2)*2 tiles = (12/2)*3 cubes = 18 cubes. Alternatively, we can equate the per-numbers in an equation $u/12 = 3/2$ solved by moving to opposite side with opposite sign, $u = 3/2*12 = 18$. Per-numbers are all over mathematics and science, e.g. meter = (meter/second)*second = speed*second.

Double-counting in the same unit creates fractions: Marking 2 tiles with a dot for each 3 tiles traveled thus creates a per-number 2 tiles/3 tiles = 2/3. Having travelled 12 tiles, we mark 2 dots 4 times. Again, this can be precited by recounting in the per-number: *T* = 12 tiles = (12/3)*3 tiles marking dots on (12/3)*2 tiles, i.e. 8 tiles with dots.

MC05: Bundle-Numbers Add Next-to or On-top, Directly or Reversed

Once counted as stacks, totals may unite next-to or on-top, iconized by a cross showing the two directions. Adding 2 3s and 4 5s next-to as 8s means adding the areas 2*3 and 4*5, called integral calculus where addition follows multiplication. Adding them on-top, first recounting must change the units to the same. This is called proportionality. Reversed addition asks e.g. '2 3s and how many 5s give 3 8s?'. Here, first the 2*3 stack is pulled away from the 3*8 stack, then recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called reversed integration or differentiation.

MC06: Next-to Addition & Subtraction of Per-numbers and Fractions is Calculus

Throwing a dice 8 times, the outcomes 1 and 6 place 4 cubes on a chess board, and the rest place 2 cubes. When ordered, we may have 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 = 8 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s. This average says that all per-numbers would be 2.6 if alike.

Per-numbers thus add by areas, i.e. by integration. Reversing the question to '2 5s + how many 3s total 3 8s' leads to the equation $2*5 + u*3 = 3*8$ solved by differentiation:

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2\text{-}T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC07. Double-Counting Sides in a Rectangle Halved by its Diagonal

Two neighboring tiles form a rectangle, that halved by its diagonal creates a right triangle with base *b*, height *h* and diagonal *d*. Recounting pair of sides produces the trigonometry formulas: $h = (h/d)*d = \sin A*d$; $b = (b/d)*d = \cos A*d$, and $h = (h/b)*b = \tan A*b$ that allows a $(b,h) = (+3,+2)$ angle to be predicted by tan-1(2/3) to give 33.7 degrees. This again allows predicting the diagonal: $h = \sin A*d$, or $2 = \sin 33.7*d$, or $d = 2/\sin 33.7 = 3.60$.

MC08. Meeting Pythagoras

Four tiles with base *b* and diagonal *d* form a squared tile. Here 4 diagonals form a square containing 4 half-tiles, i.e. 2 tiles. Consequently $d*d = 2*b*b$, or $d^2 = b^2 + b^2$.

A tile-pair has base *b*, height *h*, and diagonal *d*. Turned 90 degrees a copy is placed on-top. Repeated three times, this creates a square with the side $b+h$. Inside we find a diagonal square and four half tile-pairs; as well as a $b*b$ square and a $h*h$ square and two tile-pairs. But 4 half tile-pairs is 2 full tile-pairs, so we see that $d*d = b*b + h*h$, or $d^2 = b^2 + h^2$, making it easy the add squares, you just square the diagonal.

The normal from the right angle divides the diagonal in *p* and *q*. Seeing *b* as $d*\cos A$, and *p* as $b*\cos A$, we get $b*b = (d*\cos A)*b = d*(\cos A*b) = d*p$. So, the extension of the normal divides the diagonal square in two parts equal to the squares of the neighboring rectangle side. Since only the angle *A* is involved this applies to all triangles with angles not above 90 degrees, thus leading to the extended Pythagoras: $a^2 = b^2 + c^2 – 2*b*c*\cos A$, etc.

MC09. The Height of an Accessible Flagpole

The point *P* forms a right triangle with a flagpole of height *h*. From a point *Q* in the distance *s* from *P*, the vertical distance to the diagonal is *k*. The angle *P* may now be found in two ways, $\tan P = h/b = k/s$, allowing the unknown height *h* to be found by moving to opposite side with opposite sign, $h = k/s*b$. Solved in a tile-system (coordinate-system), the diagonal is a line passing through three points with the coordinates (0,0), (s,k) and (b,h) providing the slope $c = \tan P = k/s$ and the equation $y = k/s*x$ that with $x = b$ gives $y = k/s*b$.

MC10. The Height of an Inaccessible Flagpole

Three points *A*, *B* and *C* with distances $AB = r$ and $BC = s$ are placed on a line towards the foot of an inaccessible flagpole with height *h*. With the flagpole, *A* and *B* form two right triangles with diagonals *d1* and *d*2. The vertical distance is *p* from *d1* to *B*, and *q* from *d*2 to *C*. The distance from *C* to the foot of the flagpole pole is *c*. Using $\tan A = p/r$ and $\tan B = q/s$, the triangles give two formulas for *h*: $h = (r+s+c)*p/r = (s+c)*q/s$. Solved for *c*, this gives a formula that inserted in the *h*-formula gives $h = p*q*r/(p*r – p*s)$.

MC11. How High the Moon?

A vertical stick with height *h* helps finding the position of the moon or sun. If the shadow has the length *s*, the angle to the sun is predicted by $\tan A = s/h$. A compass helps finding a direction line segment north with the same length as the shadow. The segment between the two has the length *a*. The angle *A* then may be predicted by the formula $\sin(A/2) = ½a/s$.

MC12. The Slope of a Tile

A folding ruler allows creating a right triangle with the bottom line following a sloped tile. A lead line placed in the distance d along the ruler from the bottom line will mark on it a distance b. The slope of the tile is the same as the top angle A, thus predicted by $\tan A = b/d$.

MC13. How Many Turns on a Steep Tile?

On a 30-degree squared tile, a 10-degree road is constructed. How many turns will there be? We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance $BC = u$. First, in the triangle BCD, the angle B is 30 degrees, and $BD = u*\cos(30)$. With Pythagoras we get $u^2 = CD^2 + BD^2 = CD^2 + u^2*\cos(30)^2$, or $CD^2 = u^2(1-\cos(30)^2) = u^2*\sin(30)^2$. Next, in the triangle ACD, the angle A is 10 degrees, and $AD = AC*\cos(10)$. With Pythagoras we get $AC^2 = CD^2 + AD^2 = CD^2 + AC^2*\cos(10)^2$, or $CD^2 = AC^2(1-\cos(10)^2) = AC^2*\sin(10)^2$. Finally, in the triangle ACB, $AB = 1$ and $BC = u$, so with Pythagoras we get $AC^2 = 1^2 + u^2$, or $AC = \sqrt{(1+u^2)}$.

Consequently, $u^2*\sin(30)^2 = AC^2*\sin(10)^2$, or $u = AC*\sin(10)/\sin(30) = AC*r$, or $u = \sqrt{(1+u^2)}*r$, or $u^2 = (1+u^2)*r^2$, or $u^2*(1-r^2) = r^2$, or $u^2 = r^2/(1-r^2) = 0.137$, giving the distance $BC = u = \sqrt{0.137} = 0.37$. So, two turns: 3.70 cm and 7.40 cm up the tile.

MC14. Rectangles as Extended Squares

A rectangle with base b and height $h = c*b$ may be called a 'c extended square'. It consists of a lower square, $b*b$, and an upper rectangle $(h-b)*b$, becoming a square also if c is 2. The two diagonals $d1$ and $d2$ are raised the angles $A1$ and $A2$ that may be predicted by $\tan A1 = h/b = c$, and $\tan A2 = (h-b)/b = h/b-1 = c-1$. The diagonal $d1$ is predicted by Pythagoras: $d1^2 = b^2 + h^2 = b^2 + c^2*b^2 = b^2*(1+c^2)$, or $d1 = b*\sqrt{(1+c^2)}$. The diagonal $d2$ is predicted by $d2 = b*\sqrt{(1+(1-c)^2)}$. Finally, the normal n to the diagonal $d1$ is predicted by $n*d1 = h*b = c*b^2$, or $n* b*\sqrt{(1+c^2)} = c*b^2$, or $n = b*c/\sqrt{(1+c^2)} = b* 1/\sqrt{(1+1/c^2)}$.

So, combining a tile with half of its neighbor will provide a rectangle as a 1.5 extended square where the diagonal angels and lengths may be predicted by proper formulas: $\tan A1 = 1.5$ giving $A1 = 56.3$, $\tan A1 = 0.5$ giving $A1 = 26.6$. Likewise with the two long diagonals: $d1 = b*\sqrt{(1+c^2)} = b*\sqrt{(1+1.5^2)}= b*1.80$; and $d2 = b*\sqrt{(1+(1-c)^2)}= b*\sqrt{(1+1.5^2)}= b* 1.12$. Finally, the normal: $n = b*1/\sqrt{(1+1/c^2)} = b*1/\sqrt{(1+1/1.5^2)} = b*0.832 = b*\sin A1$.

MC15. The Golden Factor Pervades Art

On a tile, a circle with center in the midpoint of an edge and passing through the opposite corners marks two points that extend the tile to one side or the other with the golden factor $\frac{1}{2}(1+\sqrt{5}) \approx 1.62$, i.e. 62%. This extends the original edge with the same factor as will extend the new edge to a length equivalent to adding an extra tile.

Likewise, with a radius half the edge the half-diagonal from the midpoint to the corner will mark a length that will divide the edge in two parts connected by the golden factor.

MC16. Meeting Pi on the Pavement

Two neighboring tiles are circumscribed by a semicircle, again circumscribed by two tiles. On the right tile, the diagonal creates two triangles enveloping a quarter of the semicircle, i.e. 180/4 degrees or pi/4. Consequently, $4*\sin(180/4) < \pi < 4*\tan(180/4)$. In other words, $\pi = n*\sin(180/n) = n*\tan(180/n) = 3.14\ldots$ for n sufficiently large.

MC17. Meeting Algebra

Half of a $b*b$ tile will extend a tile upwards to a $h*b$ playing card. Removing from a $h*h$ square two playing cards, and adding the bottom right tile that has been removed twice will leave the square (h

$– b)^2 = h^2 – 2*a*b + b^2$. And, removing from a $h*b$ playing card the bottom $b*b$ tile will leave the top $(h – b)*b = h*b – b*b$.

Four playing cards are arranged to form a $(h+b)*(h+b)$ square. Inside we find a $h*h$ square, a $b*b$ square and two playing cards, so, $(h+b)*(h+b) = h*h + b*b + 2*h*b$.

Pulling away a $b*b$ tile from the $h*h$ square leaves a $(h–b)*h$ and a $(h–b)*b$ rectangles that add up to a $(h–b)*(h+b)$ rectangle. Consequently, $(h+b)*(h–b) = h^2 – b^2$.

To solve the quadratic equation $x^2+6x+8 = 0$ we use four tiles forming a square, labeling the first side x and the next 6/2. The $(x+6/2)$ square now contains two $6/2*x$ rectangles and two squares, x^2 and $(6/2)^2$ split in two parts, 8 below and $(6/2)^2-8$ above if possible, all disappearing except for last part. So $(x+6/2)^2 = (6/2)^2 – 8 = 1$ giving $x = -6/2 \pm 1 = -2$ and -4. Looking instead at $x^2+bx+c = 0$ gives the solution $x = -b/2 \pm \sqrt{((b/2)^2 – c)}$.

MC18. Predicting Change

Two $b*b$ tiles form a $h*b$ playing card that is extended with a tape on-top and to the left to show that a change in h and b, Δh and Δb, will give a change in the area, $\Delta(b*h) = \Delta b*h + b*\Delta h$, or with per-numbers, $\Delta(b*h)/(b*h) = \Delta b/b + \Delta h/h$. Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision. Here we neglect the upper right tape-corner, which is allowed for sufficiently small changes, giving $(b*h)'/(b*h) = b'/b + h'/h$. So, with $y = x^n$ we get that $dy/y = n*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$.

MC19. Following Change Formulas e.g. when Playing Golf

Tiles form a coordinate system to move in. Person A starts a (+1,+1) trip in (0,3). Person B starts a (+1,-2) trip in (0,9). Predict where they meet. Person A starts a (+1,+s) trip in (0,0) where s decreases with 1 from +4 to -4. Person B starts a (+1,+s) trip in (10,0) where s increases with 1 from -4 to +4. Person A starts a (+1,+s) trip in (0,½) where s is doubled from ½ the first 4 steps, then halved the next 6 steps. Person B starts a (+1,+s) trip in 0,0 where s decreases with 1 from +3 to -3, then increases with 1 from -3 to +3. Person A and B start a (+1,+s) trip in (0,0) and (2,0) wanting to end closest to a golf hole in (10,0).

MC20. The Saving Formula

A saving combines a deposit amount a with an interest percent r, illustrated by two tiles, K1 and K2. K2 receives a one-time deposit a/r, and each period its interest amount $a/r*r = a$ is transferred to K1 after K1 has received its own interest amount. After n periods, K1 will contain a saving A growing from a deposit amount a and an interest percent r. But, at the same time, K1 will contain the total interest percent R of the initial amount a/r in K2, so $A = a/r*R$, or $A/R = a/r$, where $1+R = (1+r)^n$. Using the doubling-time as the period, a 1$ deposit will after 5 doubling periods save 31$ = 5$ deposit + 26$ compound interest.

MC21: Having Fun with a Tile System and with Bundle-bundle Squares

A dozen recounts as 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed in a tile system, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. With a per-number 2G/3R, a dozen R may be changed to 2G+9R, 4G+6R, 6G+3R, and 8G, thus traveling along a line sloping down with the per-number. With Bundle-Bundle squares we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas: $n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

MC22. Pascal's Triangle

A triangle of tiles consists of 1 tile in column 1, 2 in column 2, etc. until column 5. Traveling the triangle with a (+1,+1) win-step or a (+1,+0) loose-step we observe how many roads lead to each tile.

Could it be predicted? Start over, but now let a coin decide the next step. Mark the tile with a short stroke. How many times did you win? Could it be predicted?

MC23. Game Theory

Two players A and B choose column and row at 2x2 tiles carrying the numbers 1, 2, 3, 4 in the top and bottom row, indicating what A pays back to B after having received a fix fee from B. Showing paper or stone means choosing the first or second strategy. Thus, if A chooses stone and B paper, A will pay back 2 to B. Which fee makes the game fair? Use cubes to show that the fee is 2.5 if 4 and 2 change places. In the first game, 3 is a stable 'saddle point' going up if A changes, and down if B changes, which they don't want. In the second game both players will be tempted to change, so both will mix strategies, but how?

MC24. Geometry with Handles

Graph theory and topology is geometry where neither distances nor angles matters, but only the relative positions between the points. A classic problem is the supply problem shown with two separated rows with 3 tiles each: How can three houses A, B and C be supplied with electricity, gas and water with no crossing wires? Hint: Connect A and B with gas and water. Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

MC25. The Electric Circuit

To work properly, a device demands energy coming from a supplier, thus creating a circuit of carriers with the demander as a sink and the supplier as a source. If the demand is 16 energy-units per second, and the supplier provides 8 units per carrier, a flow of 2 carriers per second is needed, enabled by a device resistance at 4, calculated as the ratio between the supply and the flow. In technical terms: If a device needs 16 Joule per second (Watt) it needs 2 electrical carriers (Coulombs) per second (Ampere) as current from a battery delivering 8 Joule per Coulomb (Volt). To realize this, the device needs the resistance 4 Ohm.

A circuit thus follows two formulas: Demand = Supply * Flow, or Watt = Volt * Ampere, or $P = U * I$; and Supply = Resistance * Flow, or Volt = Ohm * Ampere, or $U = R * I$.

Built into the device, the resistance cannot change, but the voltage can. Supplying a (4ohm, 16watt) device from a 4volt instead of an 8volt battery will give the current 4volt/4ohm or 1 ampere, supplying 4volt*1ampere = 4watt instead of 16watt, i.e. only a quarter of what is needed.

Supplying a (4ohm, 16watt) device from a 16volt instead of an 8volt battery will give the current 16volt/4ohm or 4ampere, supplying 16volt*4ampere = 64watt instead of 16watt, i.e. 4 times what is needed.

With a 12volt battery, a 3ohm device will produce the current 12volt/3ohm or 4ampere supplying 12volt*4ampere = 48watt. With a 1ohm it will be 12ampere and 144watt.

Using a 12volt battery to supply both a (3ohm, 48watt) and a (1ohm, 144watt) device, the total resistance 4ohm will produce the current 12volt/4ohm or 3ampere which on the first device will use 3ohm*3A = 9volt supplying 9volt*3A = 27W ≈ 60% of what is needed; and on the next device will use 1ohm*3A = 3volt supplying 3volt*3A = 9W ≈ 6% of what is needed. Thus, the bigger consumer receives the smaller part. Consequently, multiple devices are connected, not serial as here, but parallel increasing the current to 16A.

Three tiles serve at simulating how 2 cups supply a device labeled (4ohm, 16watt) with energy from an 8volt battery. LEGO-bricks serve as energy-units, and a slow metronome tells when a 'second' has passed. From the battery, 8ers are placed in 2 cups that move to the devise to deliver 2*8 units at the time signal, and then move back empty to refill.

In the case of a 4volt battery, 1 cup carries a 4er and delivers 1*4 watt, only 25% of what is needed. In the case of 1ohm, 8cups deliver 8*4 = 32watt.

With a 12volt battery supplying first a (3ohm, 48watt) and then a (1ohm, 144watt) device, 3 cups each supply 9 to the first device needing 48, and 3 to the second device needing 144 before returning to refill.

DISCUSSION AND RECOMMENDATION

This paper asked 'What kind of mathematics may grow from tiles?' The background was the phenomenological observation, that Many presents itself to children as bundle-numbers with units as e.g. 2 3s, thus having squared tiles as 1 1s or bundle-bundles as the unit, which allows geometry and algebra to go hand in hand from grade 1.

With units, a core question is how to change it, traditionally leading to proportionality in its classical regula-de-tri form multiplying before dividing, or to its modern form doing the opposite by first finding the per-unit-number. And thus postponed until after teaching all four operations from addition to division.

Recounting turns this order around. First division pushes away bundles to be stacked by multiplication to be pulled away by subtraction in order to find unbundled singles that become decimals, fractions or negative numbers depending on where they are placed. Recounting produces a formula $T = (T/B)*B$ present all over mathematics and science, and showing there things: How to change units, how to solve equations by moving to opposite side with opposite sign, and how per-numbers must be multiplied before being added.

Double-counting in two units produces per-numbers, becoming fractions with like units. As geometrical representations of bundle-numbers, squares and rectangles lead directly to double-counting the sides in rectangles halved by their diagonals, thus allowing trigonometry and tile or coordinate geometry to precede traditional plane geometry.

Addition comes last in two forms, on-top needing proportionality to change units, and next-to adding areas as integral calculus, also occurring when adding per-numbers and fractions with units to avoid mathematism. And reversed addition leads directly to differentiation.

Quadratic expressions, equations and functions also relate to tiles in a natural way, as does differential equations through change formulas directing trips through a tile system

So, it turns out that the core of mathematics springs from tiles once we accept the two-dimensional bundle-numbers children develop while adapting to Many before school. Education needs not teaching 'metamatism' as the only means leading to the end goal, Mastery of Many. Tiles will show the way directly in a concrete way including also outdoor and STEM education; and will perhaps be able to answer the Cinderella question with a yes, there is a hidden unnoticed alternative that makes the prince dance. Tile-mathematics may offer a new Kuhnian paradigm that will finally create a mathematics for all. This paper has taken a first step in an experiential learning cycle by designing more than a score of micro-curricula to be tested inside and outside classrooms in the hope that after several cycles of redesigning they will become scores making math dislike evaporate.

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A BUNDLE COUNTING TABLE

A guide to bundle-counting in pre-school.

Bundle-counting clarifies that we count by bundling, typically in tens.

Example 01. Counting Mikado Sticks

The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches

The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option

Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches

The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

05. PROPOSALS FOR PAPERS TO THE CTRAS 2020 CONFERENCE, INVITATION TO CO-AUTHORSHIP

STEM PREVENTS A GOAL-DISPLACEMENT THAT MAKES MATHEMATICS A GOAL INSTEAD OF A MEANS

Asking what is the purpose of mathematics education, US and UK mathematics educators say “to learn school mathematics”. Others say “to learn set-based mathematics as defined by university mathematics.” Focusing on competences leads to saying “to learn mathematical competences” or “to master mathematics”. Seldom, if ever, is heard that the goal is “to master many” or “to develop the number-language that children bring to school.”

Sociological imagination (Bauman, 1990) may prevent a goal displacement where a means becomes a goal instead. Historically, the Pythagoreans chose the word ‘mathematics’ meaning ‘knowledge’ in Greek as a common name for their knowledge about Many in space and time and by itself: astronomy, music, geometry and arithmetic. And today in North America, mathematics is still a common name for geometry and algebra, showing their outside goals in their original meanings, earth-measuring in Greek, and reuniting in Arabic. Integration and differentiation also name their tasks directly, to integrate small changes, and to differentiate a total change in small changes.

To avoid a goal displacement, mathematics must de-model (Tarp, 2019) its core ingredients: digits, operations, equations, fractions, functions etc. to allow primary school develop the flexible bundle-numbers children bring to school by teaching, not numbers to add, but numbering totals by counting, recounting and double-counting, where recounting 8 fingers in 2s as 8 = (8/2)*2 leads directly to the recount-formula *T = (T/B)*B* with per-numbers that solve equations, that occur in most STEM-formulas typically predicting proportionality, and that become fractions when double-counting in the same unit.

Liberated from its goal displacement, mathematics education may have its own communicative turn as in the 1970s (Widdowson, 1978) such that from now on both the word- and the number-language are taught and learned through their use and not through their grammar, thus allowing all students to model outside quantities as to levels, change and distribution.

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STOP TEACHING WRONG NUMBERS AND OPERATIONS, START GUIDING CHILDREN’S MASTERY OF MANY

Learning means adapting the inside brain to outside nature and culture. Vygotsky prioritizes culture and wiser-brain teaching, Piaget nature and peer-brain learning.

Adapting to Many, children answer the question ‘How many?’ with bundle-numbers as T = 2 3s containing two digits: 3 is a quantity-number in space, also called a cardinal-number taking on positive integer values; 2 is a counting-number in time taking on also decimal, fractional, and negative values as T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

Quantity-numbers may add, and so may counting-numbers, but not in between. So, digits must be categorized before adding. Digits are not numbers but operators, needing a multiplier to become a number, T = 2 3s = 2*3, as seen when writing numbers fully as polynomials, as e.g. $T = 345 = 3*B^2 + 4*B + 5*1$

So, teaching digits as numbers is teaching wrong numbers. And bundle-numbers need not to be taught since children bring them to school, that should guide them to develop their number-language by learning that digits are icons with as many strokes as they represent.

operations are icons also, rooted in the counting process: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

recounting 8 in 2s gives a recount-formula: 8 = (8/2)*2, or $T = (T/B)*B$, used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically predicting proportionality.

Later recounting between digit-units and tens leads to tables, and to equations when asking e.g. T = 4 6s = ? tens, and T = 42 = 4.2 tens = ? 7s.

So, childhood education should guide children develop the quantitative competence they bring to school using Kolb's experiential learning cycles.

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TO SUPPORT STEM, TRIGONOMETRY AND COORDINATE GEOMETRY SHOULD PRECEDE PLANE GEOMETRY

Halved by its diagonal c, a rectangle becomes a right triangle ABC with base b and height h. Using the recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, mutual recounting gives trigonometry: $h = (h/c)*c = \sin A*c$, $b = (b/c)*c = \cos A*c$, $h = (h/b)*b = \tan A*b$.

Splitting the diagonal in $c1$ and $c2$ by the triangle-height produces two triangles where $\cos A = c1/b = b/c$, making $b^2 = c*c1$, and $\cos B = c2/h = h/c$, making $h^2 = c*c2$, thus giving the Pythagoras rule $h^2 + b^2 = c^2$.

Finding $\sqrt{70}$ means squeezing 7 tens until becoming a square $(8+t)^2$ situated between 8^2 and 9^2. And having four parts as shown by two playing cards placed like an L: 8^2, and $8*t$ twice, and t^2. Neglecting t^2, we get the equation $8*t = (70-8^2)/2 = 3 = (3/8)*8$, solved by recounting 8 in 3s, giving $t = 3/8 = 0.375$, so $8.375^2 = 70{,}14 = 70$ approximately.

In a coordinate system, a circle with center in the origin and radius r gets the equation $x^2 + y^2 = r^2$, else $(\Delta x)^2 + (\Delta y)^2 = r^2$. In a horizontal right triangle, moving along the diagonal will change x and y with Δx and Δy. Recounting Δy in Δx gives $\Delta y = (\Delta y/\Delta x)*\Delta x = m*\Delta x = \tan A*\Delta x$ that allows drawing lines from tables.

Intersection points between lines are predicted by a linear equation solved by technology or by moving to opposite side with opposite sign.

Intersection points between lines and circles or parabolas are predicted by a quadratic equation $x^2 + b*x + c = 0$, solved by two L-placed playing cards showing that

$(x+t)^2 = x^2 + 2*x*t + c + (t^2 – c)$ where the first three terms disappear with $t = b/2$.

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A FRESH-START YEAR10 (PRE)CALCULUS CURRICULUM

Often precalculus suffers from lacking student knowledge. Three options exist: make mathematics non-mandatory, choose an application-based curriculum; or, to rebuild student self-confidence, design a fresh-start curriculum that also includes the core of calculus by presenting integral calculus first.

Writing a number out fully as a polynomial, e.g. T = 345 = 3*B^2 + 4*B + 5 shows the four ways to unite numbers, resonating with the Arabic meaning of the word algebra, to reunite: addition and multiplication unite changing and constant unit-numbers into totals; and next-to-block-addition (integration) and power unite changing and constant per-numbers, all having reverse operations that split totals into parts.

Addition, multiplication, and power are defined as counting-on, repeated addition and repeated multiplication. As reverse operations, x = 7-3 is defined as the number that added to 3 gives 7, thus solving the equation x+3 = 7 by moving to opposite side with opposite sign. Likewise, x = 7/3 solves x*3 = 7, the factor-finder (root) x = 3√7 solves x^3 = 7, and the factor-counter (logarithm) x = log3(7) solves 3^x = 7, again moving to opposite side with opposite sign.

Hidden brackets allow reducing a double calculation to a single: 2+3*x = 14 becomes 2+(3*x) = 14, solved by x = (14-2)/3. Next transposing letter-equations as *T = a+b*c^d* really boost self-pride.

Future behavior of 2set unit-number tables is predicted by linear, exponential, or power models assuming constant change-number, change-percent, or elasticity.

1-4set per-number speed tables are modeled with lines, parabolas and double-parabolas, allowing technology to calculate the distance covered, thus introducing integral calculus, that also occurs when adding per-numbers in mixture-problems, and when adding percent in cross tables generated by statistical questionnaires.

Trigonometry comes from mutual double-counting sides in a rectangle halved by its diagonal, and is used to model distances to far away points, bridges, roads on hillsides, motion down an incline, and jumps from a swing.

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TO SUPPORT STEM, CALCULUS MUST TEACH ADDING BUNDLE-NUMBERS, PER-NUMBERS AND FRACTIONS ALSO

Created to add locally constant per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier. Difference research searching for hidden differences finds that the answer is yes.

Integral calculus occurs in grade one when performing next-to addition of bundle-numbers as e.g. T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg, again with differential calculus coming from the reverse question: 2kg at 3$/kg + 4kg at ? $/kg = 6kg at 5 $/kg. Here the per-number graph is piecewise constant c, i.e. there exists a delta-interval so that for all positive epsilon, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus also added by their areas, and never without units.

In high school adding per-numbers occurs when the meters traveled with varying m/s speed P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P, is found by slicing the area thinly so that its change may be written as $dA = P*dx$ in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find $A' = dA/dx = P$.

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CONFLICTING GRAND THEORIES CREATE 2X3X2 DIFFERENT MATHEMATICS EDUCATIONS

As part of institutionalized education, mathematics falls under the focus of the three grand theories, philosophy, sociology and psychology, discussing different kinds of mathematics, of education and of learning; and recommending appropriate means to institutional goals. However, is the goal to master mathematics first, as a means to later master many; or to master many directly if mastering mathematics proves difficult?

As to learning, psychology sees coping coming from brains adapting to outside nature and culture, but which is more important? Vygotsky points to culture, mediated by a more knowledgeable wiser-brain, a teacher. Piaget points to nature, automatically creating inside schemata that accommodate if meeting outside resistance from nature or from peer-brain communication.

As to mathematics, philosophy has three conflicting views: Pre-modern mathematics is inspired by the Pythagoreans seeing mathematics as knowledge about Many in space and time and by itself as expressed in astronomy, music, geometry and arithmetic; and as part of the three basic Rs: reading, writing and 'rithmetic called reckoning in Germanic countries. Modern mathematics needs no outside examples for its concepts. Alternatively, postmodern scepticism sees mathematics as a number-language abstracting inside concepts from outside examples, and parallel to the word-language.

As to institutions, sociology recommends imagination to prevent a goal displacement making a means a goal instead. As to education, two conflicting views exits. One sees the student as a servant of the state forcing its population to choose between different multiyear tracks from upper secondary school, and forcing students back to start if changing track. One sees the state as a servant of the student by helping students to uncover and develop their personal talent in self-chosen half-year blocks after puberty. So, two different learning forms, three different mathematics forms, and two different education forms create 2x3x2 different ways of conducting mathematics education.

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REPLACING STEAM WITH STE3M WILL INCLUDE ALSO ECONOMICS AND ENGLISH

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of physical quantities. Statistics post-dicts unpredictable quantities by setting up probabilities for future behavior, using factual or fictitious numbers as median and fractals or average and deviation. Including economics and English in STEM opens the door to statistics also. Art may be an appetizer, but not a main course since geometry and algebra should be always together and never apart to play a core role in STEM.

Macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheet for simulations using different parameters.

Microeconomics describes equilibriums in the individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable cost, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determines the equilibrium price. Market surveys leads to statistics, as does insurance. In the households, spending comes from balancing income and transferals with saving and tax. In a bank, income come from simple and compound interest, from installment plans as well as risk taking. At a stock market, courses fluctuate. Governments must consider quadratic Laffer-curves describing a negative return of a tax-raise. To avoid units, factories use variations of Cobb-Douglas power elasticity production functions for modeling.

In English sentences may be analyzed on a word level as to the frequency of subjects, verbs, direct and indirect objects, predicates, and unspecified words.

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THE POWER OF PER-NUMBERS

Uniting unit-numbers as 4$ and 5$, or per-numbers as 6$/kg and 7$/kg or 6% and 7%, we observe that addition and multiplication unite changing and constant unit-numbers into a total, and integration and power unite changing and constant per-numbers. Reversely, subtraction and division split a total into changing and constant unit-numbers, and integration and power split a total into changing and constant per-numbers.

Recounting 8 in 2s as 8 = (8/2)*2 creates a recount-formula *T = (T/B)*B*, saying 'From T, T/B times, T may be pushed away'; and used to change units when asking e.g. 2 6s = ? 3s, giving the prediction T = (2*6/3)*3 = 4*3 = 4 3s.

Recounting 8 in 2s also provides the solution u = 8/2 to the equations as u*2 = 8 = (8/2)*2; thus solving most STEM-equations, since the recount-formula occurs all over. In proportionality, $y = c*x$; in coordinate geometry as line gradients, $\Delta y = \Delta y/\Delta x = c*\Delta x$; in calculus as the derivative, $dy =$

*(dy/dx)*dx = y'*dx*. In science as meter = (meter/second)*second = speed*second, etc. In economics as price formulas: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

With physical units, recounting gives per-numbers bridging the units. Thus 4$ per 5kg or 4/5 $/kg gives T = 15kg = (15/5)*5kg = (15/5)*4$ = 3$; and T = 16$ = (16/4)*4$ = (16/4)*5kg = 20kg. With like units, per-numbers become fractions.

Trigonometry occurs as per-numbers when mutually recounting sides in a rectangle halved by its diagonal, $a = (a/c)*c = \sin A*c$, etc.

Modeling mixtures as 2kg at 3$/kg + etc, unit-numbers add directly, but per-numbers P add by the area A under the per-number graph, found by slicing it thinly so that the change may be written as $dA = P*dx$ in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find the per-number $A' = dA/dx = P$.

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MIXING DESIGN AND DIFFERENCE RESEARCH WITH EXPERIENTIAL LEARNING CYCLES ALLOWS CREATING CLASSROOM TEACHING FOR ALL STUDENTS

International tests show that not all students benefit from mathematics education. This poor-performance-problem raises a Cinderella question: is there a hidden difference that can make the Prince dance? If so, design research can create Kolb's experiential learning cycles to adapt a given micro-curriculum so that all students may benefit.

In primary school, difference research searching for hidden differences has identified several alternatives: Digits are icons. Numbers are double-numbers with bundles as units, e.g. T = 2 3s. Flexible bundle-numbers have over- and underloads, e.g. T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 tens, and ease operations as e.g. 329/ 7 = 32*B*9/ 7 = 28*B*49 /7 = 4*B*7 = 47, or 23*8 = 2*B*3*8 = 16*B*24 = 18*B*4 = 184.

Operations are icons also where division is a broom pushing away bundles, multiplication a lift stacking bundles, subtraction a rope pulling away stacks to find unbundled, and addition the two ways to unite stacks, on-top and next-to.

Changing units may be predicted by a recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, or, and used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically predicting proportionality: meter = (meter/sec)*sec = speed*sec.

In middle school, double-counting leads to per-numbers becoming fractions with like units, and adding by their areas as integral calculus. In algebra, factors are units placed outside a bracket. Trigonometry occurs when mutually double-counting sides in a rectangle halved by its diagonal.

In high-school, redefining inverse operations allows equations to be solved by moving to opposite side with opposite sign. And adding per-numbers by areas allows introducing integral calculus before differential calculus.

Designing and redesigning micro-curricula as experiential learning cycles allows teachers perform design research in their own classroom, to be reported as master projects first, and later perhaps as PhD projects including more details.

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FROM PLACE VALUE TO BUNDLE-BUNDLES: UNITS, DECIMALS, FRACTIONS, NEGATIVES, PROPORTIONALITY, EQUATIONS AND CALCULUS IN GRADE ONE

Traditionally, a multi-digit number as 2345 is presented top-down as an example of a place value notation counting ones, tens, hundreds, thousands, etc.; and seldom as four numberings of unbundled, bundles, bundle-bundles, bundle-bundle-bundles, etc., to provide a bottom-up understanding abstracted from concrete examples, which would introduce exponents in primary school as the number of bundle-repetitions. Counting ten fingers in 3s thus introduces bundle-bundles: T = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles, the unbundles singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles thus leading to fractions, T = 7 = 2 1/3 3s; or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may also be included in the counting sequence: 0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s may be written as a recount-formula: 8 = (8/2)*2, or $T = (T/B)*B$, used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically expressing proportionality.

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

At the end of grade one, recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s.

References

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SOCIOLOGICAL IMAGINATION DESIGNS MICRO-CURRICULA FOR EXPERIENTIAL LEARNING CYCLES

Forced by peer review to focus on existing research, many education research articles fail to be validated in the classroom by observing if its educational goal is reached. However, the peer review crisis creates a need for a different research meeting its proper genre demands: reliable data and valid findings to a research question.

To help student brains adapt to the outside world, mathematics education must decide if its goal is to master inside mathematics as a means to later master outside quantity, thus risking what sociology calls a goal displacement (Bauman, 1990) where a means becomes a goal instead; or to master quantity directly if first mastering contemporary university mathematics becomes too difficult to many students.

Many curriculum reforms include competences. But again, we must ask: is the goal to obtain inside mathematical competence, or outside quantitative modeling competence?

A learning-by-doing curriculum calls for experiential learning cycles as described by Kolb's learning theory (Kolb, 1983) being adapted e.g. in the new Vietnamese curriculum; and containing cyclic phases. First micro-curriculum A is taught and validated if meeting its expected goals, next systematic observations gather reliable data as to which goals are met, and which are not, then reflections modifies the micro-curriculum into version B. Then plan B is taught, etc.

Combined with design research (Bakker, 2018), experiential learning cycles allows teachers to become action learners or action researchers in their own classroom reporting their work in master or

PhD papers. To meet the genre demands of research, the data gathered must be reliable, and the findings must be tested for validity. In design research, reliability comes by making systematic observations through notes, interviews, questionnaires etc. And testing validity here means holding on to the strong parts of the actual micro-curriculum and changing the weak parts.

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CONCRETE STEM SUBJECTS ALLOW MATHEMATICS LEARNING BY MODELING AND PEER-BRAIN TEACHING

Traditionally, mathematics is considered one of the core subjects in education because of the many 'applications of mathematics'. This phrasing leads directly to the view that "of course mathematics must be learned first before it can be applied by others". Consequently, mathematics teaches the operation order addition, subtraction, multiplication and division with cardinal numbers, later expanded to integers, rational and real numbers, again followed by expressions including also unspecified numbers.

Talking instead about outside roots leads to the opposite view that "of course, mathematics must be learned through its outside roots, also constituting its basic applications". This 'de-modeling' view resonates with the fact that historically, the Pythagoreans chose the name mathematics, meaning knowledge in Greeks, as a common label for their four areas of knowledge about Many in time and pace, in time, in space and by itself: astronomy, music, geometry and arithmetic. Later the Arabs added algebra with polynomial numbers created by systematic bundling. Here the outside roots are evident through the original meanings of geometry and algebra: earth-measuring and reuniting.

So, mathematics grew and may still grow from counting, recounting and double-counting bundles, and from applying science, describing forces pumping motion in and out of matter when having the same or opposite directions.

Working in groups with science applications allows students to learn through peer-brain teaching instead of through wiser-brain teaching. As to matter, tasks could be to find its mass, its center, its density, and the heat transfer under collision between visible macro-matter and invisible micro-matter, applied in steam power, or when placing ice-cubes in water.

As to motion, tasks could be to describe traveling with constant or changing speed horizontally or on an incline, vertical motion, projectile orbits; and circular motion, swings or see-saws on a market place. As well as how to use electrons to store or transport motion and information.

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THE SIMPLICITY OF MATHEMATICS DESIGNING A STEM- BASED CORE CURRICULUM FOR REFUGEE CAMPS

Numbers as 2345 evade the place value notation if seen as numbering unbundled, bundles, bundle-bundles, bundle-bundle-bundles. Here exponents occur as the number of bundling-repetitions, e.g. when counting ten fingers as T = ten = 3B1 3s = 1BB1 3s.

Stacking bundles in blocks, the unbundled singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles leading to fractions, T = 7 = 2 1/3 3s, or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may be included in the counting sequence: 0B1, 0B2, …, 0B7, 1B-2; 1B-1, 1B0, 1B1, …, 9B7, 1BB-2, 1BB-1, 1BB.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s gives a recount-formula, 8 = (8/2)*2 or T = (T/B)*B, that changes unit from 1s to 2s (proportionality), that gives the equation u*2 = 8 the solution u = 8/2 (moving to opposite side with opposite sign), and that shows that per-numbers as 8/2 must be multiplied to areas before being added (integral calculus).

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

Recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s. Recounting in different units gives per-numbers bridging the units, becoming fractions with like units, and adding by areas. Mutually recounting sides in a block halved by its diagonal gives trigonometry.

References

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CALCULATION MODELS, FACT OR FICTION

As qualitative literature, also quantitative literature has the genres fact and fiction when modeling real world situations.

Fact is ‘since-then’ calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. ‘since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20’. Fact models can be trusted once the numbers and the formulas and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter.

Fiction is ‘if-then’ calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. ‘if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20’. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.

This paper looks at three infection models, the standard logistic model and two alternatives, one formulated as a differential equation, one as a difference equation.

The models all assume that the population change is proportional to the population itself thus giving a doubling factor that is assumed to decrease with the number of infected as the first model assumes, or with time as the two others assume.

However, where the standard model cannot be validated since infection data are difficult or impossible to achieve, reliable data from the number of hospital beds points to the other fiction models. Here, the scientific principle of simplicity, known as Occam's razor, points to the difference equation, easy to set up in a spreadsheet.

It may be simple but it provides important information: its high degree of elasticity recommends a gradual reduction of the two central factors, group size and meeting time, instead of a complete shutdown.

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06. REMEDIAL MATH MICRO-CURRICULA – WHEN STUCK IN A TRADITIONAL CURRICULUM

Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

BACKGROUND

A city council is challenged by the economical and organizational costs of many teachers seeking different jobs after only a few years of service, especially in the case for math teachers. Seeing a valuable resource lost, the council tries persuading the teachers to stay and give mathematics a second chance. However, most teachers recline arguing that the teaching material present at the school and at conferences and seminars are very much alike.

Consequently, the city council brought their concern to a university professor asking if not other ways to teach primary math was available. The professor witnessed the author's off-Broadway cafeteria poster at the ICME 2016 showing how to cure math dislike with flexible bundle numbers and cup-writing, accepting the three ways children count 5 in 2s, using overload and underload besides the traditional way (Tarp, 2017a)

T = I I I I I = II I I I = 1B3 = II II I = 2B1 = II II II = 3B-1 2s.

Flexible bundle-numbers and cup-writing provides alternatives to tradition calculations:

65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92

65 – 27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38

7x 48 = 7x 4]8 = 28]56 = 33]6 = 336

336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 4

Inspired by a paper on mastering many (Tarp, 2018b), the professor asked the author to design a list of remedial math micro-curricula meant for teachers stuck in traditional curricula allowing them to see unnoticed alternatives. Furthermore, the professor wanted the suggestion to have the form of a scientific paper, so that it may be published in a journal. This paper then is the reaction to the demand.

INTRODUCTION

Research in mathematics education has grown since the first International Congress on Mathematics Education, ICME, in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish center for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'. Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

SOCIAL THEORY LOOKING AT MATHEMATICS EDUCATION

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

MATHEMATICS, BEFORE AND NOW

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3 = 6' stating that 2 3s can always be re-counted as 6 1s.

DIFFERENCE RESEARCH LOOKING AT MATHEMATICS EDUCATION

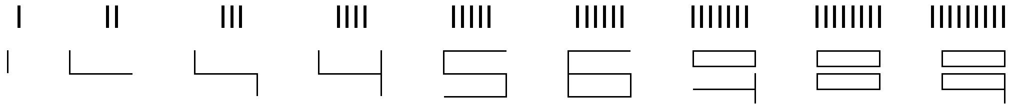
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' (Tarp, 2017b) is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a list of remedial micro-curricula for classes stuck in a traditional mathematics curriculum. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

MEETING MANY CREATES A ‘COUNT-BEFORE-ADDING’ CURRICULUM

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:



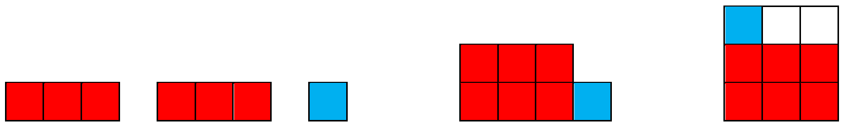
1 2 3 4 5 6 7 8 9

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T, we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

7 / 3	2.some
7 – 2 x 3	1

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A TYPICAL MATHEMATICS CURRICULUM

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precede bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, let us use difference research and imagination to uncover or invent remedial micro-curricula for classes stuck in the tradition.

REMEDIAL MICRO-CURRICULA FOR CLASSES STUCK IN THE TRADITION

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to use two-dimensional block-numbers and bundle-counting, recounting in the same and in a different unit, and calculator prediction before next-to and on-top addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as '8 counted in 2s' instead of as '8 divided between 2'; and to rewrite the number as '10 or 5 times less something' and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of 'mathe-matism' true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations 'move to the opposite side with the opposite sign': In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' A hymn can be created: "Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as

the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to 'pre-dict' a number, instead statistics can 'post-dict' its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5th root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon,

the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio □y/□x can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Inspirational purposes have led to the creation of several DrAlTarp YouKu.com, SoKu.com videos, and MrAlTarp YouTube videos: Deconstructing Fractions, Deconstructing Calculus, Deconstructing PreCalculus Mathematics, Missing Links in Primary Mathematics, Missing Links in Secondary Mathematics, Postmodern Mathematics, PreSchool Math.

CONCLUSION

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, one alternative presents itself directly for future studies creating a paradigm shift (Kuhn, 1962): to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining the misery coming from teaching meta-matism. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research should search for differences and test if they make a difference. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a is-sentence, trust the subject since it exists, but doubt the predicate, it is a verdict that might be gossip. So, maybe we should stop teaching essence and instead start letting learners meet and experience existence.

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07. DECONSTRUCTED, CALCULUS RISES IN 3 VERSIONS FOR PRIMARY, MIDDLE AND HIGH SCHOOL

EXTENDED ABSTRACT

"Calculus is hard - and therefore only for the few, and only late in high school".

This tradition has established itself worldwide (Rasmussen et al, 2014; Bressoud et al, 2016). Which motivates the core sociological question "Is calculus hard by nature, or by choice?" (Bauman, 1990).

Postmodern deconstruction may provide an answer by using 'Difference Research' (Tarp, 2018) searching for hidden differences that may make a difference.

Thus, to deconstruct calculus education means asking the philosophical questions "Which ontological fact lies behind the epistemological construct calculus? How does this fact present itself phenomenologically to the user? And what psychological consequences does this have for making calculus a tool for all students?"

Algebraically, Hindu-Arabic numbers as 456 = 4**B*^2 + 5**B* + 6*1 are sums of monomials, i.e. a sum of areas geometrically. They show that there are four ways to unite numbers: number-addition, multiplication, power, and area-addition; and that only monomials add, thus making simple numbers operators, that may be multiplied, but needing a unit-factor to add.

In this way, area-addition roots primary school calculus if allowing children to add what they bring to school, bundle-numbers or area-numbers as 2 3s. Here, adding two areas when asking "2 3s + 4 5s = ? 8s" roots integral calculus, that reversed roots differential calculus when asking "2 3s + ? 5s total 3 8s", answered by removing the 2*3 area before counting the rest in 5s, i.e. by letting subtraction precede division:

(3*8 – 2*3)/5 = ΔArea/5.

Later, in middle school, double-counting a quantity in two different units leads to per-numbers becoming fractions with like units: 2$/5kg, and 2$/5$ = 2/5 = 40/100 = 40%.

As with simple numbers, also per-numbers and fractions are operators to be multiplied to areas before adding, now as the area under a per-number graph; and again, rooting differential calculus when reversed.

Examples: "2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg"; and "1/2 of 30$ + 4/5 of 60$ = ? of 90$".

An *a* by *b* area shows the geometrical meaning of per-numbers. Multiplied, *a*b* gives the area. Divided, their per-number *b/a* gives the diagonal's turning angle, tan*A* = *b/a*, as well as its steepness, Δ*y*/Δ*x* = *b/a*.

Another example of adding per-numbers is speed: "2sec at 3m/sec + 4sec at 5m/sec = 6sec at ? m/sec".

Observing that a ball falls with accelerated speed makes high school physics formulate speed-questions as "5sec at 0m/sec increasing gradually to 50m/sec totals how many meters?"

Here, the speed per-number, *s*, is not piecewise, but locally constant. So again, the answer is the area under the per-number *s*-graph, now split into many small slices to add up. But how to add many small numbers? Very simple, if the numbers may be written as differences making all middle terms cancel out and leaving only the difference between the terminal and initial numbers.

So, finding the area under a locally constant *s*-graph roots differentiation as a difference-equation: Find a formula *F* so that the slice *s**Δ*x* can be written as a difference Δ*F*: *s**Δ*x* = Δ*F*, or *s* = Δ*F*/Δ*x*? Alternatively, we may say that the area always changes with the last slice, ΔArea = *s**Δ*x*, or *s* =

$\Delta A/\Delta x$. So, also high school differentiation lets subtraction precede division as in primary and middle school calculus.

Again, the area gives the answer. In an x by x area, a small change in the sides, dx, will make the area change with two slices x*dx, and a very small corner dx^2, so d(x^2) ≈ 2x*dx, or d/dx(x^2) = 2x. Likewise, in an x by x^2 area, d/dx(x^3) = 3x^2.

Once created to find the area under a per-number graph, differential calculus may also be applied for optimization purposes since the per-number gives the steepness of the diagonal in the dy by dx area.

We can formalize the difference between piecewise and locally constancy by observing that a variable y is globally constant c if their difference is less than any positive number epsilon.

Likewise, y is piecewise constant c if an interval delta exists so that I y-c I < epsilon in delta.

Finally, y is locally constant c if for any epsilon an interval delta exists so that I y-c I < epsilon in delta. So, to go from piecewise to locally constant, we just interchange epsilon and delta.

Another traditional learning obstacle is that calculus builds on pre-calculus, seen by many teachers as the most difficult course to teach since here all lacking student knowledge surface.

However, here a difference making a difference is the 'Algebra Square' showing the four ways to reunite and split-into constant and changing unit-numbers and per-numbers: Addition/subtraction unites/split-into changing unit-numbers; and multiplication/division unites/split-into constant unit-numbers. Whereas power/root&log unites/split-into constant per-numbers; and integration/differentiation unites/split-into changing per-numbers (Tarp, 2012).

Thus, where the focus of calculus is to unite and split-into changing per-numbers, the focus of pre-calculus is to unite and split-into constant per-numbers.

Or, rephrased from a change-perspective we may say that where pre-calculus is about constant change, calculus is about changing predictable change, and statistics is about changing unpre- but post-dictable change.

Reversing the exponential formula for constant change-percentage, $y = b*a$^x, equations as and 5 = 2^x and 5 = x^3 root a logarithm as a factor-counter, and a root as a factor-finder.

Consequently, calculus is not hard by nature, but by choosing to neglect its three outside ontological roots: reuniting and splitting-into bundle numbers and piecewise or locally constant per-numbers. And to neglect that only numbers with units, i.e. monomials, add meaningfully. A fact that is not accepted until calculus.

These roots present themselves in a natural way to the learners having therefore little problems learning to do calculus if allowed to meet existence before essence by choosing a Piagetian over a Vygotskian approach.

Thus, in education, primary and middle school calculus as well as presenting integral before differential calculus in high school remains to be tested and thoroughly researched e.g. experientially (Kolb, 1983) using design research (Bakker, 2018).

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08. TOWARDS TRUTH, BEAUTY AND GOODNESS IN MATHEMATICS EDUCATION

PURPOSE. WHAT MAY BE DIFFERENT IN MATH EDUCATION

Pausing for a moment at the entry to the third decennium of the 21. century we may ask, what can and what cannot be different in mathematics education, leading on to the question: how will a civilized mathematics education for the rest of the century look like? To get an answer, we may rephrase the question from behind: how to make mathematics education comply with the three classical virtues, truth and beauty and goodness.

To use a suitable methodology, difference-research (Tarp, 2018) recommends looking for hidden differences, e.g. from the viewpoint of the three grand theories, philosophy and sociology and psychology, typically absent in research except for the latter. Here philosophy and psychology may inform the discussion on truth and beauty, and sociology may address the question on goodness.

What we find is a third way to mastering Many respecting the quantitative competence and mastery of Many, children develop before coming to school. This alternative to the classical and set-based version may fulfil the dream of mathematics education for all students, once developed and tested.

RESEARCH DESIGN. GRAND THEORY LOOKS AT MATHEMATICS EDUCATION

Within philosophy, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber (1930) was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologycally, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB + 4*B + 5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a totalitarian rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a 'pris-pital', a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

FINDINGS. MEETING MANY, CHILDREN USE BUNDLE-NUMBERS TO COUNT WITH UNITS

How children master Many can be seen at preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 4 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as T = 7 = 1B5 = 2B3 = 3B1 = 4B1; or even as 1BB3 or 1BB1B1. Also, children love to count in 3s, 4s, and in hands.

Children thus show an unnoticed quantitative competence before they come to school. So why not develop instead of rejecting the core mastery of Many that children bring to school?

CONCLUSION

We asked: how to make mathematics education comply with the three classical virtues, truth and beauty and goodness. Using sociological imagination (Mills, 1959) on pre-school observation we can suggest that the following differences may be worthwhile testing.

It is truth and beauty:

To reserve names for what exists. And since the end-goal of mathematics education is to master many, many should be present in each statement as the total T, thus using full sentences with a subject, a verb and a predicate in both the word-language and the number-language.

To transform a collection of sticks into an icon for a digit with five sticks in the 5- icon etc.

To use the word bundling when using a counting sequence to answer the question 'How many?' to find the degree of many in a given total: 0B1, 0B2, …, 0B9, 1B0, 1B1, etc.

To use other bundle-sizes than ten to meet bundle-of bundles e.g. when counting ten fingers in 3s as 0B1, 0B2, 0B3 or 1B0, 1B1, …2B2, 2B3 or 3B0 or 1BB0B0, 1BB0B1. Thus, allowing ten and hundred and thousand to be called bundle and bundle of bundles and bundle-of-bundles-of-bundles.

To show that the unbundled may be accounted for in three different ways using a decimal, a fraction or a negative number as in T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

To see a total as a flexible rectangular tile with a height that increases if squeezed, and decreases if expanded to hold the same total. And to see a total transformed into a square thus showing approximate values of a square root.

To see the operation 'push-away' named division iconized as a broom brushing away bundles; the operation 'stack bundles' named multiplication iconized as a lift; the operation 'pull-away' a stack named subtraction iconized as a rope; the operation 'uniting stacks' named on-top or next-to addition iconized as a cross showing the 2 directions; and the operation of bundling bundles iconized as a hat.

To see the process of (re)counting a total of 8 in 2s written as T = 8 = (8/2)*2 that, with letters for unspecified numbers, becomes a recount formula, T = (T/B)*B, also called a proportionality formula present all over mathematics and science, and able to let a calculator predict the result of a recounting before performing it.

To see integral calculus coming from next-to addition of two stacks asking e.g. 2 3s + 4 5s = ? 8s, thus letting multiplication precede addition; and to see differential calculus coming from reversing next-to addition of two stacks thus letting subtraction precede division. And to include units when adding.

To see that double-counting creates per-numbers as 3$/4kg allowing problems to be solved by recounting in the per-number: 20kg = ?$; 20kg = (20/4)*4kg = (20/4)*3$ = 15$.

To see the algebra square showing the four ways to 2x2 numbers, constant and changing unit- and per-numbers, together with the four ways to unite them, addition and multiplications and power and integration, as well as the five ways to split them, subtraction and division and root and logarithm and differentiation. And thus, to see the problem of backward or reversed calculation formulated as an equation with the answer predicted by the reverse operations.

To use a tile divided with horizontal and vertical rubber bands to multiply multi-digit numbers as well as algebraic expressions.

To see the quadratic equation solved by two identical tiles turned a quatre round.

To see the Pythagorean theorem coming from rearranging 4 identical tiles.

To see the multiplication formula of differential calculus appearing as the shadows of a tile.

To see the different change formulas as special examples of the number formula T = a*x^2+b*x+c.

To use two tiles to show the most beautiful formula in mathematics: when saving, the ratio between the final and single $-and %-input are identical.

To treat integral before differential calculus, using that if adding differences, middle terms disappear.

It is goodness:

To accept and develop the flexible-bundle numbers children bring to school.

To allow totals to be present simultaneously in time and in space as bundle-numbers and tiles.

To postpone addition to after division, multiplication and subtraction has changed a pile into a total.

To allow calculus in primary school as direct and reversed next-to addition of two bundle-numbers; and in middle school as adding per-numbers asking 2kg at 3$/kg + 4kg at 5$/kg total 6kg at ?$/kg.

To treat fractions as, not numbers, but operators needing numbers to become numbers, i.e. as per-numbers with like units.

To parallel technical words with rooted words: function/formula, root/factor-finder, logarithm/factor-counter, divided by/counted in, continuous/locally constant, differentiable/locally linear, equation/reversed calculation, fraction/per-number, etc.

Originality and Recommendation

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T =

2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many should be an overall goal of a curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

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09. A REFUGEE CAMP MATH CURRICULUM

The name ‘refugee camp curriculum’ is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses on proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about ‘refugee camp mathematics’ thus allows locating a setting where children do not have access to normal education, thus raising the question ‘What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional learning materials?’. This motivates another question ‘How much mathematics can be learned as ‘finger-math’ using the examples of Many coming from the body as fingers, arms, toes and legs?’

Focus 01. Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Focus 02-04. Counting Ten Fingers, Sticks, Cubes in Various Ways

Focus 05-07. Counting a Dozen Finger-parts, Sticks, Cubes in Various Ways

Focus 08. Counting Numbers with Underloads and Overloads.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Etc. until Focus 30

Conclusion. A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional bundle numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, needing parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

10. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

ABSTRACT

In an isolated covid-19 household we may ask: How much mathematics is a child able to learn by itself? Or even: How much mathematics can we learn from a child isolated form traditional schooling?

As a matter of fact, amazingly much I learned first from our three Korean foster girls, and now from my two grandchildren. Changing from teaching to learning mode you observe that meeting many, children bundle and count with units. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they may say '5 3s and 3'; and when asked "How many 4s?" they may say '5 4s less 2'; and, integrating them next-to each other, they typically say '2 7s and 4'. Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. And they easily accept that 8 recounted in 2s means 8/2 times pushing away 2, giving 8 = (8/2)x2 leading directly to the core mathematical formula of proportionality, T = (T/B)xB, present all over STEM. Furthermore, they typically propose a horizontal stroke as an icon for a rope pulling away the stack to look for unbundles, to be placed next-to the stack as a decimal, or on-top counted in bundles, or reporting whet is needed for an additional bundle, e.g. T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s = 3.-2 3s.

And when asked to unite the two blocks 2 3s and 3 4s, they do so both on-top after recounting the blocks in the same unit, and next-to as areas thus performing integral calculus. Corona-isolation thus shows that core university mathematics as linearity and calculus is present at the child before school begin. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

The paper designs two question-guided curricula, one for counting, and one for adding.

MEETING MANY, CHILDREN BUNDLE AND COUNT WITH UNITS

How children master Many I observed from my three Korean foster girls. Asked "How old next time?", a 3year old would say "Four" and show 4 fingers; but would react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they would say '5 3s and 3'; and when asked "How many 4s?" they would say '5 4s less 2'; and, integrating them next-to each other, they typically said '2 7s and 4'.

Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

MATERIALS FOR QUESTIONGUIDED COUNTCURRICULUM

Typically, a 'mediating curriculum' sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a 'developing curriculum' sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and stacks with inside two-dimensional bundle-numbers. This calls for different materials from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018, 2020) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

QC01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the 'learning opportunity (L.O.) is to change many ones to one icon with as many sticks or strokes as it represents if written less sloppy. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

QC02, counting sequences: "How to count fingers?" Here the L.O. is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H".

QC03, icon-counting: "How to count fingers by bundling?" Here the L.O. is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an 'inside bundle-number' describing the 'outside stack-number'. Turning over a two- or three-dimensional stack or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

QC04, calculator-prediction: "How can a calculator predict a counting result?" Here the L.O. is to see the division sign as an icon for a broom pushing away bundles: 7/2 means 'from 7, push away bundles of 2s'. The calculator says '3.some', thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles. Finally, the subtraction sign iconizes a rope pulling away the stack to look for unbundled singles. By showing '7-3*2 = 1' the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional L.O. is to write 8 = (8/2)*2 as a 'recount-formula' T = (T/B)*B, saying "From T, T/B times B can be pushed away", to predict counting and recounting.

QC05, recounting in another unit: "How to change a unit?" Here the L.O. is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another L.O. is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

QC06, unbundled becomes decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the L.O. is to see that the unbundled occur in three ways: Next-to the stack as a stack of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside stack 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

QC07, prime or foldable units: "Which stacks can be folded?" Here the L.O. is to examine the symmetry of a stack. The stack T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd.

QC08, finding units: “What are possible units in T = 12?”. Here the L.O. is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

QC09, recounting from tens to icons: “How to change unit from tens to icons?” Here the L.O. is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

QC10, recounting from icons to tens: “How to change unit from icons to tens?” Here the L.O. is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking ‘T = 3 7s = ? tens’. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional L.O. uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3*(ten less 3) = 3*ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. So, ‘less less 1’ means adding 1.

QC11, recounting stack-sides. “How to recount sides in a stack halved by its diagonal?” Here, in a stack with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

QC12. On squared paper a point has an out-number x and and up-number y, A(x,y). The per-number Δy/Δx allows moving on a line.

With A(2,5) and B(4,6), the line per-number is Δy/Δx = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = (Δy/Δx)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

MATERIALS FOR QUESTION GUIDED ADD CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

QA01, next-to addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?” Here the L.O. is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

QA02, reversed next-to addition: “If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?” Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

QA03, on-top addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?” Here the L.O. is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

QA04, reversed on-top addition: “If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?” Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

QA05, adding tens: “With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?” Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

QA06, subtracting tens: “If T1 = 23 and T2 add to T = 71, what is T2?” Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

QA07, multiplying tens: “What is 7 43s recounted in tens?” Here the L.O. is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

QA08, dividing tens: “What is 348 recounted in 6s?” Here the L.O. is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349/6 = 34B9/6 = 30B49/6 = (30B48 +1) /6 = 58 + 1/6.

QA09, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the L.O. is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining local constancy by epsilon and delta.

QA10, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the L.O. is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

QA11, solving the quadratic equation x^2 + 6x + 8 = 0. Two playing cards have the width 3 and the height x + 3. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that (x+3)^2 = x^2 + 2*3*x + 3^2, or, (x+3)^2 = x^2 + 6*x + 8+1, or (x+3)^2 = 1 since x^2 + 6x + 8 = 0. So x = -3 ± 1 = -4 and -2.

QA12, finding common units: “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factoring:

T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) *2ab.

DISCUSSION

Meeting Many makes children bring flexible bundle-numbers to school with core math as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social ‘colonization’ of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child’s existing mastery of Many.

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11. SELF-EXPLANATORY LEARNING MATERIAL TO IMPROVE LEARNERS' MASTERY OF MANY

In an isolated covid-19 household we may ask: How to create simple material supporting the children in improving their mastery of Many?

This workshop is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional bundle-numbers with units, neglected by the school's 1-dimensional line-names making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance.

CHILDREN SHOW MASTERY OF MANY BEFORE SCHOOL

In an isolated covid-19 household we may ask: How to create simple material supporting the children in improving their mastery of Many?

This workshop-design is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but, held together two by two, protests by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional bundle-numbers with units, neglected by the school's 1-dimensional line-names making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance (Tarp 2018, 2020). In the workshop you will need 12 sticks, 12 cubes, a pegboard with rubber bands, squared paper and a pencil.

BRIDGE OUTSIDE EXISTENCE TO INSIDE ESSENCE

E01. Many become icons.

Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many sticks or strokes as it represents. Repeat with cubes.

Sticks and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc. A looking glass finding nothing when looking for more will iconize zero. If used as bundle-size, ten has no icon but is reported as 1 bundle or 1B or 1B0.

E02. Flexible bundle-counting roots negative numbers.

Count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s.

Including bundles, ten fingers may be bundle-counted in fives as '0Bundle1, 0B2, 0B3, 0B4, 1B0, 1B1' that also may be counted as 0B6 or 2B less 4 or 2B-4 with an overload or an underload, thus writing 6 in three ways as a 'flexible bundle-number':

T = 6 = 0B6 = 1B1 = 2B-4

Using flexible bundle-numbers, ten fingers may also be counted in 5s as '1B-4, 1B-3, etc.'

Counting on from ten, we meet 'Viking-counting' where eleven is 'one-left', twelve is 'two-left', and thirteen is 'three-ten', while 'three-twenty' becomes twenty-three.

Counting in scores (twenties) from forty, the Danish Viking-descendants still count: half-three-scores, three-scores, half-four-scores, four-scores, and half-five-scores for ninety. Unable to understand the half-notion, the French instead counts over when expressing 87 as ‘4 scores and 1 ten and 7.’

E03. Flexible bundle-counting roots polynomials.

Bundle-counting ten fingers in 3s, nine becomes 3B or 1BB, 1bundle-bundle, called hundred when using ten-bundles. And ten becomes 3B1 3s or 1BB0B1 3s, leading on to the general number-formula or polynomial T = ten = 1*B^2 + 0*B + 1*1 3s, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to stack-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Counting ten fingers in 2s, eight becomes 1BBB, called thousand when using ten-bundles.

Bundle-counting thus allows meeting power before the other operations which allows replacing the place-value-system with bundle-writing: T = 345 = 3BB4B5.

E04. Recounting as flexible bundle numbers will ease traditional calculations.

T = 53 = 5B3 = 4B13 = 6B-7 tens

65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

E05. Counting creates icons.

Count 7 cubes in 2s by 3 times pushing away 2s with a phone iconized as a division sign, predicted by a calculator as ‘7/2 = 3.some’. Stack the bundles by a lift iconized as a cross, 3x2. To look for unbundled, pull away the stack with a rope iconized as a subtraction sign, predicted by a calculator as ‘7-3x2 = 1’. So T = 7 = 3 2s & 1. So operations predict.

E06. Placing the unbundled roots decimals, negative numbers and fractions.

Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:

T = 7 = 2.1 3s = 3.-2 3s

Placing the unbundled instead on-top of the stack counted in bundles roots fractions:

T = 7 = 2 1/3 3s

Counting in tens, T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens.

E07. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a recount-formula T = (T/B)*B.

Recounting 8 in 2s by 8/2 times pushing away 2s is predicted on a calculator as T = 8 = (8/2)*2, which with unspecified numbers becomes a recount-formula T = (T/B)*B, (from T, T/B times, push away Bs) using a full number-language sentence with a subject, a verb and a predicate.

This recount ‘proportionality’ formula occurs all over mathematics and science: when relating proportional quantities as y = c*x; in trigonometry as sine and cosine and tangent, a = (a/c)*c = sinA*c; in coordinate geometry as line gradients, □y = (Dy/□x)*□x = c*□x; in calculus as the derivative, dy = (dy/dx)*dx = y’*dx; speed in science: □s = (□s/□t)*□t = v*□t.

E08. Recount from tens to icons

OUTSIDE, to answer '40 = ? 5s' we use a pegboard or a squared paper to transform the stack 4.0 tens to 8.0 5s. So decreasing the base will increase the height. INSIDE, we formulate an equation to be solved by recounting 40 in 5s:

u*5 = 40 = (40/5)*5, so

u = 40/5 giving 40 = 8*5 = 8 5s

Notice that recounting gives the equation solution rule 'move to opposite side with opposite calculation sign'.

E09. Recount from icons to tens

OUTSIDE, to answer '4 7s = ? tens' we use a pegboard or a squared paper to transform the stack 4 7s to 2.8 tens. So, increasing the base will decrease the height.

INSIDE: oops, with no calculator ten-button we can't use the recount-formula? Oh, we just multiply, thus creating multiplication tables.

Using flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

E10. Double-counting in two physical units.

Double-counting in two physical units gives a 'per-number' as 2m per 3sec or 2m/3sec or 2/3 m/sec. To answer the question 'T = 6m = ?sec' we just recount 6 in the per-number:

T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

Double-counting in the same unit, per-numbers become fractions: 2m per 3m = 2 per 3 = 2/3, and 2 per 100 = 2/100 = 2%.

To answer the question '20 per hundred is ? per 400' we just recount 400 in 100s: T = 400 = (400/100)*100 giving (400/100)*20 = 80.

To answer the question '20 per 400 is what per hundred?' we just recount 100 in 400s: T = 100 = (100/400)*400 giving (100/400)*20 = 5.

STEM formulas contain per-numbers:

meter = (meter/sec)*sec = velocity*sec,

kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter;

force = (force/square-meter)*square-meter = pressure*square-meter;

energy = (energy/sec)*sec = Watt*sec;

energy = (energy/kg)*kg = heat * kg.

Lego-bricks: number = (number/meter)*meter = density*meter.

E11. Mutual double-counting the sides in a stack with base b and height a, halved by its diagonal c, creates per-numbers:

a = (a/c)*c = sinA*c

b = (b/c)*c = cosA*c

a = (a/b)*b = tanA*b

$\pi \approx$ n*sin(180/n)

Draw a vertical tangent to a circle with radius 5. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/5 with tangent of the angle on a calculator.

E12. On squared paper a point has an out-number x and an up-number y, A(x,y). The per-number $\Delta y/\Delta x$ allows moving on a line.

With A(2,5) and B(4,6), the line per-number is $\Delta y/\Delta x = (6-5)/(4-2) = ½$. Changing position to C(8,y) gives $\Delta y = (\Delta y/\Delta x)^* \Delta x = ½*(8-2) = 3$, and y = 5+3 = 8, giving C(8,8).

E13. Next-to addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s? Here we see that next-to addition OUTSIDE means adding by areas where multiplication precedes addition. INSIDE, the recount-formula predicts the result. Next-to addition is called integral calculus.

E14. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

E15. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s? Here we see that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

E16. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

E17. Adding tens: If T1 = 23 and T2 = 48, what is T1+T2 when added as tens? Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

E18. Subtracting tens: If T1 = 23 and T2 add to T = 71, what is T2? Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

E19. Multiplying tens: What is 7 43s recounted in tens? Here we see that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

E20. Dividing tens: What is 348 recounted in 6s? Here we see that recounting a total with overload often eases division:

T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58;

T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

E21. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what? Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

E22. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg? Here we see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a 'locally constant' situation, subtracting per-numbers is called differential calculus.

GRAND THEORY HOLDS CONFLICTING CONCEPTIONS ON CONCEPTS

Within philosophy, Platonism and Existentialism discuss if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget discuss if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

DISCUSSION

The physical fact Many makes children bring flexible bundle-numbers to school containing core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many.

References

Tarp, A. (2018). Mastering many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education 11*(1), 103-117.

Tarp, A. (2020). De-modeling numbers, operations and equations: from inside-inside to outside-inside understanding. *Ho Chi Minh City University of Education Journal of Science 17*(3), 453-466.

12. WORKSHOP EXERCISES IN FLEXIBLE BUNDLE-NUMBERS

E01. Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many strokes as it represents. Repeat with cubes transforming 3 1s to 1 3s.

E02. Bundle-count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s: Using ‘flexible BundleNumbers’, T = 6 = 0**B**6 = 1**B**1 = 2**B**-4 **5s** (overload, standard, underload). And 0**B**1 = 1**B**-4, 0**B**2 = 1**B**-3, … **5s**

E03. Bundle-count ten fingers in 3s using bundle-bundles. Then in 2s. T = ten = 1**BB**0**B**1 = 101 **3s**.

Write traditional numbers as flexible BundleNumbers: T = 53 = 5**B**3 = 4**B**13 = 6**B**-7 tens

E04.

Flexible BundleNumbers ease Operations

65 + 27 = ? =	6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92
65 – 27 = ? =	6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38
7* 48 = ? =	7* 4**B**8 = 28**B**56 = 33**B**6 = 336
336 /7 = ? =	33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

E05. With cubes, transform the three OUTSIDE parts of a counting process, PUSH & LIFT & PULL, into three INSIDE operation-icons: division / & multiplication x & subtraction -.

Five counted in **2s**: I I I I I (push away **2s**) II II I (lift to stack) II II I (pull to find unbundles ones) II II I.

E06. Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:	T = 7 = 2.1 **3s** = 3.-2 **3s**
Placing the unbundled instead on-top of the stack counted in bundles roots fractions:	T = 7 = 2 1/3 **3s**

Recount traditional numbers: T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens

E07. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a **recount-formula** **T = (T/B)*B** (from T, T/B times, push away Bs) coming from recounting 8 in 2s by 8/2 times pushing away 2s as predicted on a calculator as T = 8 = (8/2)*2, thus using a full number-language sentence with a subject, a verb and a predicate.

OUTSIDE: T = I I I I I; T counted in **2s**: ~~II~~ ~~II~~ I; T – 2x2: = ~~II~~ ~~II~~ I ; INSIDE:

5/2	2. some
5 – 2x2	1

E08. Recount from tens to icons (decreasing the base will increase the height)

OUTSIDE, to answer the question ‘40 = ? **5s**’, on squared paper transform the stack 4.0 **tens** to **5s**.

INSIDE, formulate an equation to be solved by recounting 40 in **5s**:

u*5 = 40 = (40/5)*5, so u = 40/5.

Notice that recounting gives the solution rule ‘move to opposite side with opposite calculation sign’.

E09. Recount from icons to tens (increasing the base will decrease the height)

OUTSIDE, to answer ‘3 **7s** = ? **tens**’, on squared paper or a pegboard change the stack 3 **7s** to **tens**.

INSIDE: oops, with no ten-button on a calculator we can’t use the recount-formula? Oh, we just multiply! Use flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

E10. DoubleCounting in two physical units

DoubleCounting in two physical units gives a ‘per-number’ as e.g. 2m per 3sec, or 2m/3sec.

To answer the question 'T = 6m = ?sec', we just recount 6 in the per-number: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec. Answer the question 'T = 12sec = ? m'.

Find formulas with per-numbers in science and mathematics.

<u>E11</u>. Mutual double-counting the sides in an axb stack halved by its diagonal c creates trigonometry: a = (a/b)*b = tanA*b, etc

Draw a vertical tangent to a circle with radius r. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/radius with tangent of the angle on a calculator.

A 12x12 square ABCD has AB on the ground and is inclined 20 degrees. From B, a straight road is to be constructed intersecting the borderline AD in the point E, inclined 5 degrees. Find the length DE. (Hint: Show that if DE = 2, then the incline of the road is 3.2 degrees).

<u>E12</u>. On squared paper a point has an out-number x and an up-number y, A(x,y). The per-number $\Delta y/\Delta x$ allows moving on a line.

With A(2,5) and B(4,6), the line per-number is $\Delta y/\Delta x$ = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = ($\Delta y/\Delta x$)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

<u>E13</u>. Next-to addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?

<u>E14</u>. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?

<u>E15</u>. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?

<u>E16</u>. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?

<u>E17.</u> E19. Multiplying tens: What is 27 43s recounted in tens? T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161

<u>E18</u>. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what?

<u>E19</u>. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?

<u>E20</u>. Solving STEM proportionality heating problems with recounting

With a heater giving 20 J in 30 sec, what does 40 sec give, and how many seconds is needed for 50J?

With 40 Joules melting 5kg, what will 60 Joules melt and what will 7 kg need?

With 3 degrees needs 50 Joules, what does 7 degrees need; and what does 70 Joules give?

With 4 deg. in 20kg needing 50 Joules, what does 9 deg. in 30 kg need? What does 70 Joules give in 40 kg?

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	~~10**B**10~~
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

13. FLEXIBLE BUNDLE-NUMBERS RESPECT & DEVELOP KIDS' OWN MATH

Abstract

In an isolated covid-19 household we may ask: How can accepting children's own flexible bundle-numbers help us improve their mastery of Many?

This poster is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. The poster gives an overview of what may be done with children's 'flexible bundle-numbers'.

01. Many exists in space as a total. Instead viewing the total in time by counting 1s allows transforming many 1s into 1 many-icon with as many strokes or sticks as it represents, stopping at ten where also counting bundles begin, so that ten = 1B0. Including units, the counting sequence then becomes 0B1, 0B2, …, 0B9 or 1B-1, 1B or 1B0, 1B1, etc. Counting in 3s, ten fingers become a total of T = 3B1 = 1BB0B1 3s. Including units makes the place value system superfluous.

02. Counting in bundles instead involves four tasks: First division iconized as a broom pushes away bundles, then multiplication iconized as a lift stacks the bundles, then subtraction iconized as a rope pulls away the stack to find unbundled, finally addition iconized as a two-way cross unite the two on-top or next-to. This natural order is the opposite of the tradition.

03. Five fingers can be flexibly bundle-counted in 2s with over- or underload: T = 5 = 1B3 = 2B1 = 3B-1 2s thus rooting negative numbers. The unbundled can be placed next-to the stack, or on-top counted in bundles, thus rooting decimals and fractions. T = 5 = 2B1 = 2.1 = 2 ½ 2s.

04. Flexible bundle-numbers ease operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

05. Recounting 8 in 2s giving 8 = (8/2)x2 generalizes into a proportionality 'recount formula', T = (T/B)*B, predicting that 'from T, T/B times, Bs can be pushed away'; and occurring all over STEM. And allowing a calculator predict the result of changing units.

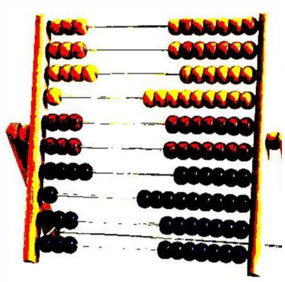
06. Recounting from tens to icons by asking '? 7s = 35' is called an equation u*7 = 35. It is easily solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking '2 7s = ? tens' is eased by using underloads: T = 2*7 = 2*(B-3) = 2B-6 = 20-6 = 14; and 6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To change units we simply recount in the per-number: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Double-counting an axb block halved by its diagonal c produces trigonometric per-numbers: a = (a/b)*b = tan A*b, etc.

09. Next-to addition geometrically means adding areas where multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed. On-top addition involves recounting to get like units, or solving equations if reversed.



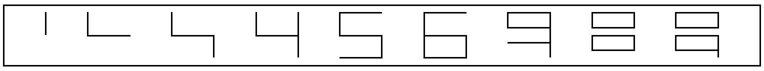
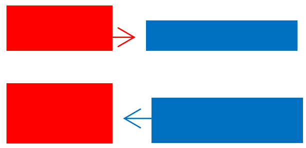
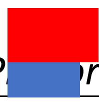
Count before you Add

MatheMatics as ManyMath

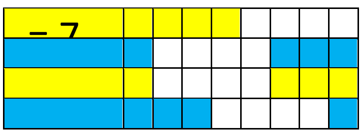
a Natural Science about MANY

MATHeCADEMY.net

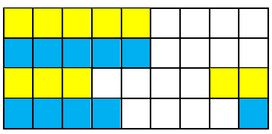
Cure Math Dislike with Kid's own **BundleNumbers** with **Units:** 2 **3s**

Count in ***Icons*** in ***Bundles***	 T = I I I I = 4 = 4 T = 7 = ~~III~~ I I I I = ~~III~~ ~~III~~ I = ~~III~~ ~~III~~ ~~III~~ = 1B4 3s or 2B1 3s or 3B-2 3s
ReCount in *same* ***Unit*** in *new* ***Unit***	**ReBundle** to create Overload or Underload T = 7 = I I I I I I I = 1B4 **3s** = 2B1 **3s** = 3B-2 **3s** T = 7 = 2B1 **3s** = 1B3 **4s** = 1B2 **5s** = 3B1 **2s** = 1BB1B1 **2s** = 11B1 **2s**
ReCounting Predicted by a ***Recount-Formula***	Push ● Lift ● Pull ● Unite: **/ X – +** 8 = ? **2s.** 8 count in **2s** 8/2 times, stack as 8/2 **2s.** **So** 8 = (8/2)x2 **T = (T/B) x B = T/B** Q: 2 **4S** = ? **5S** 2x4/5 1.some 2x4 – 1x5 3 A: 2 **4s** = 1B3 **5s**
ReCount in ***Tens*** from ***Tens***	3 **7s** = ? **tens** Answer: 3x7 = 21 = 2B1 **tens** ? **7s** = 3 **tens** Answer: (30/7) x 7 = 4B2 **7s** 
DoubleCount in *PerNumbers* in *PerFive, 3/5* in *PerHundred, %*	With 4$ per 5kg or 4/5 $/kg, T = 20kg = (20/5) x 5kg = (20/5) x 4$ = 16$ 3/**5** = 3$/**5**$ of 200$ = ?$. 200$ = (200/5) x **5**$ gives (200/5) x 3$ = 120$ 70**%** = 70$/100$ of 300$ = ?$. 300$ = (300/100) x 100$ = gives (300/100) x 70$ = 210$
Add *NextTo* *OnTop*	T = 2 **3s** + 4 **5s** = 3B2 **8s**  *gration* T = 2 **3s** + 4 **5s** = 1B1 **5s** + 4 **5s** = 5B1 **5s** *P*  *rtionality*
Multiply, Divide *BundleWriting*	**7 x** 63 = **7 x** 6B3 = 42B21 = 44B1 = 441 245 **/7** = 24B5 **/7** = 21B35 **/7** = 3B5 = 35

Abacus in 2 modes: T = 7 = 2B1 **3s**



Geometry-mode



Algebra-mode

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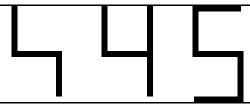
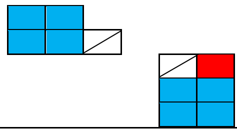
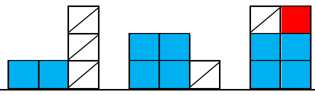
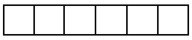
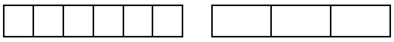
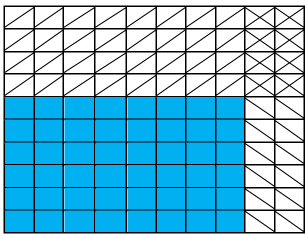
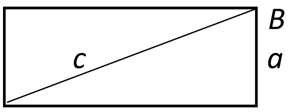
MATHeCADEMY.net	*Teaching Teachers to Teach MatheMatics as* *ManyMath* PYRAMIDeDUCATION **CATS**: **C**ount & **A**dd in **T**ime & **S**pace

Flexible BundleNumbers

Develops when **Kids** adapt to Many

Outside & Inside Math

Digits as ICONS III IIII IIIII		3 4 5
Operations as ICONS	Push ● Lift ● Pull ● Unite	/ X – +
Count Fingers in 5s using BundleCounting & BundleNumbers	I●●●● II●●● III●● IIII● 卌 卌 I●●●●	T = 0*B*1 = 1*B*-4 **5s** T = 0*B*2 = 1*B*-3 **5s** T = 0*B*3 = 1*B*-2 **5s** T = 0*B*4 = 1*B*-1 **5s** T = 1*B*0 = 1*B*0 **5s** T = 1*B*1 = 2*B*-4 **5s**
Unbundled creates Decimals & Fractions & Negative Numbers IIIII ➔ ǁ ǁ I		T = 5 = 2*B*1 **2s** = 2.1 **2s** T = 2 1/2 **3s** T = 3*B*-1 2s = 3.-1 **2s** T = 1*BB* 0*B*1 (T = *p**x^2 + *q**x + *r*)
ReCount in Same Unit creates Flexible Numbers ▌▌▌▌▌III ➔ 53	5: ǁ I I I ǁ ǁ I ǁ ǁ ǁ 	T = 1*B*3 **2s** *Overload* T = 2*B*1 **2s** *Standard* T = 3*B*-1 **2s** *Underload* T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 **tens**
Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7 x 48 = ? = 336 /7 = ? =	6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38 7 x 4*B*8 = 28*B*56 = 33*B*6 = 336 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48
ReCount in New Unit  6 = ? **2s** *ReCount-Formula:*	 6 = (6/2) x 2 T = (T/***B***) x ***B***	T = 5 = (5/2) x 2 = ? = 2*B*1 **2s** 5/2 2.some 5 - 2*2 1
ReCount: Tens to Icons ▌▌▌IIIII = ? **7s**	3*B*5 **tens** = *u**7	*u**7 = 35 = (35/7)*7 so *u* = 35/7
ReCount: Icons to Tens 6 **8s** = ? **tens** 		T = 6 **8s** = 6 x 8 = (***B***-4) x (***B***-2) = ***BB*** - 4***B*** - 2***B*** - - 8 = 10***B*** - 6***B*** + 8 = 4*B*8 = 4.8 **tens** = 48
DoubleCount gives PerNumbers	2$ per 3kg = 2$/3kg	T = 6$ = (6/2) x 2$ = (6/2) x 3kg = 9kg
Like Units: Fractions 5% of 40	5$/100$ of 40$	T = 40$ = (40/100) x 100$ gives (40/100) x 5$ = 2$
DoubleCount a Block halved by its Diagonal	 *A* *b* *C*	*a* = (*a*/*c*)**c* = sin *A***c* *a* = (*a*/*b*)**b* = tan *A***b* π = *n**tan(180/*n*) *for n large* *c***c* = *a***a* + *b***b*

14. FLEXIBLE BUNDLE-NUMBERS RESPECT & DEVELOP KIDS' OWN MATH

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total *T* by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. *T* = 2 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', *T* = (*T/B*)**B*, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c;$ in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', sticks say 2*B*2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation *u**7 = 35. It is easily solved by recounting 35 in 7s: *u**7 = 35 = (35/7)*7. So *u* = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking '2 7s = ? tens' is eased by using underloads: *T* = 2*7 = 2*(*B*-3) = 20-6 = 14; and 6*8 = (*B*-4)*(*B*-2) = *BB* - 4*B* - 2*B* -- 8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Next-to addition geometrically means adding by areas, so multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed.

09. On-top addition means using the recount-formula to get like units. Changing units is also called proportionality, or solving equations if reversed.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Tarp, A. (2020). De-modeling numbers, operations and equations: from inside-inside to outside-inside understanding. *Ho Chi Minh City University of Education Journal of Science 17*(3), 453-466.

15. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

I. MEETING MANY, CHILDREN BUNDLE AND COUNT WITH UNITS

How children master Many I observed from my three Korean foster girls. Asked "How old next time?", a 3year old would say "Four" and show 4 fingers; but would react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they would say '5 3s and 3'; and when asked "How many 4s?" they would say '5 4s less 2'; and, integrating them next-to each other, they typically said '2 7s and 4'.

Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

II. MATERIALS FOR QUESTION GUIDED COUNT CURRICULUM

Typically, a 'mediating curriculum' sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a 'developing curriculum' sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and stacks with inside two-dimensional bundle-numbers. This calls for different materials from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018, 2020) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

QC01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the 'learning opportunity (L.O.) is to change many ones to one icon with as many sticks or strokes as it represents if written less sloppy. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

QC02, counting sequences: "How to count fingers?" Here the L.O. is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H".

QC03, icon-counting: "How to count fingers by bundling?" Here the L.O. is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an 'inside bundle-number' describing the 'outside stack-number'. Turning over a two- or three-dimensional stack or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

QC04, calculator-prediction: "How can a calculator predict a counting result?" Here the L.O. is to see the division sign as an icon for a broom pushing away bundles: 7/2 means 'from 7, push away bundles of 2s'. The calculator says '3.some', thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles. Finally, the subtraction sign iconizes a rope pulling away the stack to look for unbundled singles. By showing '7-3*2 = 1' the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional L.O. is to write 8 = (8/2)*2 as a ‘recount-formula’ T = (T/B)*B, saying “From T, T/B times B can be pushed away”, to predict counting and recounting.

QC05, recounting in another unit: “How to change a unit?” Here the L.O. is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another L.O. is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

QC06, unbundled becomes decimals, fractions or negative numbers: “Where to put the unbundled singles?” Here the L.O. is to see that the unbundled occur in three ways: Next-to the stack as a stack of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside stack 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit..

QC07, prime or foldable units: “Which stacks can be folded?” Here the L.O. is to examine the symmetry of a stack. The stack T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd.

QC08, finding units: “What are possible units in T = 12?”. Here the L.O. is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

QC09, recounting from tens to icons: “How to change unit from tens to icons?” Here the L.O. is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

QC10, recounting from icons to tens: “How to change unit from icons to tens?” Here the L.O. is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking ‘T = 3 7s = ? tens’. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional L.O. uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3*(ten less 3) = 3*ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. So, ‘less less 1’ means adding 1.

QC11, recounting stack-sides. “How to recount sides in a stack halved by its diagonal?” Here, in a stack with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

QC12. On squared paper a point has an out-number x and and up-number y, A(x,y). The per-number Δy/Δx allows moving on a line.

With A(2,5) and B(4,6), the line per-number is Δy/Δx = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = (Δy/Δx)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

III. MATERIALS FOR QUESTION GUIDED ADD CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school;

and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

QA01, next-to addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" Here the L.O. is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

QA02, reversed next-to addition: "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

QA03, on-top addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the L.O. is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

QA04, reversed on-top addition: "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

QA05, adding tens: "With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

QA06, subtracting tens: "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

QA07, multiplying tens: "What is 7 43s recounted in tens?" Here the L.O. is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

QA08, dividing tens: "What is 348 recounted in 6s?" Here the L.O. is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349/6 = 34B9/6 = 30B49/6 = (30B48 +1) /6 = 58 + 1/6.

QA09, adding per-numbers: "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" Here the L.O. is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

QA10, subtracting per-numbers: "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the L.O. is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

QA11, solving the quadratic equation x^2 + 6x + 8 = 0. Two playing cards have the width 3 and the height x + 3. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that $(x+3)^2 = x^2 + 2*3*x + 3^2$, or, $(x+3)^2 = x^2 + 6*x + 8+1$, or $(x+3)^2 = 1$ since $x^2 + 6x + 8 = 0$. So $x = -3 \pm 1 = -4$ and -2.

QA12, finding common units: "Only add with like units, so how add $T = 4ab^2 + 6abc$?". Here units come from factoring:

$T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) *2ab$.

IV. DISCUSSION

Meeting Many makes children bring flexible bundle-numbers to school with core math as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many.

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16. FLEXIBLE BUNDLE-NUMBERS RESPECT & DEVELOP KIDS' OWN MATH

I. CHILDREN SHOW MASTERY OF MANY BEFORE SCHOOL

In an isolated covid-19 household we may ask: How to create simple material supporting the children in improving their mastery of Many?

This workshop-design is based on the observation that when asked ‘How old next time?’, a 3year old will say 4 showing 4 fingers; but, held together two by two, protests by saying ‘That is not 4. That is 2 2s’, thus rejecting the predication ‘four’ by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional bundle-numbers with units, neglected by the school’s 1-dimensional line-names making some children count-over by saying ‘twenty-ten’. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance (Tarp 2018, 2020). In the workshop you will need 12 sticks, 12 cubes, a pegboard with rubber bands, squared paper and a pencil.

II. BRIDGE OUTSIDE EXISTENCE TO INSIDE ESSENCE

E01. Many become icons.

Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many sticks or strokes as it represents. Repeat with cubes.

Sticks and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc. A looking glass finding nothing when looking for more will iconize zero. If used as bundle-size, ten has no icon but is reported as 1 bundle or 1B or 1B0.

E02. Flexible bundle-counting roots negative numbers.

Count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s.

Including bundles, ten fingers may be bundle-counted in fives as ‘0Bundle1, 0B2, 0B3, 0B4, 1B0, 1B1’ that also may be counted as 0B6 or 2B less 4 or 2B-4 with an overload or an underload, thus writing 6 in three ways as a ‘flexible bundle-number’:

T = 6 = 0B6 = 1B1 = 2B-4

Using flexible bundle-numbers, ten fingers may also be counted in 5s as ‘1B-4, 1B-3, etc.’

Counting on from ten, we meet ‘Viking-counting’ where eleven is ‘one-left’, twelve is ‘two-left’, and thirteen is ‘three-ten’, while ‘three-twenty’ becomes twenty-three.

Counting in scores (twenties) from forty, the Danish Viking-descendants still count: half-three-scores, three-scores, half-four-scores, four-scores, and half-five-scores for ninety. Unable to understand the half-notion, the French instead counts over when expressing 87 as ‘4 scores and 1 ten and 7.’

E03. Flexible bundle-counting roots polynomials.

Bundle-counting ten fingers in 3s, nine becomes 3B or 1BB, 1bundle-bundle, called hundred when using ten-bundles. And ten becomes 3B1 3s or 1BB0B1 3s, leading on to the general number-formula or polynomial T = ten = 1*B^2 + 0*B + 1*1 3s, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to stack-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Counting ten fingers in 2s, eight becomes 1BBB, called thousand when using ten-bundles.

Bundle-counting thus allows meeting power before the other operations which allows replacing the place-value-system with bundle-writing: T = 345 = 3BB4B5.

E04. Recounting as flexible bundle numbers will ease traditional calculations.

T = 53 = 5B3 = 4B13 = 6B-7 tens

65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

E05. Counting creates icons.

Count 7 cubes in 2s by 3 times pushing away 2s with a phone iconized as a division sign, predicted by a calculator as ‘7/2 = 3.some’. Stack the bundles by a lift iconized as a cross, 3x2. To look for unbundled, pull away the stack with a rope iconized as a subtraction sign, predicted by a calculator as ‘7-3x2 = 1’. So T = 7 = 3 2s & 1. So operations predict.

E06. Placing the unbundled roots decimals, negative numbers and fractions.

Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:

T = 7 = 2.1 3s = 3.-2 3s

Placing the unbundled instead on-top of the stack counted in bundles roots fractions:

T = 7 = 2 1/3 3s

Counting in tens, T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens.

E07. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a recount-formula T = (T/B)*B.

Recounting 8 in 2s by 8/2 times pushing away 2s is predicted on a calculator as T = 8 = (8/2)*2, which with unspecified numbers becomes a recount-formula T = (T/B)*B, (from T, T/B times, push away Bs) using a full number-language sentence with a subject, a verb and a predicate.

This recount ‘proportionality’ formula occurs all over mathematics and science: when relating proportional quantities as y = c*x; in trigonometry as sine and cosine and tangent, a = (a/c)*c = sinA*c; in coordinate geometry as line gradients, □y = (□y/□x)*□x = c*□x; in calculus as the derivative, dy = (dy/dx)*dx = y’*dx; speed in science: □s = (□s/□t)*□t = v*□t.

E08. Recount from tens to icons

OUTSIDE, to answer ‘40 = ? 5s’ we use a pegboard or a squared paper to transform the stack 4.0 tens to 8.0 5s. So decreasing the base will increase the height. INSIDE, we formulate an equation to be solved by recounting 40 in 5s:

u*5 = 40 = (40/5)*5, so

u = 40/5 giving 40 = 8*5 = 8 5s

Notice that recounting gives the equation solution rule ‘move to opposite side with opposite calculation sign’.

E09. Recount from icons to tens

OUTSIDE, to answer ‘4 7s = ? tens’ we use a pegboard or a squared paper to transform the stack 4 7s to 2.8 tens. So, increasing the base will decrease the height.

INSIDE: oops, with no calculator ten-button we can't use the recount-formula? Oh, we just multiply, thus creating multiplication tables.

Using flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

E10. Double-counting in two physical units.

Double-counting in two physical units gives a 'per-number' as 2m per 3sec or 2m/3sec or 2/3 m/sec. To answer the question 'T = 6m = ?sec' we just recount 6 in the per-number:

T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

Double-counting in the same unit, per-numbers become fractions: 2m per 3m = 2 per 3 = 2/3, and 2 per 100 = 2/100 = 2%.

To answer the question '20 per hundred is ? per 400' we just recount 400 in 100s: T = 400 = (400/100)*100 giving (400/100)*20 = 80.

To answer the question '20 per 400 is what per hundred?' we just recount 100 in 400s: T = 100 = (100/400)*400 giving (100/400)*20 = 5.

STEM formulas contain per-numbers:

meter = (meter/sec)*sec = velocity*sec,

kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter;

force = (force/square-meter)*square-meter = pressure*square-meter;

energy = (energy/sec)*sec = Watt*sec;

energy = (energy/kg)*kg = heat * kg.

Lego-bricks: number = (number/meter)*meter = density*meter.

E11. Mutual double-counting the sides in a stack with base b and height a, halved by its diagonal c, creates per-numbers:

a = (a/c)*c = sinA*c

b = (b/c)*c = cosA*c

a = (a/b)*b = tanA*b

$\pi \approx$ n*sin(180/n)

Draw a vertical tangent to a circle with radius 5. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/5 with tangent of the angle on a calculator.

E12. On squared paper a point has an out-number x and and up-number y, A(x,y). The per-number $\Delta y/\Delta x$ allows moving on a line.

With A(2,5) and B(4,6), the line per-number is $\Delta y/\Delta x$ = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = ($\Delta y/\Delta x$)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

E13. Next-to addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s? Here we see that next-to addition OUTSIDE means adding by areas where multiplication precedes addition. INSIDE, the recount-formula predicts the result. Next-to addition is called integral calculus.

E14. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

E15. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s? Here we see that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

E16. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

E17. Adding tens: If T1 = 23 and T2 = 48, what is T1+T2 when added as tens? Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

E18. Subtracting tens: If T1 = 23 and T2 add to T = 71, what is T2? Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

E19. Multiplying tens: What is 7 43s recounted in tens? Here we see that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

E20. Dividing tens: What is 348 recounted in 6s? Here we see that recounting a total with overload often eases division:

T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58;

T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

E21. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what? Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

E22. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg? Here we see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a 'locally constant' situation, subtracting per-numbers is called differential calculus.

III. GRAND THEORY HOLDS CONFLICTING CONCEPTIONS ON CONCEPTS

Within philosophy, Platonism and Existentialism discuss if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget discuss if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

IV. DISCUSSION

The physical fact Many makes children bring flexible bundle-numbers to school containing core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social ‘colonization’ of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child’s existing mastery of Many.

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17. FINALLY, MATHEMATICS MAY BECOME WHAT IT WAS, A NATURAL SCIENCE ABOUT MANY

ABSTRACT

Background: In an isolated covid-19 household we may ask: How much mathematics is a child able to learn by itself? Or even: How much mathematics can we learn from a child isolated form traditional schooling? Objectives: Is there a different third 'post-setcentric' mathematics education that allows children develop their own number-language? Design: Difference research is chosen as a method, inspired by sociological imagination. Setting and Participants: In a kindergarten, children are asked to perform and discuss different counting problems. Data collection and analysis: The uncovered counting-curriculum is compared with a traditional curriculum. Results: In the children's own counting-curriculum digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Double-counting in two units creates per-numbers. Addition occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas. Conclusions: Instead of colonizing it, mathematics education should develop and learn from the number-language children bring to school allowing counting, recounting and double-counting to take place before adding on-top and next-to.

BACKGROUND

Normally, learning mathematics means going to class and follow the teacher's instruction. However, if all of a sudden there is no class to go to, is learning mathematics still possible? To a certain degree, perhaps, if the instruction comes on the internet or by mail. But in homes without an internet, will learning have to stop until it is again possible to return to the traditional routines?

These are some of the many questions asked under the 2020 pandemic crated by the corona virus covid19. A routine has been locked down, and even an institutionalized one, so how will parent be able to replace the professionalism of the institution? After all, we create institutions precisely to more effectively reach social goals through professionally trained employees instead of trying to do so individually, thus allowing us to do what we are best at to the common benefit of society.

Consequently, when an institution closes down, we may fear that we will no more be able to reach its goal. And, if it is a good institution allowing most learners to reach the goal, we should be sad and long for the reopening of the institution.

On the other hand, if several students fail to reach the goal, we could be hopeful. Maybe there are other ways to reach the goal that the institution cannot see for various reasons? So, in this case, a lockdown allows rethinking the institutional goal and consider other ways to the goal.

So, let this be our research question: are there different ways than the institutionalized to the end goal of mathematics education? To get an answer we use difference research (Tarp, 2018) searching for differences that might make a difference.

DIFFERENCE RESEARCH

Difference research uses sociological imagination described by Mills (1959) as the core of sociology. Bauman (1990, 84) agrees and warns against a goal displacement where "The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right."

So, what is the end goal of mathematics education? Is it mathematics itself which seems to be a self-reference; or is mathematics just a means on the way to reaching another final goal?

Here, the STEM subjects, Science and Technology and Engineering and Mathematics, indicate that mathematics is a tool using numbers and formulas to predict other numbers. And since numbers describe the physical fact Many, it looks like mathematics itself is but a means to get to the real end goal, to master Many as it occurs in time and space. And thus, to be replaced by other means if not obtaining that. Which raises the question: are there different forms of mathematics?

MATHEMATICS, STABLE UNTIL THE ARRIVAL OF SET

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a set-derived 'meta-matics' defining its concepts top-down as examples from abstractions instead of bottom-up as abstractions from examples. Thus, with the arrival of calculus, a function was defined bottom-up as a placeholder for an unspecified formula, i.e. as an abstraction from formulas containing both known and unknown numbers; but with the arrival of set theory, this definition was turned around so that today a function is defined as an example of a subset in a set-product where first-component identity implies second-component identity.

Russell, however, showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

This 'setcentrism' (Derrida, 1991) thus transformed classical grounded 'many-math' into a modern self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

Thus, today mathematics is taught in at least two forms, a pre-setcentric version favored in North America who rejected the New Math movement; and a setcentric version in the rest of the world. To see if there might also be a different third post-setcentric way of learning to master Many we may see how preschool children react to different counting tasks (Tarp, 2018).

MEETING MANY, CHILDREN BUNDLE IN BLOCK-NUMBERS TO COUNT AND SHARE

Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And also using a full sentence as in the word-language with a subject 'that' and a verb 'is' and a predicate '2 2s', which abbreviated makes a formula a number-language sentence 'T = 2 2s'.

Likewise, children use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 more'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 more'.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2*B*2 or 3*B* less2. Or, if counting in 3s, as 1*B*7, 2*B*4, 3*B*1, or 4*B*less2. Some even see that 3 bundles are the same as one bundle-of-bundles, 3*B* = 1*BB*.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0*B*2, 0*B*3, 0*B*4 no 1*B*0, 1*B*1, 1*B*2, 1*B*3, 1*B*4 no 2*B*0, 2*B*1, 2*B*2.

And children don't mind using 'bundle-writing' with a full sentence containing a subject, a verb and a predicate as in the word-language: *T* = 8 = 1*B*5 = 2*B*2 = 3*B*-1 3s. Some might even write *T* = 8 = 3*B*-1 = 1*BB*-1 3s.

Also, children smile when they see that, counting in hands, *T* = 5 = 1*B*0 5s, thus realizing that ten is written as 10 because ten becomes 1*B*0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering '8/2', a calculator predicts that this can be done 4 times; they thus meet the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering '4 x 2', a calculator predicts they have a total of 8; they thus meet the multiplication sign as an icon for a lift lifting up 2s.

And again, they smile when they see that entering '8 – 4 x 2', a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; they thus meet the subtraction sign as an icon for a rope pulling away the 4 2s.

In this way children see that counting involves three processes: pushing away, lifting up, and pulling away that can be performed by a broom, a lift, and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription 'From the total 8, 8/2 times, 2s can be pushed away' may be shortened to a calculation formula '8 = 8/2 x 2', later with unspecified numbers becoming a core mathematical formula expressing proportionality, the recount-formula '*T* = (*T*/*B*) x *B*.

Exposed to counting, children adapt a natural way to the three basic operations division, multiplication and subtraction; and typically enjoy using a calculator, or even the recount-formula, to predict the counting result. We may call this the child's natural 'counting curriculum'.

So why not develop instead of rejecting the core mastery of Many that children bring to school? And that links inside concepts with outside examples thus allowing a two-way connection, to model the outside inside, and to de-model (Tarp, 2020) the inside outside. And let us try to give this developing curriculum a form that can be practiced outside institutionalized education.

A Developing Curriculum for Home Learning

The developing curriculum aims at connecting inside mental thinking about outside physical actions by connecting inside bundle-numbers with outside blocks or tiles.

As material we will use tiles on a floor, on a chessboard, on a pegboard, or on squared paper. As well as twelve matches and cubes, and a folding ruler. Including the arm, we accept that each hand may have 6 'fingers' to count. We report on squared paper with a pencil.

MC01. Counting by Iconizing and Bundling

Outside, Many exists physically as a total in space. Seeing instead the total in time by counting 1s will allow transforming many 1s into 1 many-icon with as many sticks or strokes as it represents. Thus, a total of five matches may be placed as a 5-icon with five sticks which may also be shown by folding a ruler five times. Inside mentally we show this by drawing five strokes on the borderline between tiles, and report by writing '*T* = 5' while saying 'The total is five'.

We stop at ten that we unite to a bundle so that ten 1s become 1 tens. This allows Many to be multi-counted in bundles and singles so that ten is 1Bundled0 or 1*B*0, eleven is 1*B*1, and twelve is 1*B*2. Where thirteen directly links to three and ten, it is different with eleven and twelve, coming from the Nordic Vikings counting ‘en-left’, and ‘twe-left’, and also bundling in scores, twen-tens.

Including bundles *B* as the unit, the counting sequence becomes 0*B*1, 0*B*2, …, 0*B*9, 1*B*0, 1*B*1, etc. Including units thus makes the place value system more understandable or even superfluous.

Counting cubes in 3s we see that 3 1s becomes 1 3s used as a unit. Here the counting sequence becomes ‘0*B*1, 0*B*2, 0*B*3 or 1*B*0; 1*B*1, 1*B*2, 1*B*3 or 2*B*0; 2*B*1, 2*B*2, 2*B*3 or 3*B*0, 3*B*1’.

Stacking the 3s we see that 3 bundles is 1 bundle-of-bundles or a 3 by 3 square, so we should instead count 9 as ‘2*B*3 or 3*B* or 1*BB* or 1*BB*0*B*0 3s’. And twelve as 1*BB*1*B*0 3s.

Likewise, counting in 2s shows 1 4s as 1*BB*, i.e. as a bundle-of-bundles called hundred when counting in tens; and 1 8s as 1*BBB*, i.e. as a bundle-of-bundles-of-bundles called thousand when counting in tens.

Instead of counting the actual number of sticks in a bundle, we may count the number missed in a full bundle. Counting fingers in 5s, we thus have two counting sequences: 0*B*1, 0*B*2, 0*B*3, 0*B*4, 1*B*; and 1*B* less4 or 1*B*-4, 1*B*-3, 1*B*-2, 1*B*-1, 1*B*.

So, bundle-counting leads to ‘flexible bundle-numbers’ recounting a total in different ways, T = 7 = 2*B*1 = 3*B*-2 = 1*B*4 3s. Here the last two may be called under-flow and over-flow numbers.

Likewise, when counting in tens where T = 53 = 5*B*3 = 6*B*-7 = 4*B*13 tens.

Using flexible bundle-numbers will often ease traditional calculations by replacing or explaining carrying:

65 + 27 = 6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92

65 – 27 = 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38

7 x 48 = 7 x 4*B*8 = 28*B*56 = 33*B*6 = 336

336 /7 = 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48

MC02. Iconizing Operations to Allow a Calculator Predict a Recounting Result

We bundle-count a total of 8 in 2s by 4 times pushing away 2s. Using an uphill stroke to iconize a broom pushing away the bundles, the action ‘from 8, push away 2s’ may be entered on a calculator as ‘8/2’, thus predicting the 4 times. The bundle may then be stacked as 4 x 2, using a multiplication lift, x, as an icon for stacking, T = 8 = 4 x 2. This allows the calculator predict that a stack of 4 2 may be recounted in 1s as 8 1s, T = 4 x 2 = 8. With both we now may rewrite 8 = 4 x 2 as 8 = (8/2) x 2, which becomes a ‘recount- formula’ $T = (T/B)$ x B if using unspecified numbers, saying ‘From T, T/B times, B can be pushed away’. Now, asking “how many 2s gives 8?” becomes an equation ‘u x 2 = 8’. With 8 = (8/2) x 2, the recount-formula directly gives the solution u = 8/2, obtained by moving a number to opposite side with opposite calculation sign.

MC03. Unbundled may Become Decimals, Fractions or Negative Numbers

Bundle-counting a total of 7 in 3s, the calculator says ‘7/3 = 2.some’, thus predicting that 2 times pushing away 3s will leave some unbundled behind. To predict how many, we use a rope to iconize pulling away something. And with ‘7 – 2 x 3 = 1’, the calculator predicts that 7 can be recounted in 3s as T = 7 = 2*B*1 = 2.1 3s, using a decimal point to separate the bundles from the unbundled. Instead placing the unbundled on-top of the stack it must be counted in 3s as 1 = (1/3) x 3, called a fraction, or we may pull away 2 from a full bundle as 1 = 1.-2 3s, called a negative number.

So, bundle-counting again leads to flexible bundle-numbers recounting a total in different ways, T = 7 = 2.1 = 2 1/3 = 3.-2 = 1.4 3s, or T = 7 = 2*B*1 = 2 1/3*B* = 3*B*-2 = 1*B*4 3s

Likewise, when counting in tens where T = 53 = 5B3 = 5 3/10B = 6B-7 = 4B13 tens

MC04. Recounting Between Icons-numbers and Tens

Recounting 2 3s in 4s, the calculator predicts the result 1B2 4s, since (2 x 3)/4 = 1.some, and 2 x 3 – 1 x 4 = 2. Recounting 20 in 4 means solving the equation u x 4 = 20. The calculator predicts the result 5 4s, since 20 = (20/4) x 4 = 5 x 4 = 5 4s.

Recounting 7 8s in tens, we cannot use the recount-formula since the calculator has no ten-button. Instead, the multiplication button gives the answer directly, only leaving out the unit and the decimal point.

Using flexible bundle-numbers on a pegboard the result becomes

T = 6 x 8 = B-4 x B-2 = BB – 4B – 2B - - 4 x 2 = (10-4-2)B + 8 = 4B8, thus showing that -x- is +.

MC05. Double-counting Creates Per-numbers

Traveling through a row of tiles may be rewarded with 2 matches per 3 tiles thus creating the 'per-number' 2/3 matches/tiles. Travelling 12 tiles thus gives 2 matches 4 times, which can be precited by recounting in the per-number: T = 12 tiles = (12/3)*3 tiles = (12/3)*2 matches = 8 matches. Alternatively, we can equate the per-numbers in an equation u/12 = 2/3 solved by moving to opposite side with opposite calculation sign, u = 2/3*12 = 8. Per-numbers are all over mathematics and science, e.g. meter = (meter/second) * second = speed * second.

Double-counting in the same unit creates fractions: Marking 2 tiles with a dot for each 3 tiles traveled thus creates a per-number 2 tiles/3 tiles = 2/3. Having travelled 12 tiles, we mark 2 dots 4 times. Again, this can be precited by recounting in the per-number: T = 12 tiles = (12/3)*3 tiles marking dots on (12/3)*2 tiles, giving 8 tiles with dots.

MC06. Bundle-Numbers Add Next-to or On-top, Directly or Reversed

Once counted as stacks, totals may unite next-to or on-top, iconized by a cross showing the two directions. Adding 2 3s and 4 5s next-to as 8s means adding the areas 2*3 and 4*5, called integral calculus where multiplication precedes addition. Adding them on-top, first recounting must change the units to the same. Reversed addition asks e.g. '2 3s and how many 5s give 3 8s?'. Here, first the 2*3 stack is pulled away from the 3*8 stack, then recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3B3 5s. Subtraction preceding division is called reversed integration or differentiation.

MC07. Next-to Addition and Subtraction of Per-numbers and Fractions is Calculus

Throwing a dice 8 times, the outcomes 1 and 6 place 4 cubes on a chess board, and the rest place 2 cubes. When ordered, we may have 5 rooms with 2 cubes per room, and 3 rooms with 4 cubes per room. When adding, the room-numbers 5 and 3 add directly as single-numbers to 5+3 = 8 rooms, whereas the per-numbers add indirectly as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2B6 8s. This average says that all per-numbers would be 2B6 if alike.

Per-numbers thus add by areas, i.e. by integration. Reversing the question to '2 5s + how many 3s total 3 8s' leads to the equation 2*5 + u*3 = 3*8 solved by differentiation:

2*5 + u*3 = 3*8, so u*3 = 3*8 – 2*5, so u = (3*8 – 2*5)/3 = 4 2/3, or u = (T2 – T1)/3 = ΔT/3

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC08. Meeting Algebra

Half of a b*b tile will extend a tile upwards to a h*b playing card. Removing from a h*h square two playing cards, and adding the bottom right tile that has been removed twice will leave the square

$(h - b)^2 = h^2 - 2*a*b + b^2$. And, removing from a $h*b$ playing card the bottom $b*b$ tile will leave the top $(h - b)*b = h*b - b*b$.

Four playing cards are arranged to form a $(h+b)*(h+b)$ square. Inside we find a $h*h$ square, a $b*b$ square and two playing cards, so, $(h+b)*(h+b) = h*h + b*b + 2*h*b$.

Pulling away a $b*b$ tile from the $h*h$ square leaves a $(h-b)*h$ and a $(h-b)*b$ rectangles that add up to a $(h-b)*(h+b)$ rectangle. Consequently, $(h+b)*(h-b) = h^2 - b^2$.

To solve the quadratic equation $x^2+6x+8 = 0$ we use four tiles forming a square, labeling the first side x and the next 6/2. The $(x+6/2)$ square now contains two $6/2*x$ rectangles and two squares, x^2 and $(6/2)^2$ split in two parts, 8 below and $(6/2)^2-8$ above if possible, all disappearing except for last part. So $(x+6/2)^2 = (6/2)^2 - 8 = 1$ giving $x = -6/2 \pm 1 = -2$ and -4. Looking instead at $x^2+bx+c = 0$ gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

MC09. Double-Counting Sides in a Rectangle Halved by its Diagonal

Two neighboring tiles form a rectangle, that halved by its diagonal creates a right triangle with base b, height h and diagonal d. Double-counting pair of sides produces the trigonometry formulas: $h = (h/d)*d = \sin A*d$; $b = (b/d)*d = \cos A*d$, and $h = (h/b)*b = \tan A*b$ that allows a $(b,h) = (+3,+2)$ angle to be predicted by tan-1(2/3) to give 33.7 degrees. This again allows predicting the diagonal: $h = \sin A*d$, or $2 = \sin 33.7*d$, or $d = 2/\sin 33.7 = 3.60$.

MC10. Meeting Pythagoras and Pi

Four tiles with base b and diagonal d form a squared tile. Here 4 diagonals form a square containing 4 half-tiles, i.e. 2 tiles. Consequently $d*d = 2*b*b$, or $d^2 = b^2 + b^2$.

A tile-pair has base b, height h, and diagonal d. Turned 90 degrees a copy is placed on-top. Repeated three times, this creates a square with the side $b+h$. Inside we find a diagonal square and four half tile-pairs; as well as a $b*b$ square and a $h*h$ square and two tile-pairs. But 4 half tile-pairs is 2 full tile-pairs, so we see that $d*d = b*b + h*h$, or $d^2 = b^2 + h^2$, making it easy the add squares, you just square the diagonal.

The normal from the right angle divides the diagonal in p and q. Seeing b as $d*\cos A$, and p as $b*\cos A$, we get $b*b = (d*\cos A)*b = d*(\cos A*b) = d*p$. So, the extension of the normal divides the diagonal square in two parts equal to the squares of the neighboring rectangle side. Since only the angle A is involved this applies to all triangles with angles not above 90 degrees, thus leading to the extended Pythagoras: $a^2 = b^2 + c^2 - 2*b*c*\cos A$, etc.

Two neighboring tiles are circumscribed by a semicircle, again circumscribed by two tiles. On the right tile, the diagonal creates two triangles enveloping a quarter of the semicircle, i.e. 180/4 degrees or $\pi/4$. Consequently, $4*\sin(180/4) < \pi < 4*\tan(180/4)$. In other words, $\pi = n*\sin(180/n) = n*\tan(180/n) = 3.14\ldots$ for n sufficiently large.

A TRADITIONAL MATHEMATICS CURRICULUM

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5.

This offers a ‘natural’ curriculum with multidigit numbers obeying a place value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7.

Division creates fractions and decimals and percentages where $1/2 + 2/3 = 7/6$. And root and log create numbers as $\sqrt{2}$ and log 3 where $\sqrt{2}*\sqrt{3} = \sqrt{6}$, and where $\log 100 = 2$.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g. that $(a+b) * (a-b) = a^2 – b^2$.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

In right-angled triangles, the sides and angles may be related with trigonometrical operations as sine, cosine and tangent where $\sin (60) = \sqrt{3}/2$.

In a calculation, changing the input x will change the output y, making y a function of x, $y = f(x)$, using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g. that $(f*g)'/(f*g) = f'/f + g'/g$. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that $\int 6*x^2 dx = 2*x^3 + c$ with c as an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

COMPARING A DEVELOPING AND A TRADITIONAL CURRICULUM

Different Numbers

The developing curriculum shows digits as icons that come from transforming many 1s into 1 icon with as many sticks or strokes as they represent if written less sloppy. This differs from the tradition seeing both digits and letters as mere symbols.

The digits also show that counting is done by bundling, since ten becomes 1bundle0 or 1*B*0 or 10 if used as bundle-size thus needing no icon itself. Which again shows that a number is a multi-counting of unbundled singles, of bundles, and of bundles-of-bundles etc. as when ten counted in 3s becomes 1*BB* & 0*B* & 1. Which again allows repeated multiplication to appear as power in early counting as wished by Dienes (1964). Furthermore, including bundles leads directly to polynomials as well as to digital counting since counting in 2s, 9 become 1*BBB* & 0*BB* &1*B* &1. This differs from the tradition considering a row of digits as one number obeying a place value system creating difficulties to some learners without physical experience with bundles and bundles-of-bundles. And the tradition presents ten as 10 because it contains 1 ten and no 1s, thus using a meaningless self-reference and silencing the word ‘bundle’

Bundle-counting typically leaves some outside unbundled singles that may be placed next-to or on-top of the stack with bundles, thus linking inside decimals and fractions and negative numbers to different outside locations from the beginning. This differs from the tradition postponing fractions and decimals to after division, and negative numbers to even later. Furthermore, bundling partly, full or too much leads to inside flexible bundle-numbers, e.g. $T = 7 = 1B4 = 2B1 = 3B\text{-}2$ 3s. Allowing overloads and underloads often eases calculations, so the traditional concept of carrying is no longer needed.

Using both sticks and cubes as physical totals allows inside numbers to be linked to outside blocks, thus keeping algebra and geometry linked from the beginning. And it links bundles-of-bundles to squares thus giving a physical meaning to the square-root of 40 by saying: squeeze a 4 by 10 tile into a square.

Double-counting creating per-numbers and fractions with like units makes both operators needing a number to become a number. This differs from the tradition not accepting per-numbers, and seeing fractions as rational numbers with no need to be multiplied before adding.

Different Operations

In the developing curriculum the operations are icons for the counting processes. Division is a broom pushing away the bundles, multiplication is a lift stacking them, subtraction is a rope pulling away a stack to find unbundled singles. And addition is a cross showing the two directions, on-top or next-to, stacks or blocks may be united once they are counted.

Linked in this way to outside physical actions, inside operations get different meanings.

The division 8/2 now means 8 counted in 2s or split in 2s, and not split in 2 as in the tradition.

The multiplication 6 x 5 means 6 5s, i.e. a 6 by 5 block that does not necessarily have to be recounted in tens as 30. And if it does, doubling the base from 5 to ten means halving the height from 6 to 3 to maintain the same total.

The link between inside numbers and outside blocks allows the multiplication table be shown on a pegboard divided by rubber bands so the numbers multiplied occur as flexible bundle numbers, e.g. 6 x 8 = (*B*-4) x (*B*-2). This pegboard multiplication leads directly on to core algebraic formulas as (a-b) x (c-d).

Addition shows the two ways to unite blocks. Here on-top addition asking 2 3s + 4 5s = ? 5s needs recounting to make the units like. And next-to addition asking 2 3s + 4 5s = ? 8s leads to integral calculus where adding the block areas means that multiplication precedes addition.

Subtraction comes in two versions. One describes what a bundle needs to be full as in *T* = 7 = 2*B*1 = 3*B*-2 3s. Another is reversed next-to addition of blocks asking e.g. 2 3s + ? 5s = 3 8s. Here the initial block must first be pulled away before recounting the rest in 5s, i.e. here subtraction precedes division which is natural as the reverse operation to integration.

With per-numbers and fractions as operators needing numbers to become numbers, adding them means adding blocks where multiplication precedes addition, i.e. adding them by areas as integral calculus. This differs from the tradition seeing no problem in transforming mathematics into mathematism by adding them without units, thus creating the fraction-paradox: 1 red apple of 2 plus 2 red apples of 3 total 3 red apples of 5, and not 7 red apples of 6 as the school teaches.

Power comes directly when bundling also bundles, and bundles-of-bundles etc. This differs from the tradition seeing power as repeated multiplication postponed to much later.

Finally, counting ten in 3s as *T* = ten = 1*BB* 0*B* 1 = 1 x *B*^2 + 0 x *B* + 1 allows meeting the basic number formula, a polynomial, from the beginning, thus also meeting the four ways to unite numbers into a total: addition, multiplication, power and next-to block-addition also called integral calculus. Distinguishing between unit- and per-numbers, the basic ‘algebra-square’ (Tarp 2018, 2020) shows that addition and multiplication unite changing and constant unit-numbers, and that integration and power unite changing and constant per-numbers. And that all have reverse operations splitting instead of uniting, subtraction and division split a total in changing and constant unit-numbers. And differentiation and root or logarithm split a total in changing and constant per-numbers, where root is a factor-finder predicting that 3√20 is the answer to the reversed power question *u*^3 = 20; and where logarithm is a factor-counter predicting that log3(20) is the answer to the reversed power question 3^*u* = 20.

As to the order, physical counting suggests division before multiplication before subtraction in order to allow counting create an inside bundle-number describing the outside transformation of a total into blocks to be united in one of two ways. This inverse order to the tradition allows formulating the recount-formula, one of the core formulas in mathematics and science, thus from the beginning giving a calculator a core role as a machine for predicting recounting results. And, it allows meeting linearity and calculus at once instead of waiting to the university level.

This all differs from the tradition teaching addition as counting-on with subtraction as its reverse, multiplication as repeated addition with division as its reverse, never linked to outside blocks needing to change base from an icon to ten, or from ten to an icon. And silencing the polynomial number-formula allowing the 4+5 ways of uniting or splitting to occur in primary school although it shows the end goal of mastering Many.

Different formulas

In the developing curriculum, a formula is a number-language sentence that along with a word-language sentence contains a subject linked to an outside physical object that via a verb is linked to an inside predicate. This allows formulas to have two roles, forward calculation as T = (8/2) x 2 = 4 x 2 = 8, or reversed calculation as u x 2 = 8 solved by recounting 8 in 2s as 8 = (8/2) x 2, thus giving the solution u = 8/2 as well as a solution method: move a number to opposite side with opposite calculation sign. This method is applicable to all examples of reversed calculations: if u + 2 = 8, then u = 8 – 2; if u x 2 = 8, then u = 8/2; if u^3 = 8, then u = 3√8; if 3^u = 8, then u = log3(8).

This differs from the tradition seeing a formula as an example of a function introduced in high school as a subset of a set-product where first-component identity implies second-component identity. And seeing an equation as an example of transforming open statements using the laws of abstract algebra: if u x 2 = 8 then (u x 2) x 2^-1 = 8 x 2^-1; so u x (2 x 2^-1) = 4; so u x 1 = 4; so u = 4 by the associative law and the definitions of an inverse and neural element.

Different Geometry

In the developing curriculum, geometry retains its original meaning, earth-measuring. This occurs when the core outside object, a rectangular a by b stack, is split in two by its diagonal c. This creates a right-angled triangle where a mutual recounting of the three sides creates the trigonometrical formulas. Which allows a wealth of stimulating outside problems to be addressed at a very early age. This differs from the tradition letting plane and coordinate geometry precede trigonometry.

Different Calculus

In the developing curriculum, calculus occurs in primary school as next-to addition of blocks, in middle school as adding piecewise constant per-numbers asking questions as 2kg at 3$/kg + 4kg at 5$/kg total 6 kg at ? $/kg. And finally, in high school, adding locally constant per-numbers defined by interchanging the epsilon and delta in the constancy definition. This differs from the tradition only teaching high school calculus and in reversed order with differential before integral calculus.

DISCUSSION AND RECOMMENDATION

This paper asked 'Are there different ways than the institutionalized to reach the end goal of mathematics education?' The answer is yes based upon the observation, that Many presents itself to children as bundle-numbers with units as e.g. 2 3s, outside occurring as rectangular blocks, which allows geometry and algebra to go hand in hand from grade 1.

The use of units makes counting and recounting a basic activity. First division pushes away bundles to be stacked by multiplication to be pulled away by subtraction in order to find unbundled singles that become decimals, fractions or negative numbers depending on their position. Recounting thus produces a formula T = (T/B)*B present all over mathematics and science, and showing there things: How to change units, how to solve equations by moving to opposite side with opposite sign, and how per-numbers must be multiplied to become totals.

Double-counting in two units produces per-numbers, becoming fractions with like units. As geometrical representations of bundle-numbers, rectangles lead directly to double-counting the sides in rectangles halved by their diagonals, thus allowing trigonometry to precede traditional plane geometry.

Addition comes last in two forms, on-top needing proportionality to change units, and next-to adding areas as integral calculus, also occurring when adding per-numbers and fractions with units to avoid mathematism. And reversed addition leads directly to differentiation.

So, it turns out that the core of mathematics springs from blocks once we accept the two-dimensional bundle-numbers children develop while adapting to Many before school. Education needs not teaching 'metamatism' as the only means leading to the end goal, Mastery of Many. Bundles will show the way directly in a concrete way based on outdoor activities; and will perhaps be able to answer the Cinderella question with a yes, there is a hidden unnoticed alternative that makes the prince dance. Bundle-mathematics may offer a new Kuhnian paradigm (1962) that will finally create a mathematics for all. This paper has taken a first step in an experiential learning cycle by designing half a score of micro-curricula to be tested inside and outside classrooms in the hope that after several cycles of redesigning they will allow math dislike to evaporate.

To claim mastering mathematics as the goal of mathematics education is a goal displacement. Mastering mathematics is but a means to the final goal of mastering Many.

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18. REPLACING ACADEMIC NUMBER SENSE WITH CHILDREN'S OWN MANY SENSE

ABSTRACT

Background: Why do we colonize children's number-language with one-dimensional place-value numbers? After all, their biological capacity to adapt to their environment make children develop a number-language of their own that may be developed instead of replaced by present pre-setcentric or set-centric curricula? **Objectives**: Is there a different third 'post-setcentric' mathematics education that allows children develop their own number-language? **Design**: Difference research is chosen as a method, inspired by four grand theories looking at mathematics education: biology, philosophy, psychology and sociology. **Setting and Participants**: In a kindergarten, children are asked to perform and discuss different counting problems. **Data collection and analysis**: The uncovered counting-curriculum is compared with a traditional curriculum and a special discussion compares number-sense with many-sense. **Results**: In the children's own counting curriculum digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Double-counting in two units creates per-numbers. Addition occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas. **Conclusions**: Instead of colonizing it, mathematics education should develop and learn from the number-language children bring to school allowing counting, recounting and double-counting to take place before adding on-top and next-to. This will finally allow the communicative turn to take place within the number-language as it did within the word-language 50 years ago.

Keywords: early childhood; number sense; counting; adding; proportionality.

POOR PISA PERFORMANCE DESPITE INCREASED RESEARCH

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Centre for Mathematics Education in Sweden.

However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD2015, p. 3).

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

MATHEMATICS AND ITS EDUCATION

The Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic.

This ‘pre-setcentric math’ was replaced by the present setcentric ‘New Math’ in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox ‘this sentence is false’, being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Consequently, continental Europe taught reckoning, called ‘Rechnung’ in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But without excelling in the PISA tests.

In their ‘Learning framework 2030’, OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the ‘Cinderella question’: with pre-setcentric and setcentric mathematics unable to deliver, is there a different third post-setcentric alternative, that may ‘make the prince dance’? Difference research (Tarp, 2018) may give an answer.

DIFFERENCE RESEARCH

Difference research uses sociological imagination described by Mills (1959) as the core of sociology. Bauman (1990) agrees by saying

> Sociological thinking is, one may say, a power in its own right, an anti-fixating power. It renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (p.16) **Rational** action (..) is one in which the end to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical. (p.79) Last but not least, the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance (p.84).

It is a general opinion that the goal of mathematics education is to learn mathematics. However, this goal is self-referring. So maybe traditional mathematics has a goal displacement hiding a different more fruitful way to the outside goal, to master Many as it occurs in space and time?

This scepticism towards a world ‘hitherto oppressive in its apparent fixity’ is the basis for postmodern and post structural thinking (Lyotard, 1984). But instead of rejecting the grand theories let us hear what they have to say about mathematics education to see if differences will appear.

BIOLOGY LOOKS AT EDUCATION

As a life science, biology sees life as built from green, grey and black cells.

Green cells form plants able to perform photosynthesis that store the energy form solar photons in carbon hydrate molecules by replacing oxygen with water in carbon dioxide molecules. To survive, plants must access light and water where they are situated since they are unable to move.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

Black cells exist individually in oxygen depleted areas on the bottom of lakes or in the stomach of animals, surviving by removing oxygen from carbon hydrate molecules, thus being transformed to carbon or carbon hydrogen or oil allowing energy to be used by machines.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms. Reptiles have one brain allowing it to transform outside information into a choice between alternative actions. Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients in life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations.

To answer this, we now turn to three other sciences: philosophy, psychology and sociology.

PHILOSOPHY LOOKS AT EDUCATION

Philosophy looks at the relation between outside existence, ontology, and inside representation, epistemology, or, in other words, the ‘it-we’ relation. Within philosophy, precedence is given to outside phenomena by empiricism, to inside rationality by rationalism, and to questioning ruling knowledge regimes by scepticism.

A controversy within philosophy began in ancient Greece where the sophists pointed out that to practice democracy, a population must be enlightened, especially about the difference between nature and choice to avoid being patronized by political choice masked as unpolitical nature. In opposition to this, Plato saw choice as an illusion since all physical is but examples of metaphysical forms only visible to philosophers educated at the Plato academy. (Russell, 1945)

Later, the Christian church changed the academies into monasteries, where some changed back into universities after the reformation; and after Newton’s discovery of physical laws controlling nature without being physical or metaphysical patronized. This inspired the Enlightenment Century installing two republics, one in North America taking over empiricism when developing ‘it is right if it works’ pragmatism, symbolic interactionism, grounded theory, and action research; and one in France taking over scepticism after seeing its republic turned over several times because of resistance from its German speaking neighbours.

Today, opposition against rationalism is seen within existentialism, claiming with Sartre (2007) that existence precedes essence. And explicated by Heidegger (1962) arguing that in defining verdict-sentences "subject is predicate", the subjects should be respected for naming what exist outside, whereas the predicates should be questioned and appealed since they represent an inside choice between alternatives in risk of being masked as nature.

In contrast to this, continental rationalism gives precedence to inside essence developed by rational universities through peer-reviewed research.

PSYCHOLOGY LOOKS AT EDUCATION

Psychology looks at cognitive aspects of learning, or, in other words, the 'it-I' relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development. Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

SOCIOLOGY LOOKS AT EDUCATION

Sociology looks at the social aspects of human interaction, or, in other words, the 'they-I' relation. Here, the philosophical controversy between outside existence and inside essence carries on as a controversy between different forms of social theory emphasizing individual agency or social structure.

Individual agency is emphasized in the first Enlightenment republic in North America showing strong resistance against institutional answers since they may lead to a goal displacement (Bauman, 1990) becoming an inside goal itself instead of staying as an inside means to an outside goal, thus suppressing the 'sociological imagination' (Mills, 1959) that might keep the answer fluid instead of fossilizing into what Weber (1930) calls an 'disenchanting Iron Cage'.

Structuralism is preeminent in continental Europe seeing established science as being a true inside representation of the outside environment if accepted by the society's knowledge institutions, the universities.

In the second Enlightenment republic in France, institutional scepticism inspired by Heidegger led to French post-structuralism where Derrida, Lyotard, Foucault and Bourdieu warn against hidden patronization in our most basic institutions: words, correctness, diagnosing discourses, curing institutions, and education especially that might make mathematics so difficult it may serve as symbolic violence to establish a new knowledge nobility.

Foucault (2006: 41) recommends criticizing modern institutions:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them.

As to education, Foucault (1995) sees schools as ‘pris-pitals’ mixing social power techniques from a prison and a hospital. Here students are ‘pati-mates’ forced to return to the same class, hour after hour, day after day, month after month for several years. Furthermore, self-reference is used to create diagnose ‘ignorant’ without specifying of what: ‘you don’t know math, so we must teach you math’. Consequently, humans are placed in a Kafkaesque (2015) situation where they, unable to have the diagnose defined, finally accept it and therefore also accept being retained and treated to be cured.

In Germany, Habermas (1981) theorizes the possibility of creating a third Enlightenment republic in post-war Germany. Inspired by Weber’s warning that rationalization carried too far might lead to an iron cage dis-enchanting the world, Habermas warns against system-worlds tending to colonize the life-world and recommends a power free communication rationality to prevent this from happening where “peers exchange views”.

Likewise, Arendt (1963) points out that by definition, institutions lack competition, forcing employees to follow order, which might lead to the banality of evil, where ordinary citizens must act evilly to keep the job.

Agency-based education sees knowledge construction as best taking place in symbolic interaction between peers exchanging views about the sentence subject in order to negotiate a common view. This resonates with Piagetian constructivism, and with philosophical existentialism giving precedence to existence; and with the construction of social knowledge by using Grounded Theory (Glaser and Strauss, 1967). Secondary education should be block-organized to support the student’s identity work through self-chosen half-year blocks with teachers teaching only one subject.

In contrast to this, structuralism sees democratically controlled educational institutions as the best way to mediate the university knowledge heritage. This resonates with Vygotskyan constructivism. Secondary education should be line-organized to supply the state with skilled academic and non-academic workers, thus forcing students to choose career line early, and to start all over if changing career.

FOCUSING ON EARLY CHILDHOOD EDUCATION

A good beginning is half done, an old saying goes. So why not look more closely into what preschool children already know and what they are taught. In accordance with this, the National Council of Teachers of Mathematics (2000) revised the Principles and Standards for School Mathematics to include attention to prekindergarten, since they appreciated the importance of providing students with a solid and deep mathematics knowledge at this stage. Later, The National Association for the Education of Young Children (NAEYC) and NCTM (2002) issued a joint position statement on teaching mathematics to young children. Furthermore, the research literature is not overwhelming in this area.

MEETING MANY, CHILDREN COUNT WITH BUNDLES AS UNITS

How children adapt to Many can be observed from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not four, that is two twos’, thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And also using a full sentence as in the word-language with a subject ’that’ and a verb ‘is’ and a predicate ‘2 2s’, which abbreviated makes a formula a number-language sentence ‘T = 2 2s’.

Likewise, children use bundle-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3 more’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, placing them next-to each other, they typically say ‘2 7s and 3 more’.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different colour for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new colour for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don’t mind writing using ‘bundle-writing’ with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering ‘8/2’, a calculator predicts they can have 4 each; they thus meet the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering ‘4x2’, a calculator predicts they have a total of 8; they thus meet the multiplication sign as an icon for a lift lifting up 2s.

And again, they smile they see that entering ‘8 – 4x2’, a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; they thus meet the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, lifting up, and pulling away, that can be performed by a broom, a lift, and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription ‘From the total 8, 8/2 times, 2s can be pushed away’ may be shortened to the calculation formula ‘8 = 8/2x2’, later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula ‘T = (T/B) x B’.

Exposed to counting, children adapt a natural way to the three basic operations division, multiplication and subtraction; and typically enjoy using a calculator, or even the recount-formula, to predict the counting result. We may call this the child’s natural ‘counting curriculum’.

A TYPICAL CONTEMPORARY MATH CURRICULUM

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5.

This offers a ‘natural’ curriculum with multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

Reverse operations create new numbers calling for additional education about operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as $\sqrt{2}$ and log 3 where $\sqrt{2}$ x $\sqrt{3}$ = $\sqrt{6}$, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about operating on such numbers, e.g. that (a+b) x (a-b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

To be followed by halving a rectangle by its diagonal to create a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent.

In a calculation, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about operating on calculations, e.g. that (f x g)' /(f x g) = f' /f + g' /g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫ 6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

A math curriculum always silences that it is not teaching mathe-matics, but 'mathe-matism', a mixture of 'meta-matics' defining a concept from above as an example of an abstraction instead of from below as an abstraction from examples; and 'mathe-matism' always true inside, but seldom outside a classroom where '2+3 = 5' meets countless counter-examples as 2 weeks + 3 days = 17 days.

Likewise, fractions are treated as numbers despite the fact that they are operators needing a number to become a number as shown by the fraction-paradox: 1 red apple of 2 plus 2 red apples of 3 total 3 red apples of 5, and not 7 red apples of 6 as the school teaches.

THE DIFFERENCE TO THE CHILD'S COUNTING CURRICULUM

Thus, typically the core of a contemporary mathematics curriculum is how to operate on specified and unspecified numbers and calculations.

Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles, bundles, and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Seldom children may say 'bundle' instead of ten, or to say 'ten-ten' or 'bundle-bundle' instead of hundred.

Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Using flexible bundle-numbers to rename 17 as 2B-3 tens and 24 as 1B14 tens is not allowed.

Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction.

The children never experience the thrill of seeing a calculator predict a recounting result.

Or the fun of recounting to a different unit predicted by the recount-formula and tested on a western ten by ten abacus.

Or the fun building a multiplication table on a pegboard using rubber bands to show the numbers as flexible bundle-numbers thus seeing that - - is + since the upper right corner is pulled away twice and therefore must be added again.:

6 x 8 = (B-4) x (B-2) = BB – 4B – 2B - - 4x2 = (10-4-2)B + 8 = 4B8 = 48

Or the powerful technique of double-counting in two different units to create per-numbers as 2$ per 5kg, allowing answering proportionality questions as ‘8$ buys ?kg’ by recounting in the per-number: T = 8$ = (8/2) x 2$ = (8/2) x 5 kg = 20 kg.

Or the fun to add two totals once they are counted as blocks to see that added on-top recounting first must make the unites the same; and that added next-to mean adding areas, i.e. integration where multiplication precedes addition, and becoming differentiation when reversed where subtraction precedes division since the start block must first be removed before counting bundles (Tarp, 2018, 2020).

In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children’s existing mastery and basic quantitative competence in counting and later adding.

DISCUSSING NUMBER SENSE AND NUMBER NONSENSE

The basic question in grade in early childhood education is: shall education be about numbering or about numbers? Shall education guide and support the development of the children’s already existing adaption to quantity, or shall education teach numbers? Shall the ‘I’ keep on adapting to the ‘it’ directly, or indirectly by having the adaption replaced by what is mediated by the ‘they’?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics in their publication ‘Principles and Standards for School Mathematics’ (2000) says: “Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense (p. 7).”

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children’s whole number arithmetic development?” (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept ‘number sense’ is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term ‘number sense’ for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or ‘more-ness’. Using the term ‘cardinality’ just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term ‘many sense’ is more precise than the term ‘number sense’. Especially since, with its reference to numbers, ‘number sense’ becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word ‘understanding’ makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense as many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the ‘missing link’, the bundle and the unit, absent in everyday use: T = 67 = 6B7 tens.

Therefore, a short definition could be: Having number-sense as many-sense means including the word ‘bundle’ as a unit for the numbers when counting totals. That will allow

- To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.
- To bridge an outside total with an inside numbering by bundling expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g. T = 2 3s.
- To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with ‘flexible bundle-numbers’ with overload, underload or fraction, i.e. as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s. And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 3 1/3, 1BB0B1, or 1BB1B-2. And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens. And 678 = 67B8 = 6BB7B8. (Tarp, 2020)
- To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift lifting up bundles into a stack, where subtraction iconizes a rope pulling away the stack to find unbundles singles, and where addition iconizes placing stacks next-to or on-top.
- To see the counting process predicted by the recount-formula T = T/(B)*B, saying ‘From the total T, T/B times, B-bundles can be pushed way’; and to use a calculator to enter ‘9/4’ giving ‘2’, and ‘9-2*4’ giving ‘1’ to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.
- To see totals as double-described both as outside stacks and as inside bundles.
- To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.
- To see a multiplication task as recounting from icons to tens, facilitated by using flexible block&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be

calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

- To see a multiplication equation 4*u = 20 as recounting from tens to icons, solved by the recount-formula: 4*u = 20 = (20/4) * 4, thus giving the solution u = 20/4 coming from moving a number to the opposite side with the opposite calculation sign.

Table 1

A counting table that includes the bundles in the number names

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in the set of rational numbers, etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is four numbers: 6, 7, 8 and bundles, which typically is ten. Or scores in France and Denmark both counting four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 67 = 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, ‘this sentence is false’, since the set of sets not belonging to itself will belong if and only if it will not.

Russell’s point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept ‘number-sense’ we should remember that sociologically, a school is a pris-pital. So, the moment you introduce a new construct you risk also introducing a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses is an example of what Foucault calls pastoral power:

The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault describes the salvation promise of the generalized church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and do exactly what we tell you. If not, you only have yourself to thank for your decline'.

Introducing diagnoses may also be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu, 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Bauman calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means, thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight 'mathematical competencies' (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002). Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

PROPORTIONALITY SHOWS A VARIETY OF MASTERING MANY

The need to change units has made the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

- Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.
- Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.
- Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.
- Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.
- Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.
- Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.
- Modeling a linear function f(x) = c*x, with f(2) = 5, f(7) = ?, and f(x) =12. Set-centrism prefers this to show the relevance of abstract algebra's group theory: Let us define a linear function f(x) = c*x from the set of kg-numbers to the set of $-numbers, having as domain DM = {x□R I x>0}. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: c*2 = 5, (c*2)*½ = 5*½, c*(2*½) = 5/2, c*1 = 5/2, c = 5/2. With f(x) = 5/2*x, the inverse function is f-1(x) = 2/5*x. So with 7kg, f(7) = 5/2*7 = 17.5$; and with 12$, f-1(12) = 2/5*12 = 4.8kg.

CONCLUSION

This paper asked if there is a third hidden alternative post-setcentric mathematics education.

The answer is yes, and it is easy to see by observing the mastery of Many children develop before being educated. So, we may want to respect and develop this further by accepting the children's two-dimensional bundle-numbers with units instead of colonizing their adaption to quantity with one-dimensional line-numbers forced upon them by setcentric or pre-setcentric mathematics education.

The time has come to realize that the goal of mathematics education is not to master mathematics first to later master many, but to master many directly, or indirectly by means of one of the different versions mathematics has had over several thousand years.

The time has come to enjoy a paradigm shift (Kuhn, 1962) in mathematics education using design research (Bakker, 2018) to explore what happens when mathematics education allows the children to develop the number-language they bring to school using inside flexible bundle-numbers always linked to outside blocks and tiles, thus connecting the physical with the mental, and connecting geometry with algebra, and modeling the outside inside or de-modeling the inside outside (Tarp, 2020). And to allow core mathematics as linearity and calculus to be performed in year one in primary school instead of postponing them to year one at the university level.

The numerous action learning (Kolb, 1984) and action research projects springing from the child's own counting curriculum may be difficult to carry out in a traditional classroom having so much mental and economical capital bound to the traditional line-numbers. But outside this orthodoxy plenty of opportunities present themselves in preschool, home education, special education, immigrant camps and refugee camps to give some examples.

The time has come where mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

After all, the prime goal of education it to allow the learner develop inside tools to master the outside world. And here the physical fact many plays a so dominant role that children already have a solid adaption to it that education then can choose to develop through proper guidance, or to reject for not being in agreement with the contemporary truth regime.

RECOMMENDATION: MATHEMATICS AS A NUMBER-LANGUAGE

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: 'This is a chair'. As does the number-language assigning numbers instead: 'the 3 chairs each have 4 legs', abbreviated to 'the total is 3 fours', or 'T = 3 4s' or 'T = 3*4'. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb by just saying '3*4 = 12'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is an auxiliary verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is a commutative operation'.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring set-based form has turned mathematics into a grammar labelling its outside roots as 'applications', used as means to dim the impeding consequences of teaching a grammar before its language.

A core question in language education is the following: should education develop further the children's own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978, and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its New-math revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish a sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing is seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift (Kuhn, 1962) in early childhood education in by developing the children's already existing many-sense created through their adaption to Many.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

Which here appears to be the curers in the form of teachers and teacher educators eager to force upon children the present 'math before applications' regime. Instead learning from children's rooted math provides an excellent learning opportunity for teachers to become decolonized and to join the children in celebrating and enjoying the enchantment of visiting a quantitative universe.

DATA AVAILABILITY STATEMENT

Data Sharing does not apply to this article as no new data has been created or analyzed in this study.

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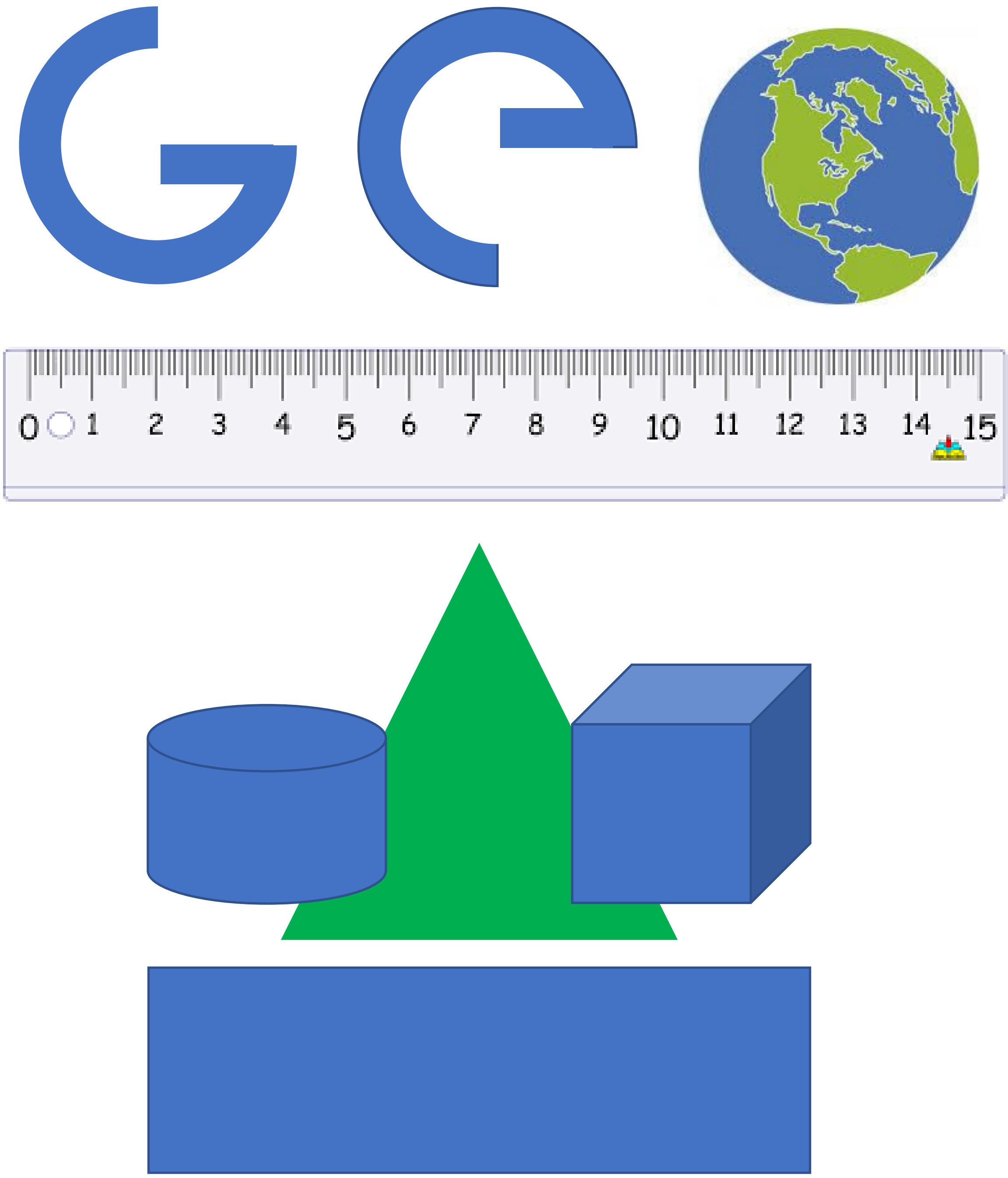

GeoMetry

Earth Measurement
from below

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19. GEOMETRY - FROM BELOW

‘Geometry from below’ means geometry as tales about a social practice, in this case about ‘earth-measurement’, to which the Greek word ’geo-metry’ can be directly translated.

The earth is what we live on and what we live on. We divide the earth between us by drawing dividing lines. This divides the soil into areas limited by lines, multi-edges, polygons.

If these boundaries disappear, it is important to be able to re-establish them, and this restoration of lines and corners requires that these can be measured.

In ancient Egypt, the Nile thus crossed its banks once a year and brought manure to the fields. After retiring, the divisions had to be re-established.

Geometry from below can be understood as the opposite of geometry from above, deducing geometry from metaphysical truths, axioms.

The following material is not a traditional textbook, but rather an activity guide with suggestions for a range of activities that the reader can perform and report.

So, the idea is that the reader builds his own textbook.

However, some proposals for definitions and rules are included.

Rules can be proved either through evidence or conviction. It is recommended to work with the last based on the task: “Try to convince someone else of the validity of the rule.” Likewise, the idea is that the reader performs his own illustrations, which is why very few illustrations are included in the material.

The following exercises should be performed both on paper, on the floor and on earth, as well as, if possible, on computer programs.

We work out a report of questions, techniques, inventions, and discoveries we meet along the way.

Jutland, Denmark, Allan Tarp

DK version January 2000, US version May 2020

http://mathecademy.net/geometry-from-below/

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GE00 LAND SHARING

In this chapter we will look at a number of traditional land-sharing problems. No solutions are provided, but the problems can hopefully serve as motivation for the following chapters.

Basic problem: How to share an area?

Exercise 1. Two people stand in random places on a restricted floor surface and are tasked with dividing it between them. represent groups of different sizes. What are the different sharing principles? Note: Random location in a room may be possible. achieved with a cube: I stand with my back and right shoulder against the wall in a corner. Then I roll a dice 4-6 times. The cube tells me how many steps I need to walk alternately forward and to the left. Same with the other participants.

Exercise 2. Perform a sharing based on the sharing principle: No matter to everyone.

Exercise 3. Perform a sharing based on the sharing principle: Just far to the limit.

Exercise 4. Perform a division based on the sharing principle: The distances to the border must be 1:2, as there are twice as many people living in the other area.

Exercise 5. Repeat exercise 1-4, but now with three people.

Exercise 6. Repeat exercise 1-4, but now with more than three people.

GE01 POLYGONS

Basic problem: How to talk about and divide areas.

Exercise 1. Set up a triangle. Measure the triangle in order to re-establish it. Delete the triangle. Re-establish the triangle on the basis of the objectives, on the one hand at any point and partly in the same place. Insert names (definitions) based on this exercise.

Exercise 2. Construct a square with skewed angles. Measure the square in order to re-establish it. Delete the square. Re-establish the square on the basis of the dimensions, on the one hand, in any place and partly in the same place.

Exercise 3. Set up a triangle. Divide it into two right triangles. Measure the right triangles in order to restore them. Re-establish the triangles on the basis of the objectives, on the one hand, at any point and partly in the same place.

Exercise 4. Set two parallel lines. Measure the distance of the lines.

Exercise 5. What is A4 paper? And what are A1, A2, A3, A5 etc?

Exercise 6. Length can be measured in many ways. Today, meters are usually used. Other targets were used in the past. Which? Is the meter system used all over the world? What is the link between meters and other targets?

Exercise 7. Can angles be measured in ways other than degrees? Why is a right angle 90 degrees and not 100 degrees?

Exercise 8. Find rules for the defined concepts and convince someone else that these rules are true.

DEFINITION 1. One point is

DEFINITION 2. A straight line is

DEFINITION 3. A polygon is

DEFINITION 4. The circumference of a polygon is

DEFINITION 5. An angle is

DEFINITION 6. A polygon's parts are

DEFINITION 7. A diagonal is

DEFINITION 8. A rectangle is

DEFINITION 9. A right triangle is

DEFINITION 10. The distance between a point and a line is

DEFINITION 11. Parallel lines are

DEFINITION 12. A normal is

DEFINITION 13. A convex polygon is

Rule 1. A polygon can be divided into

Rule 2. A triangle can be divided into

GE02 TRIANGLES

Basic problem: How are triangles designated and calculated?

Exercise 1. Construct a triangle and enter names for the different components of a triangle. How many targets can we only achieve in order to uniquely construct a triangle? How many different triangle types are there?

Exercise 2. A triangle can be divided by different lines. Name some of these and discuss what sharing principles the different lines might embodies.

Exercise 3. Find rules for the named lines in practice 2 and convince someone else that these rules are true.

Exercise 4. Construct an equilateral triangle. Do special rules apply to straight-legged triangles?

Exercise 5. Construct an equilateral triangle. Do special rules apply to equilateral triangles?

Exercise 6. Cut out a triangle and hang it in a corner. Draw the load line, how is it? Switch to the other corners. Where is the center of gravity of the triangle?

Exercise 7. The foot point of a dividing line in a triangle is the intersection of the line with a side. Do special rules apply to foot points for heights, medians, angular halves and bisectors?

Exercise 8. A triangle can be wrapped in a rectangle in different ways. Which rectangle has the smallest circumference?

Exercise 9. Construct a triangle ABC. Move A to A* without changing the circumference. What will be the new points?

DEFINITION 1. A triangle or a three-corner is

DEFINITION 2. The three angles of a triangle are

DEFINITION 3. The three sides of a triangle are

DEFINITION 4. An acute triangle is

DEFINITION 5. An obtuse triangle is

DEFINITION 6. In triangle ABC the height ha is

DEFINITION 7. In triangle ABC the median ma is

DEFINITION 8. In triangle ABC, the angular bisector bA is

DEFINITION 9. In triangle ABC the bisector na is

DEFINITION 10. An SSA triangle is a triangle in which two sides and one angle are known. Similarly, an SAA and an SSS triangle are defined.

Rule 1. In a triangle is the angle sum.

Rule 2. In a triangle, the heights intersect

Rule 3. In a triangle, the angular half-lines intersect

Rule 4. In a triangle, the medians intersect

Rule 5. In a triangle, the bisectors intersect

Rule 6. In a triangle, the intersection of the heights has the following property:

Rule 7. In a triangle, the intersection of the angular halves has the following property:

Rule 8. In a triangle, the intersection of the medians has the following property:

Rule 9. In a triangle, the intersection of the bisectors has the following property:

GE03 RIGHT TRIANGLES

Basic problem: How is a right triangle designated and calculated?

Exercise 1. Construct a rectangle and divide it into two right triangles using a diagonal. Introduce names for the different components of the right triangle.

Exercise 2. Construct a rectangle and divide it into two right triangles using a diagonal. Measure the angles. Measure the diagonal and side lengths, partly in cm, and partly by diagonal lengths (as a percentage of the diagonal). Compare to the calculator's sin and cos button.

Exercise 3. Construct a rectangle and divide it into two right triangles using a diagonal. Measure the angles. Measure the diagonal and side lengths, partly in cm, partly in floor lengths (as a percentage of the floor, i.e. the horizontal side). Compare to the calculator's tan button.

Exercise 4. Set on millimeter paper a quartz circle with a radius of 10 cm, or set on the floor a quartz circle with a radius of 1 meter. Sign a series of right triangles ABC with A in the center of the circle, B on the circular arc and C on the 0-degree line. A shall pass degrees 10, 20, 30 up to 80. Table for each A value the length of BC and AC as well as the BC/AC ratio. Compare to the calculator's sin, cos and tan button.

Exercise 5. Construct a right triangle from abandoned dimensions. What's the minimum number of goals we can settle for? How many different types of right triangles are there? we can measure up to the unknown goals, but can we also count for the unknown goals?

Exercise 6. Construct a random triangle and divide it into two right triangles. Measure the parts in the right triangles and calculate them afterwards. Finally, enter the dimensions of the original triangle's parts, and check when measuring.

Exercise 7. Construct a known triangle and divide it into two right triangles. Measure the parts in the right triangles and calculate them afterwards.

Exercise 8. Set off a known SAA triangle. Measure and calculate the other parts.

Exercise 9. Set off a known SSA triangle. Measure and calculate the other parts.

Exercise 10. Set off a known rectangle. Measure and calculate the length and angles of the diagonal.

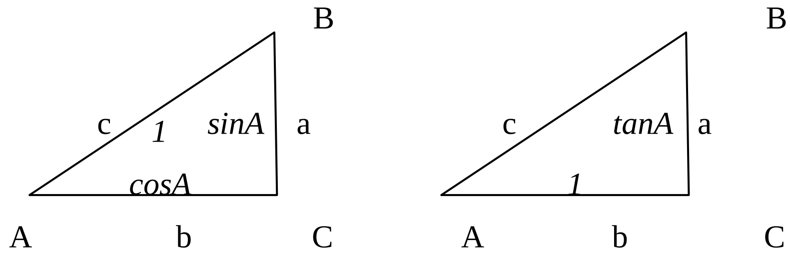
Exercise 11. Construct using the different known angle sizes of the seaweed, e.g. 27°, 42° and 133°.

Exercise 12. Construct a rectangle and construct new rectangles with the same diagonal. How are they?

Exercise 13. Use a PC spreadsheet to set up the different types of triangular calculation.

GREEK DEFINITIONS: In a right triangle, the two short sides are called catheter, and the long hypotenuse.

ARABIC DEFINITIONS: Triangle ABC is right with C as the right angle. b is horizontal and a vertical. The floor side b is called the cosine side as seen from A. The wall side a is called the sine side or the tangent side as seen from the A. Sine and cosine sides are indicated as a percentage of the sloping wall c, or with c as a unit of measurement ("recounting" in c's: a = a/c*c = sinA*c). The tangent side is indicated as a percentage of the floor b.



sinA = a/c

cosA = b/c

tanA = a/b

The calculation problem

In a right triangle there are three unknowns. Calculation therefore requires three equations.

The Greeks knew only two, one angle- and one area equation (Pythagoras):

A+B = 90 a^2 + b^2 = c^2

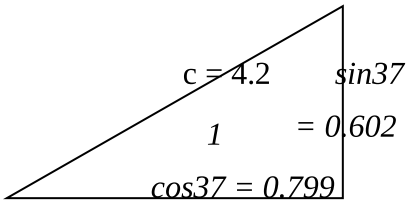
The Arabs knew three, one angle and two side equations:

A+B = 90 a = c*sinA b = c*cosA

The following calculations use the two sides of the triangle: the outer side with the actual numbers and the inside with the percentages. Alternatively, we could use equation schemes, see the appendix.

Example 1. *Given SAA, Find SSA.*

In triangle ABC is C = 90°, A = 37° and c = 4.2. Find a = ?, b = ? and B = ?



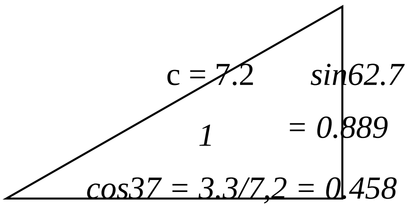
B = ? = *90-37 = 53*

a = ? = *4.2*sin37 = 2.5*

37 = A b = ? = *4.2*cos37 = 3.4* C = 90

Example 2. *Given SSA, Find SAA.*

In triangle ABC, C = 90°, b = 3.3, and c = 7.2. Find a = ?, A = ? and B = ?



B = ? = *90-62.7 = 27.3*

a = ? = *7.2*sin62.7 = 6.4*

INVcos 0.458 = 62.7° = A = ? b = 3.3 C = 90

GE04 DIFFERENT APPLICATIONS

Problem 1: How is the height of something high, for example, a flagpole?

Problem 2: How to move a corner thing?

Problem 3: How are shortcuts calculated?

Problem 4: How does one thing move fastest from a point in one area to a point in another area when we move at different speeds in the two areas?

Problem 5: How are the different openings calculated at a door that is ajar?

Problem 6: How to build a path up a steep mountainside?

Problem 7: How steep can a ladder be set so as not to break glass or ice?

Problem 8: How are astronomical distances calculated?

Problem 9: How should a triangular bridge over a river be sized?

Problem 10: During a swing, when do we have to jump off to get the furthest away?

Exercise 1. Determine the height of a tall thing (a flagpole) in two different ways: the lightweight, where we can get all the way to the thing, and the hard one where we can't. Determine in the same way the width of a wide thing (a house wall or a "river"). If possible, check the calculations by measurement.

Exercise 2. Set off once with the width 2 meters, which forms a crack of 90 degrees. A stick should be turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 1 meter must be turned around the corner without lifting. What is the maximum length of the box? If necessary, use the other objectives.

Exercise 3. Once with the width 3 meters forms a crack of 90 degrees and continues with a width of 4 meters. A stick should be turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 2 meters must be turned around the corner without lifting. What is the maximum length of the box?

Exercise 4. A treasure can be found by first traveling 4 meters to the east, then 2. 5 meters to the south, and finally 3. 1 meter to the west. Find a shortcut to the treasure.

Exercise 5. A treasure can be found by first traveling 4 meters to the east in 32 degrees northerly direction, then 2. 5 meters to the south in 68 degrees westerly direction, and finally 3. 1 meter to the west in a 14-degree northerly direction. Find a shortcut to the treasure.

Exercise 6. A boat rows over a 50-meter-wide river at a speed of 20 meters/minute. The current in the river is 10 meters/minute. What angle should you praise for landing right across the face on the opposite side? How long does the trip take?

Exercise 7. Construct a straight line and two points on either side of the line. Go from one point to the other with steps equal to 1 shoe length. 1 second corresponds to 1 shoe length in one area and 2 shoe lengths in the other. Find the fastest route. Could the fastest route be figured out? If necessary, use a PC spreadsheet to calculate different routes. A rule applies to the approach and refraction angle at the point on the boundary line which the fastest route passes. Which?

Exercise 8. Open a door ajar. There are now three openings, one perpendicular to the wall, one parallel to the wall and one parallel to the door. How big are these openings? How much will 10 degrees extra increase these openings by?

Exercise 9. Tip a plate 30 degrees and sign up a way up that can rise no more than 20 degrees (a hairpin bend). Repeat the exercise with other degree numbers. How much does the gravitational pull of a car increase when the rise of the road increases 10 degrees?

Exercise 10. Place a heavy book on a scale. Tilt it to different positions. What happens to the weight? Could this result be calculated? How much is the pressure against a vertical hand supporting it?

Exercise 11. Construct and load a triangular bridge and control the pressure against the surface by placing the bridge on two scales. Can these numbers be calculated?

Exercise 12. Distances in space cannot be measured, but must be calculated. How is the radius of the earth calculated? How far is it to the moon? What is the radius of the moon? How far is the sun? What is the radius of the sun?

Exercise 13 (difficult). A person sits in a swing that is suspended in ropes that are 3 meters long. Pull out the swing so that it is in height 1 meter above the bottom point. What angle does this correspond to?

If the swing is released, the speed v can be calculated from the formula: v^2 = 19.6*h, where h is the distance from the maximum height in the outer position. The speed consists of a horizontal and a vertical part.

Where do we have to jump off to get the furthest away?

(After the jump, the horizontal part of the speed will be unchanged, while the vertical part will grow downwards by 9.8 m/s every second.)

DEFINITION 1. One incident angle is

DEFINITION 2. A reflective angle is

Rule 1. The fastest route between two points in two different areas with speed v1 and v2 will meet the wrestling law at the crack point between the two areas:

sin (incident angle)/sin (reflective angle) = v1/v2.

GE05 AREA, SURFACE AREA, COVERAGE

Basic problem: How is the size of an area designated and calculated?

Exercise 1. A piece of squared A4 paper covers a certain area. How can the extent, area, coverage, extent of this surface be measured?

Exercise 2. A piece of squared A4 paper is divided by a transverse diagonal. What is the area of a right triangle?

Exercise 3. A piece of squared A4 paper is divided by lines into three triangles. What is the area of an ordinary triangle? Does this rule also apply to blunt-angled triangles?

Exercise 4. Construct a triangle and determine its area.

Exercise 5. Construct a triangle and divide it into 2 parts with the same area. In 3 parts. In 4 parts.

Exercise 6. Same as 5, but now the divider must be a normal to the baseline.

Exercise 7. A height can be difficult to measure accurately. Can a triangle area be calculated solely from sides and angles?

Exercise 8. An angle can be difficult to measure accurately. Can the area of a triangle be calculated solely from sides?

Exercise 9. Find some other surface areas that are used elsewhere and at other times.

Exercise 10. Use a PC spreadsheet to set up the different types of area calculation.

DEFINITION 1. An area unit is a square (a tile) of 1x1. 1m x 1m = 1m^2

Rule 1. A rectangle with side lengths a and b has an area of A =

Rule 2. A right triangle with catheter a and b has an area of A =

Rule 3. A triangle with a height and a surface g has an area of A =

Rule 4. Triangle ABC area can be calculated using the sine formula: A = 1/2*a*b*sinC.

Rule 5. The area of Triangle ABC can be calculated using the Herons formula:

A^2 = s*(s–a)*(s–b)*(s–c) where s = 1/2*(a+b+c) = half the circumference.

GE06 SHAPE CHANGE 1: SCALING

Basic problem: How is magnification or shrinking of a triangle is termed and calculated?

Exercise 1. When we walk away from a triangle, it seems to resize, but not shape. Construct a triangle ABC (it may be right). Set triangle AB*C* where the side lengths are half the size. What's the angles? What applies to the sides? What is the area?

Exercise 2. Construct a triangle ABC and enlarge it by 20% to 120% (scaling percentage and factor) to triangle AB*C*. Draw through B and C lines parallel to b and c (parallel transversals). These lines cut a in B** and C**. How big are the triangles ABC, AB*C*, BB*B** and CC*C** relative to each other?

Exercise 3. Construct a triangle ABC. Reseal it, but now in the 1:2 ratio. What is the scaling percentage and factor?

Exercise 4. Construct a triangle on a floor. Draw the triangle of paper in the 1:10 ratio.

Exercise 5. Construct a triangle on paper. Draw the triangle of a floor in the 1:10 ratio.

Definition 1. In a triangle, all side lengths are made k times as large. The number k is then called scaling factor, and k-1 is called scaling percent.

Rule 1. Scaling a triangle preserves the size of the angles and the direction of the sides.

Rule 2. If two triangles are similar, one is a scaling of the other.

Rule 3. In the triangle ABC and AB*C* we know that AB* = k*AB (AB* "re-counted" in AB's: AB* = (AB*/AB)*AB = k*AB) and AC* = k*AC.

Triangle AB*C* will then be a triangle of ABC. BB* = (k-1)*AB (provided k>1).

GE07 CHANGE OF FORM 2: LAND CONSERVATION

Basic problem: How can a shape change shape without changing the area (property exchange)?

Exercise 1. Construct a triangle ABC. Construct triangles A*BC with the same area. How will the A* items lie?

Exercise 2. Construct a right triangle ABC. Set triangles A*B*C with the same area. How will points A* and B* lie?

Exercise 3. Set off a rectangle ABCD. Construct rectangles AB*C*D* with the same area. How will the b*, c* and d* points lie?

Exercise 4. Construct two rectangles with the same area, increasing the longest side by 30%.

Exercise 5. Construct two right triangles with the same area, increasing the height by 25%.

Exercise 6. Construct two triangles with the same area, reducing the height by 40%.

Exercise 7. Set off a rectangle ABCD. Construct a square AB*C*D* with the same area.

Exercise 8. Construct a right triangle ABC. Construct the area a^2, as well as a rectangle with the area a^2 if one side length is c. Set the area b^2, as well as a rectangle with the area b^2, if a side length is c. What is the total area of these two rectangles?

Definition 1. A rectangle with the sides b and c must be changed to a rectangle with the sides a and s to preserve the area: a*s = b*c or a/b = c/s.

Definition 2. A rectangle with the sides b and c must be changed to a square with the side s to preserve the area: s^2 = b*c or a/s = s/b. s is then called a medium proportional to a and b, or a geometric average of a and b.

Definition 3. In triangle ABC, the BC side is enlarged by the factor k to B*C. The line through B parallel to B*A cuts AC into A*. A*C is then called a k-incision for AC and BC.

Rule 1. A fourth proportional s to a, b and d is a d/a cut to a and b.

Rule 2. A triangle retains area by

Rule 3. A right triangle retains area at

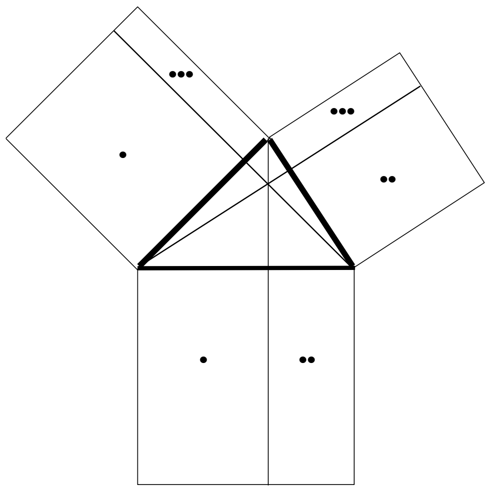
Rule 4. A rectangle retains area at

Rule 5. An mean proportional s to a and b can be constructed as the height of the semicircle arc above the a+b at the divider point.

GE08 OUTER SIDE SQUARES OF THE TRIANGLE, PYTHAGORAS

Basic problem: How can we make sure a triangle is right?

What applies to the outer side squares of a triangle?



Exercise 1. Construct a pointed triangle ABC with outer side squares. Set the three heights. The heights will divide the outer side squares into 3x2 parts. What is the case with these parts?

Exercise 2. Construct a right triangle ABC with outer side squares. Set the height. The height will divide one of the outer side squares into two parts. What is the case with these parts?

Exercise 3. Construct a blunt-angled triangle ABC with outer side squares. Set the three heights. The heights will divide the outer side squares into 3x2 parts. What is the case with these parts?

Exercise 4. Draw a triangle with heights and outer squares on cardboard. Cut out and road as a means of conviction.

Exercise 5. Use the land conservation technique as a means of conviction.

Exercise 6. Place a rectangle on a floor and use the diagonal to control the angles.

Exercise 7. Find some full numbers a, b, and c that meet the requirement a^2 + b^2 = c^2.

Exercise 8. Make of a string a closed loop with knots to check if an angle is right.

Exercise 9. Use a PC worksheet to set up diagonal calculation in a rectangle.

Definition 1. Triangle ABC's outer side squares are the squares a^2, b^2 and c^2.

Rule 1. In a triangle, the heights divide the outer squares of the triangle into one's "resettlement areas".

Rule 2. In triangle ABC is c^2 = a^2 + b^2 - 2*a*b*cosC

Rule 3. In the right triangle ABC (C right) is c^2 = a^2 + b^2 (Pythagoras' theorem)

Rule 4. In triangle ABC, C is right if c^2 = a^2 + b^2

GE09 Non-right triangle

Basic problem: How are the unknown parts calculated in a non-right triangle?

Exercise 1. Set off a known SSS triangle. Measure and calculate the three unknown angles.

Exercise 2. Set off a known SSA triangle. Measure and calculate the three unknown parts.

Exercise 3. Set off a known SAA triangle. Measure and calculate the three unknown parts.

Comment. An AAA triangle is only a AA triangle, as the last angle cannot be selected freely, but is determined by the other two. Thus, there is no information on a side, whereby the triangle becomes an SAA triangle.

Rule 1. In triangle ABC, the cosine relationships apply:

a^2 = b^2 + c^2 - 2*b*c*cos A

b^2 = a^2 + c^2 - 2*a*c*cos B

c^2 = a^2 + b^2 - 2*a*b*cos C,

Rule 2. In triangle ABC, the sinus relations apply:

a/sinA = b/sinB = c/sinC (watch out for the angles! why?)

GE10 Outer base lengths and base angles of the triangle

Basic problem: How can a triangle be measured from an outer base length? How can a triangle be re-established and calculated on the basis of external base dimensions?

Exercise 1. Construct a triangle ABC. Set off outside the triangle (possibly on the other side of the "river") an outer base line with known length KL. How can point A be determined from KL?

Exercise 2. Remove triangle ABC and restore it based on the outer base dimensions.

Exercise 3. How can triangle ABC's outer base dimensions (lengths or angles) form the basis for calculations of the triangle's parts, heights and area? Check by measurement.

Exercise 4. Use a PC spreadsheet to set up the calculation of the triangle's parts, heights, and area based on the outer base dimensions.

Definition 1. The base lengths of point A in relation to a base line KL are the lengths of the AK and AL line parts.

Definition 2. The base angles of point A in relation to a base line KL are the angles AKL and ALK.

Rule 1. The base angles can be calculated from the base lengths as follows:

AKL =

ALK =

Rule 2. The base lengths can be calculated from the base angles as follows:

h = height from K =

AK =

AL =

Rule 3. A right triangle ABC is considered from an outer baseline KL with length k.

Base lengths of the corners are referred to as:

KA = a1, KB = b1, KC = c1, LA = a2, LB = b2, LC = c2.

Base angles of the corners are referred to as:

AKL = A1, ALK = A2, BKL = B1, BLK = B2, CKL = C1, CLK = C2

The sides of the triangle can be calculated as follows:

a =

b =

c =

GE11 SQUARES

Basic problem: How are squares are designated and calculated?

Exercise 1. Construct different types of squares and name them.

Exercise 2. Find rules for a square's angles.

Exercise 3. Find rules for a square's diagonals.

Exercise 4. Find rules for a square area.

Exercise 5. Set a square and divide into two equal parts. Divide a square into three equal parts.

Exercise 6. Clip a square and find its center of gravity. Could we figure and make it to the center of gravity?

Definition 1. A trapeze is a square where a set of facing sides are parallel. A parallelogram is a square where both sets of facing sides are parallel. A rhombus is a parallelogram where all sides are equally long. A rectangle is a square where all angles are right. A square is a rectangle where all sides are equally long.

Rule 1. In a square, the angular sum

Rule 2. A trapeze with height h and parallel sides a and b has an area of A =

Rule 3. A parallelogram with height h and surface g has an area of A =

GE12 SPATIAL ANGULAR SHAPES, SURFACE AND VOLUME

Basic problem: How do describe spatial figures?

Exercise 1. Find different types of spatial shapes from everyday life and name them.

Exercise 2. Choose a box from everyday life. Measure the sides and diagonals. Calculate surface and volume, diagonal lengths and diagonal angles. If possible, check the volume by filling the box with water or sand, which can be poured into a cylinder glass that can measure volume.

Exercise 3. Construct a box according to given dimensions, partly of paper, partly of cardboard and partly on computer. If possible, check by calculating and measuring diagonal lengths.

Exercise 4. Choose an slanted cut-off box from everyday life, for example, a box that is cut off. Measure sides and angles. Calculate surface and volume. Measure and calculate diagonal lengths and diagonal angles.

Exercise 5. Set up a paper and floor tent. Measure or calculate side and diagonal lengths, angles, surface and volume.

Exercise 6. Construct a pyramid of paper and on the floor. Measure or calculate side and diagonal lengths, angles, surface and volume.

Exercise 7. A closed box with square bottoms must hold 1 liter. Describe different options in table or formula. Find the corresponding surface. Which box has the least surface. If necessary, use a PC spreadsheet.

Exercise 8. Same as exercise 7, but now the box is open at one end.

Exercise 9. Same as exercise 7, but now the box is open at both ends.

Exercise 10. A closed box with a square bottom shall have an outer surface of 1 m^2. Describe different options in table or formula. Find the corresponding volume. Which box has the largest volume. If necessary, use a PC spreadsheet.

Exercise 11. Same as exercise 10, but now the box is open at one end.

Exercise 12. Same as exercise 10, but now the box is open at both ends.

Exercise 13. Same as exercise 7-9, but now with a prism with an equilateral triangle as the base surface.

Exercise 14. Same as exercise 10-12, but now the prism has an equilateral triangle as the base surface.

Exercise 15. Same as exercise 7-9, but now with a pyramid with square surface instead of a box.

Exercise 16. Same as exercise 10-12, but now with a pyramid with square surface instead of a box.

Exercise 17. Same as exercise 7-9, but now with a pyramid with an equilateral triangular surface instead of a box.

Exercise 18. Same as exercise 10-12, but now with a pyramid with equilateral triangular surface instead of a box.

Exercise 19. Same as exercise 7-9, 13, 15 and 17, but now the material for the end surface is twice as expensive as aside. Minimum cost is requested.

Exercise 20. Where is the center of gravity of a box? How many degrees can the box be tilted before it topples over? Try and rain out.

Exercise 21. What is a polyhedron and what is a regular polyhedron?

Definition 1. A plane is

Definition 2. The angle between two planes is the angle between the normal of the plans.

Definition 3. A polyhedron or a multi-face is

Definition 4. A regular polyhedron is

Definition 5. A prism is

Definition 6. A fair prism is

Definition 7. A box is

Definition 8. A cube or cube is

Definition 9. A pyramid is

Definition 10. The volume or volume unit is a cube of 1x1x1. 1 liter = 1dm^3.

Rule 1. A box with side lengths a, b and c has the volume V =

Rule 2. A box of surface G and height h has the volume V =

Rule 3. A pyramid with a base G and height h has the volume V =

Rule 4. A box with side lengths a, b and c has the surface O =

Rule 5. A box with side lengths a, b and c has diagonal length D =

Rule 6. A roll with a side length a has diagonal length D =

Rule 7. The number of regular polyhedrons is

Rule 8. There are many rules for regular polyhedrons:

GE13 ROUND SHAPES: CIRCLES, CYLINDERS, SPHERES, ETC.

Basic problem: How are round plane and spatial shapes calculated and calculated?

Exercise 1. Find a series of circular things from everyday life. How can we find the center? Measure the circumference and diameter with a sewing thread. How many times can the diameter be around the circle?

Exercise 2. Draw a circle on paper. Cut it into narrow "Pizza pieces." If these are placed alternately in contrast, a rectangle-like figure will appear. What is the relationship between the two sides? What can the rectangle tell us about a circle's circumference and area.

Exercise 3. What is the circumference and area of a pie section and a circle section?

Exercise 4. What is the surface and volume of a sphere? Road the garbage of a mandarin and way then 1 cm^2 of the garbage. Way an apple and way behind a cube of 1 cm^3 carved from the apple.

Exercise 5. What is the surface and volume of a sphere? Way a plastic sphere. Cut a piece of 1 cm^2 and way it.

Exercise 6. What is the surface of a bullet? Take a mandarin or an orange. Make an equator cut and two on each other perpendicular polar cuts. This divides the surface into eight equal round triangles, each containing an equilateral triangle. Show that the approximate area of these triangles is A = 8*1.41*(1.5*r)^2/2 = 4* 3.17*r^2.

Exercise 7. What is the surface and volume of a sphere? Take a mandarin or an orange. Place three circular cuts: at the equator, at 30° and at 60° north latitude. The parts can be flattened like a circle and the chops can be flattened like circular rings. What is the total area of the two circular rings and the top? Arrow the mandarin in boats and place them alternately opposite. This produces something resembling part of a box. Assess the total volume of the boats.

Exercise 8. What is the surface of a cylinder? Make a cylinder out of an A4 paper. What is radius and height?

Exercise 9. What is the volume of a cylinder? Make a small cylinder of paper. What is radius and height? Fill the cylinder with sand and then pour the sand into a measuring glass.

Exercise 10. What is the surface and volume of a leg, a head, one, human body?

Exercise 11. Cut a slice out of a circle and fold it to form a fussy house (a cone). What is the surface and volume of a cone? Fill the cone with sand and then pour the sand into a measuring glass.

Exercise 12. Compare the volume between a cylinder, a hemisphere and a cone of the same height.

Exercise 13. Cut a triangular and square pyramid out of a piece of paper. What is the surface and volume of a pyramid? Fill the pyramid with sand and then pour the sand into a measuring glass.

Exercise 14. A closed cylinder must hold 1 liter. Describe different options in table or formula. Find the corresponding surface. Which cylinder has the least surface. If necessary, use a PC spreadsheet.

Exercise 15. Same as exercise 14, but now the cylinder is open at one end.

Exercise 16. Same as exercise 14, but now the cylinder is open at both ends.

Exercise 17. A closed cylinder shall have an outer surface of 1 m^2. Describe different options in table or formula. Find the corresponding volume. Which cylinder has the most volume. If necessary, use a PC spreadsheet.

Exercise 18. Same as exercise 17, but now the cylinder is open at one end.

Exercise 19. Same as exercise 17, but now the cylinder is open at both ends.

Exercise 20. Same as exercise 14-15, but now with a cone instead of a cylinder.

Exercise 21. Same as exercise 17-18, but now with a cone instead of a cylinder.

Exercise 22. Same as exercise 14-16 and 20, but now the material for the end surface is twice as expensive as aside. Minimum cost is requested.

Exercise 23 (difficult). A tube with diameter 1 m is filled by three equal tubes with which diameter?

Exercise 24. Place a bicycle lamp on the rim of a bicycle wheel. What is called the curve that the lamp will form if the bike ride is seen from the side (best seen in the dark)? This curve can also be produced on paper by letting a circle move along the edge of the paper. From the front, a circular motion will be a swing up and down. Is it reasonable for physics to describe the fluctuation u by a harmonic oscillation as u = R*sin(wt)? What does it say? R, w and t too?

Exercise 25. How can we make gear for a bike? When is a bike easy to tread? When will a pedal turn move the bike the most?

Exercise 26. Construct a triangle and place it in a circle. How is the center of the circle? Can we calculate the radius in this circle (the circumscribed circle of the triangle)?

Exercise 27. Construct a triangle and place a circle in it. How is the center of the circle? Can we calculate the radius in this circle (the inscribed circle of the triangle)?

Exercise 28. Under what conditions can a square be wrapped in a circle that touches all four corners? How is the center of the circle? Can we calculate the radius in this circle (the square's rewritten circle)?

Exercise 29. Under what conditions can a square wrap a circle that touches all four sides? How is the center of the circle? Can we calculate the radius in this circle (the square's inscribed circle)?

Exercise 30. Find some space targets that are used in other locations and at other times.

Exercise 31 (difficult). What can be understood by the outer piping circles of a triangle and how are they constructed? What can be said about radius and center?

Definition 1. A circle with radius r and center C is

Definition 2. A pie slice is

Definition 3. A circle section is

Definition 4. A sphere with radius r and center C is

Definition 5. A cylinder is

Definition 6. A cone is

Rule 1. In a circle, the circumference-diameter ratio is always $\pi = 3.1416$

Rule 2. A circle with radius r has the circumference O =

Rule 3. A circle with radius r has the area A =

Rule 4. From the circular arc, the diameter will always be seen at a right angle.

Rule 5. A sphere with radius r has the surface O =

Rule 6. A sphere with radius r has the volume V =

Rule 7. A cylinder with a radius r and height h has the volume V =

Rule 8. A cone with the base surface radius r and height h has the volume V =

Rule 9. A cylinder with a radius r and height h has the diagonal D =

GE14 GEOMETRY ON A SPHERE SURFACE

Basic problem: How to designate and calculate points, distances, angles, triangles, areas, etc. on a sphere surface?

Exercise 1. Look at a globe. How are points on the surface described (London, Paris, Rome, etc.)?

Exercise 2. What is the distance between two points and how is it (London-New York, Paris-Tokyo, etc.)?

Exercise 3. When are three points on a straight line (Oslo, Prague, Cairo)?

Exercise 4. Madrid-Paris-Bangkok makes up a triangle. What are the angles? What is the area?

Exercise 5. Can we count on the answers to the questions in exercise 2-4?

Definition 1. A grand circle goes through two diametrically opposite points. A pole circle or meridian circle passes through the two poles. An equator circle stands perpendicular to a pole circle midway between the poles.

Definition 2. On a sphere surface with center C and poles N and S, the equator circle and a standard pole circle are loaded. The intersections of the circles are called A and A*. The coordinates of a point P are determined as follows: through the point, a pole circle is placed that cuts the equator circle into P*. The width of the point is angle P*CP with one of the indications north or south. The length of the point is angle P*CA with one of the indications west or east.

Definition 3. The distance between two points P and Q means the smallest of the arc lengths of the grand circle through P and Q.

Definition 4. Three points A, B and C determine three large circle arcs AB, AC and BC. If these large circle arches do not coincide, they will delineate a spherical triangle ABC as well as another 7 spherical triangles.

Definition 5. The spherical exces e for a spherical triangle ABC is degree surplus above 180 degrees: e = A+B+C-180.

Rule 1. For a right spherical triangle ABC applies:

sinA = sina/sinc,

cosA = tanb/tanc,

tanA = tana/sinb

Rule 2. For a crooked spherical triangle ABC applies:

sina/sinA = sinb/sinB = sinc/sinC,

cosa = cosb*cosc + sinb*sinc*cosA

Rule 3. On a sphere with radius r, a spherical triangle ABC has the area e*4πr^2/720

GE15 MOVING: TURNING, MIRRORING AND PARALLEL DISPLACEMENT

Basic problem: How to talk turning, mirroring and parallel displacement of a triangle?

Exercise 1. Set up a triangle. Make a copy of the triangle. With the copy, perform each of the operations turning, mirroring and parallel displacement. Then set up definitions for these operations.

Exercise 2. Set up a triangle. Make a copy of the triangle and place it randomly a distance from the original triangle. Describe how the two triangles can be brought to cover each other.

Exercise 3. Put a triangle on the floor. Shift the triangle 3 meters to the south and 2 meters to the north and turn it 40 degrees.

Definition 1. If triangle ABC is turned to triangle AB*C* the angle of rotation will be angle BAB*.

Definition 2. If triangle ABC is turned about AB to triangle ABC*, ABC* is said to be a reflection of ABC on the mirror axis AB.

Definition 3. A triangle ABC is moved to triangle A*B*C*. If the sides do not change direction, a parallel displacement is indicated, the size of which is indicated by the length AA* and if the direction is specified, for example, if the direction is indicated. at angle A*AC.

Rule 1. Any movement can be performed on one or more of the operations of turning, mirroring and parallel displacement.

GE16 BLUEPRINTS

Basic problem: How is a spatial shape transferred to a plane drawing that can be used to construct the shape?

Exercise 1. Build a character into LEGO bricks. Draw the shape as a blueprint with both front, side, top vision (FST drawing).

Exercise 2. Draw the shape from exercise 1 as a blueprint in skewed vision on ISO(metric) paper. From where should the shape be seen to be correctly enrolled?

Exercise 3. Build a shape out of bricks, some of which are round (hemispheres, cylinders, cones, etc.) and some triangular. Draw the shape as FST drawing. Also draw the figure on ISO paper.

Exercise 4. Make an FST drawing of a LEGO brick shape. Build the shape from the blueprint. Draw the shape on ISO paper.

Exercise 5. Make an FST drawing of a shape with round bricks. Build the shape from the blueprint. Draw the shape on ISO paper.

Exercise 6. Make a drawing on ISO paper of a LEGO brick shape. Build the shape from the drawing. Draw the shape as FST drawing.

Exercise 7. Make a drawing on ISO paper of a shape with round bricks. Build the shape from the drawing. Draw the shape as FST drawing.

Exercise 8. we can get work in a shipyard if we can read an ISO drawing of piping. Take a clip, copper wire or similar and bend it at 90-degree angles as a model of a piping. Sign on ISO paper and let others try to reconstruct the model based on the drawing.

Exercise 9. A kitchen is set on the wall of a room with cm-dimensions 250 x 250 x 250 (height x width x depth). At the bottom is a 25 x 250 x 50 socket. Upstairs on the left is a 150 x 50 x 50 cabinet, and on the right four 50 x 50 x 50 cabinets. Above these is with a clearance of 50 cm suspended four 50 x 50 x 25 wall cabinets. Make a blueprint of the kitchen. Make a drawing of the kitchen on ISO paper. (If necessary, use the more realistic numbers 30, 60, 180 and 300 cm.)

Exercise 10. Make an FST drawing of a house. Build the house in cardboard from the blueprint. Draw the house on ISO paper.

Exercise 11. As 10, but now the opposite.

DEFINITION 1. Place a cube so that it rests on the 4-side. The cube can be seen in three different ways: the 1-Side then indicates the front vision of the cube, the 2-side side vision and the 3-side top-vision of the cube.

DEFINITION 2. From an (isometric) skewed view of a spatial figure, all three visions are seen simultaneously. A shape is viewed from a point of view that is below the figure on the left. A skewed vision can be recorded on ISO(metric) paper, where one's distances are drawn the same. we are therefore disregarding the impact of perspective.

GE17 SHADOWS AND PROJECTION

Basic problem: How to calculate shadow shapes

Exercise 1. Put a book on a table under a lamp. Turn the book upwards in 10 degrees. Measure each time the angle of rotation and shadow length. Can the shadow length be calculated?

Exercise 2. Make a triangle of a thumb stick. Place a pocket lamp and triangle on a table so that the shade falls on a wall. Move the flashlight in a 20 cm spurt and measure each time the distance to the triangle and the parts of the shadow triangle. Can the shading triangle parts be calculated?

Exercise 3. Make a triangle of a thumb stick. Place a pocket lamp and triangle on a table so that the shade falls on a vertical plate at the end of the table. Turn the plate in a 10-degree spur. Measure each time the angle of rotation and the parts of the shadow triangle. Can the shading triangle parts be calculated?

GE18 PERSPECTIVE DRAWING

Basic problem: How does a spatial shape transfer to a plane drawing that shows what's being seen?

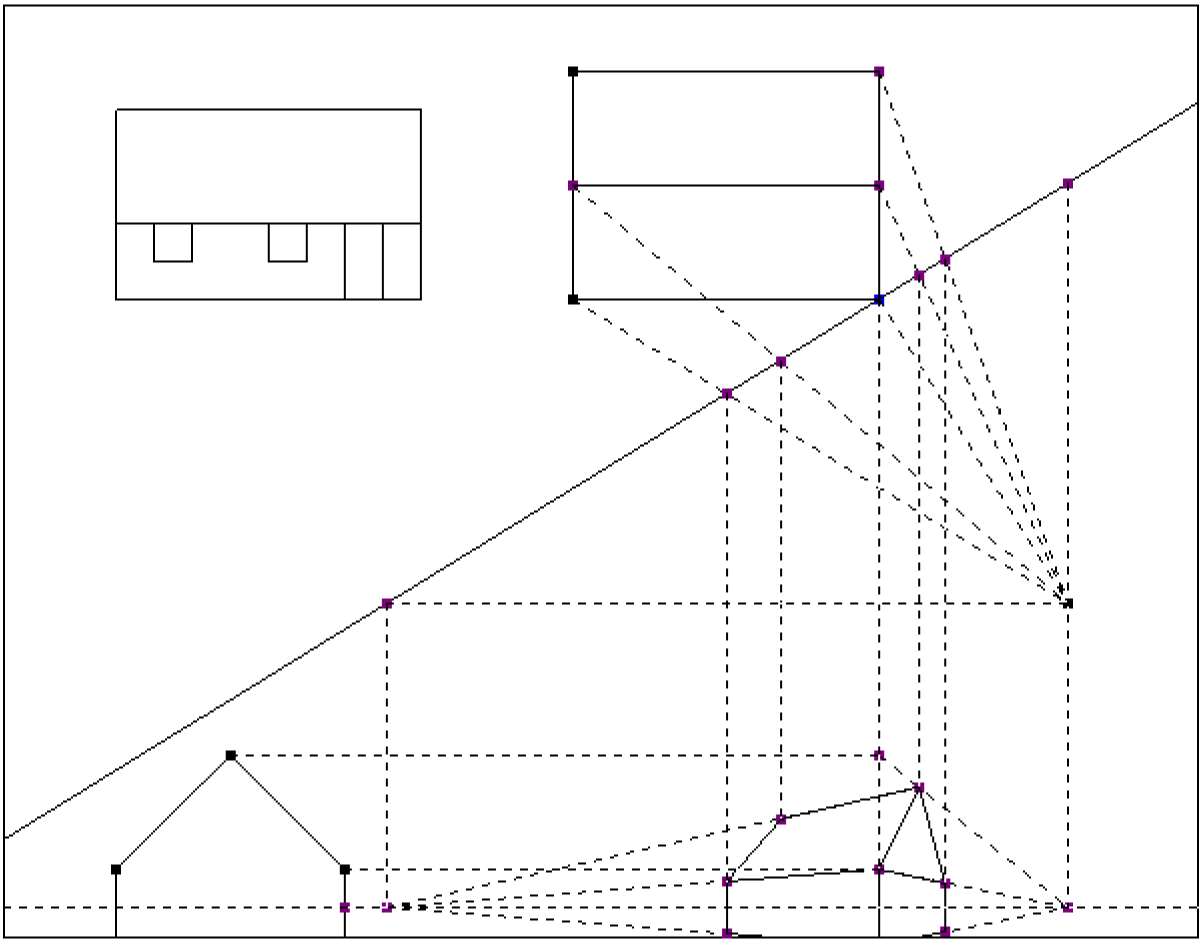
Exercise 1. Look along a surface on which there are parallel lines, such as lines on a brick wall. Move the eye in a vertical direction first up, then down. Which lines are horizontal and which fall and which increase? Move the eye in a horizontal direction first towards, then away from the surface. How does the steepness of the surface's lines change? What are the terms "horizontal line" and "disappearance point of parallel lines"?

Exercise 2. View a box-shaped thing through a window and copy the boxes' lines onto a piece of transparent plastic laid on the window. Mark the position of the eye on the plastic. Then copy the drawing onto a sheet of paper laid on top of the plastic. Sign horizontal line and vanishing points.

Exercise 3. The same exercise as exercise 2, this time, just used Albrecht Dürer's approach: Consider a box-shaped thing through a grid with horizontal and vertical lines, and copy the box lines onto a piece of A4 paper with the same number of horizontal and vertical lines as the grid. Mark the location of the eye. Sign horizontal line and vanishing points.

DEFINITION 1. Two points P and Q cast the shadows P' and Q' on a line l.

Front view Top view



Side view Perspective

Exercise 4. A house has meters 3 x 12 x 9 (height x width x depth). On top of the house there is a normal saddle roof that is 4 m high in the middle. Make an FTS drawing of the house. Use Top vision and side view to make a perspective drawing of the house seen in height 1 m from a position located 3 meters to the right and 6 meters in front of the house.

Exercise 5. Draw a tile floor seen from an eye level of 1.5 m when the horizontal distance to the tile floor is 1 m.

Exercise 6. Draw the shapes from the working drawing chapter as a perspective drawing.

Exercise 7. Draw the kitchen from the chapter on the drawing as a perspective drawing from a position that is 1 meter inside and 1 meter up in the kitchen and 1.25 meters from the wall of the closet.

Exercise 8. Draw a perspective drawing of a grid of 10 equally tall vertical bars with the same distance between them. Leave the first rod 10 cm high and let the horizontal line pass through the center of the rod. Set up two series of measurements from the drawing: one that measures the length of the rods and one that measures the distance from the vanishing point to the rods. What can be said about the two series of figures? Can the grid be perceived as a fractal?

Exercise 9. Draw a box from the supermarket in perspective (e.g. washing powder).

Exercise 10. Draw a house in perspective.

Exercise 11. Explore how we can draw in perspective on a PC.

NOTE. A perspective drawing takes into account the fact that a fixed length view decreases with increasing distance. A spatial figure is said to be drawn in perspective if all the points of the shapes are construct on a drawing plane perpendicular to the direction of vision using a point projection from the pupil of the eye (or a photographic film).

DEFINITION 1. A drawing plane is a plane perpendicular to the eye's direction of vision. A horizontal line at eye level is called the horizon line. The point on the horizon next to the eye is called the eye point. All points in the drawing plan are assumed to have the same distance from the eye.

DEFINITION 2. Parallel lines perpendicular to the direction of vision are drawn parallel. Otherwise, they are drawn so that they run together in the vanishing point of the lines on the horizon line.

COMMENT. Parallel planes: Subjects in the drawing plan are drawn from distance. Subjects in plans parallel to the drawing plan are drawn from distance, albeit on a different scale, determined by comparing top vision and side vision.

Rule 1. If a tile is drawn in perspective, its center point can be found as the intersection of the diagonals.

GE19 PARABOLAS AND PARABOLOIDS

Basic problem: Why are signals being transmitted in all directions being collected in the same direction?

Exercise 1. Set a line and a point outside the line. Select all points that are the same distance to the point and line. This shape is called a parable. The normal to the line through the point is called the axis of the parable.

Exercise 2. Construct a parable. Send a light ray (any laser light) in parallel to the axis. Examine the ray corridor after reflection on the side of the parable. Exercise can be performed by drawing or using light rays reflected in mirrors set up as keys in the reflection points.

Exercise 3. Try to prove that axis parallel rays pass the focal point of the parabola.

Exercise 4. Cut a parable in wood and use this to make a dish in clay or else. Place tin foil in the dish and try to use the dish as the sender and recipient of light.

Exercise 5. Borrow a car light from an auto-scrapper. Is it a dish?

Exercise 6. For a level clip in a cone, the following shapes can appear: Circle, ellipse, parable, and hyperbola (the so-called cone sections). How should the clips be laid in the different cases?

DEFINITION 1. A parable consists of all the points that have the same distance to a straight line (guideline) and a point (focal point). The normal to guide the guideline through the focal point is called the axis of the parable. A Paraboloid is produced by rotating a parable 360 degrees on its axis.

DEFINITION 2. An ellipsis consists of all the points whose distances to two given points (focal points) constitute a constant sum.

DEFINITION 3. A hyperbola consists of all the points whose distances to two given points (focal points) represent a constant differential.

Rule 1. At a parable, axis parallel rays will gather at the focal point, and rays emitted from the focal point will move axis parallels.

Rule 2. In the case of an ellipse, rays emitted from the focal point

Rule 1. At a hyperbola, rays emitted from the focal point

GE20 LIGHT BEND IN WATER AND IN GLASSES

Basic problem: How to calculate the bending of light rays when passing through water or glass?

Exercise 1. Fill a box-shaped glass with water. Allow a ray of light (any laser ray) to fall slanted onto the side of the glass. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 2. Let a light ray (possibly laser ray) fall slanted onto the side of a solid transparent glass box. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 3. Let a ray of light (possibly laser ray) fall slanted onto the side of a triangular transparent glass prism. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 4. Fill a cylinder-shaped glass with water. Allow a ray of light (any laser ray) to fall slanted onto the side of the glass. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the angle of a rainbow.

Exercise 5. Allow a ray of light (any laser ray) to fall perpendicular to a lens. Mark with pins the ray passage through the glass and out on the other side. Perform the test five times when the ray hits the lens in the middle, halfway and all the way out to both sides. Measure the different angles and compare with the law of wrestling from physics.

DEFINITION 1. A ray of light hits a wall. The angle between the normal and incoming ray of the wall is called the angle of approach. The angle between the normal and outgoing ray of the wall is called the outage angle. The angle between the normal of the wall and the broken ray is called the refraction angle.

Rule 1. On reflection, the incident ray angle = reflected ray angle.

Rule 2. In the case of refraction, sin(incident ray angle)/sin(reflected ray angle) = refractive index.

GE21 REPEATING FIGURES, FRACTALS

Basic problem: What aggregated figure comes from by figure repetition?

Exercise 1. Construct a square. Construct a new square whose side length is the diagonal of the previous square. Repeat this process many times.

Exercise 2. Construct a square. Construct a new square whose diagonal is the side length of the previous square. Repeat this process many times.

Exercise 3. Like 1 and 2, but now with a rectangle instead.

Exercise 4. Construct a right triangle ABC. Set a new right triangle one-angled with the first that has the short catheter as hypotenuse. Repeat this process.

Exercise 5. Construct a right triangle ABC. Set a new right triangle one-angled with the first that has the long catheter as hypotenuse. Repeat this process.

Exercise 6. Construct a right triangle ABC. Construct a new right triangle one-angled with the first that has the hypotenuse as the long catheter. Repeat this process.

Exercise 7. Construct a right triangle ABC. Construct a new right triangle one-angled with the first that has the hypotenuse as the short catheter. Repeat this process.

Exercise 8. Divide a right triangle using the height of the hypotenuse. What is the scaling factor? Repeat this process. Reverse the process.

Exercise 9. A horizontal line divides into two lines, each forming an angle of 30 degrees with the original line and half the length. Repeat this process many times.

Exercise 10. Make a number row starting with 0 and 1, with the next indent being the sum of the previous two. What are the properties of these Fibonacci numbers?

Exercise 11. Then divide a line AB with the point C/ AC = AC/BC (elevation, golden cut). What is the AC/CB scaling factor?

Exercise 12. Construct a square of the line AB. Let P be the golden cut in AB. Repeat the construction, but now from AP.

Exercise 13. Make a spiral with ray lengths $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, etc.

Exercise 14. Halves a square with side length 1 dm. Halves one part.

Exercise 15. Find some fractals yourself, e.g. in the program GeomeTricks.

Exercise 16. Find some fractals in nature.

DEFINITION 1. C is the golden cut in the line AB if AB/AC = AC/BC, i.e. if AC is mean proportional between AB and BC.

GE22 GEOMETRY OF AN OATMEAL BOX

Basic problem: Which geometry is hiding in an oatmeal box?

Suggestions for questions:

1. What is the surface and volume of an oatmeal box?

2. What is the smallest assault containing the given volume?

3. What is the largest volume that can be contained in the given surface?

4. What is the length and angles of the box's four diagonals?

5. Where is the center of gravity of the box? How much can the box be tilted without tipping over? (Full, half-filled)

6. Make a cylinder, a sphere, a cone, a pyramid, a tetrahedron with the same volume. What is the surface?

7. Make a cylinder, a sphere, a cone, a pyramid, a tetrahedron with the same surface. What's the volume?

8. Tilt the box so it looks like a steep mountain wall. Install a "hairpins" way up the box. What is the gravitational acceleration in this position?

9. Tilt the box to a specific position. How much is the load on the substrate reduced?

10. Draw the box, partly as blueprints, partly in perspective.

11. Draw the side of the box in scaled-down versions so that the wide side becomes diagonal on the wide side of the new box. What is the scaling factor? Repeat this process.

12. What is the smallest corner around the box?

13. If we can travel twice as fast on the wide side as on the narrow side, what is the fastest journey between two given points on the two sides?

14. Place 2-3 boxes on the dining table as islands in a large sea, and divide the sea between the islands.

GE23 NEWSPAPER GEOMETRY

Basic problem: Which geometry is hiding on a newspaper side?

Suggestions for questions:

1. What are the goals on a newspaper side? What are the relationships between these objectives? What is understood by Al, A2, A3, A4, A5, A6, etc. Paper?

2. Measure and calculate the diagonal (at least two ways) and its angles.

3. The diagonal divides the side into two triangles. Measure and calculate their height and elevation angles.

4. Calculate the area of the triangles in three different ways.

5. The height divides the triangle into two new triangles. How are these compared to the original? What are the scaling factors?

6. These new triangles can again be divided into new triangles. Repeat this split a few times and enjoy the resulting fractals.

7. As 2-6, but now with the newspaper folded once.

8. Calculate the right edge of the newspaper side from the lower left corner A when neither of the two center lines may be crossed.

9. A cracked corridor can be illustrated by three newspaper sides arranged appropriately in relation to each other. What is the length of the longest rod (bending ruler) that can travel around the corner? At what angle does the rod bump against both walls. Show that the angle can be calculated by the equation tanv^3 = a/b, where a and b are the two walking widths.

10. Draw the cheapest route between the two opposite corners when it is twice as expensive on one half. Show that sini/sinb = 1/2 where in and b is the angle of approach and refraction at the center line.

11. Fold a newspaper and open it 30 degrees. Sign a way up the newspaper that rises 20 degrees.

12. In two different ways, a newspaper side can be rolled around and form a cylinder. The surface is the same, is the volume also the same? What is the largest volume that can be formed by a newspaper side with the same area?

13. Fold the newspaper side and repeat the previous exercise. Will the volume be halved?

14. Fold the short side towards the long. Fold the remaining part to form a triangle. There will now be three triangles. What are their scaling factors?

GE24 SUPPLEMENTS: EQUATION SCHEMAS

For calculations based on formulas, it is possible to use an equation schema that says:

What to figure out	Which equation to use
Which numbers are used	How the equation is reshaped

Example. In triangle ABC is C = 90°, A = 37° and c = 4.2. Find a and b and B.

a = ?	sinA = a/c
A = 37	c*sinA = a
c = 4.2	4.2*sin37 = a
	2.53 = a

b = ?	cosA = b/c
A = 37	c*cosA = a
c = 4.2	4.2*cos37 = a
	3.35 = a

B = ?	A+B = 90
A = 37	B = 90-A
	B = 90-37
	B = 53

Routine tasks

Tasks **Answers**

	A	B	C	A	B	C		A	B	C	A	B	C
1			3.917	33.3		90		2.151	3.274			56.7	
2			6.519	42.4		90		4.396	4.814			47.6	
3	2.534			23.8		90			5.745	6.279		66.2	
4	3.772			21.4		90			9.625	10.338		68.6	
5	2.707		4.628			90			3.754		35.8	54.2	
6	3.883		6.265			90			4.917		38.3	51.7	
7	7.502	3.611				90				8.326	64.3	25.7	
8	8.588	8.093				90				11.801	46.7	43.3	

Non-right triangles **Answers**

	A	B	C	A	B	C		A	B	C	A	B	C
9	1.075			57.2		72.4			0.985	1.219		50.4	
10	2.212			42.5		59.6			3.201	2.824		77.9	
11		4.104		66.9		72.6		5.812		6.029		40.5	
12		4.915		21.5		86.4		1.893		5.155		72.1	
13			2.165	35.1		68.6		1.337	2.259			76.3	
14			3.041	38.4		76.2		1.945	2.847			65.4	
15	1.748	3.562				68.6				3.346	29.1	82.3	
16	1.433	4.346				88.4				4.538	18.4	73.2	
17		3.078	4.892	60.8				4.326				38.4	80.8
18		3.938	3.706	59.3				3.787				63.4	57.3
19	4.298		5.027		42.9				3.477		57.3		79.8
20	4.861		4.439		81.8				6.097		52.1		46.1