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Content

As operators, per-numbers are multiplied before adding as areas 1010

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Abstract

In our matter-with-forces-creating-motion nature quantities often occur with constant per-numbers as is the case in our human made economy. Education thus must ask how to teach per-numbers. So, after identifying examples of per-numbers, this paper reflects on their nature, and on how to teach them at different levels including the perspectives from the three grand theories. A recommendation follows a discussion between the math-before-many tradition and its reversed alternative.

Observing per-numbers

The gravity force first condenses stars as highly accelerated atoms where collisions between electrons and anti-electrons, 'small big bangs', produce light radiating energy, joule, into the space, thus pumping in joule per second per square meter into the third planet allowing water to exist in all three forms, solid and liquid and gas, with different density, kilogram per cubic meter, all depending on the local temperature. This allows gravity create cycles of water that after evaporation on low warm places is pushed up to condense again on high cool places, to be pulled down to cold rocks. Here freezing water increases its density and pulverizes some rocks into sand and minerals dissolved in water with a concentration, moles per liter. The sand follows the water down in rivers creating flatland between the rocks and the sea. Here greens cells use the light's joules per second to replace oxygen with water thus transforming carbon dioxide molecules into carbon hydrate molecules storing some joule per kg, or joule per mole. The green cells' batteries allow grey cells release the joule again by the opposite process. By removing oxygen from green cells decomposing under water, black cells transform carbon hydrate into oil or gas or coal with different joules per kg. This allows creating power plants were the pressure difference between a heated container transforming water into steam and a cooled container condensing steam to water again allows a steam-windmill to create an electrical current carrying joules to machines needing joules per second, delivered by enough electrons per second coming from a source supplying joules per electron.

The human condition thus is the fact that I need my daily bread, and that I need to produce it myself, directly or indirectly by exchanging products on a market. So, I need joules per day which I get by producing items per day for myself, or for exchanging on a market where different items the cost per item differs, and where the exchange may be directly or indirectly via dollars. Yearly, my incoming dollars are taxed with some dollars per 100 dollars to allow the state to transfer dollars to persons unable to work themselves.

In short, humans are part of three cycles: an ecological cycle with green cells, an economic cycle with other producers, and a communicative cycle with a word-language combining letters to words and qualitative sentences, and with a number-language combining digits to numbers that together with operations form predicting formulas for quantitative totals, which are united or split, and that may be constant or changing.

As to uniting totals, multiplication and addition unite constant and changing unit-numbers, and power and integration unite constant and changing per-numbers, that become fractions or percentages with like units, both being operators needing numbers to become numbers.

The four ways to unite totals, multiplication and addition and power and area-addition, are seen in the polynomial way to multi-count a given total as numbers of unbundled, of bundles, of bundle-bundles, etc., as T = 345 = 3 BBs and 4 Bs and 5 1s = 3*B^2 + 4*B + 5*1 where B stands for bundles, and BB stands for bundle-bundles. A bundle typically is ten, although the Romans bundled in 5s, and the Vikings bundled in scores or twenties, meaning two tens.

As to splitting totals, division and subtraction split into changing and constant unit numbers, and root or logarithm and differentiation split into constant and changing per-numbers.

Here we focus on integration and differentiation, together called calculus, created to unite, or to split into, changing per-numbers that may by piecewise constant or locally constant (continuous).

Reflecting on per-numbers

Per-numbers as 4$ per 5 kg, or 4$/5kg, or 4/5 $/kg, bridge $- and kg-numbers by recounting them in the per-number using the recount-formula T = (T/B)*B (Tarp, 2018):

T = 20$ = (20/4)*4$ = (20/4)*5kg = 25kg

Per-numbers also occur in mixture-problems as 2kg at 3$/kg + 4 kg at 4$/5kg = (2+4)kg at ? $/kg. The unit-numbers add directly. But as operators, the per-numbers are multiplied to become unit-numbers before adding as areas to give (2*3+4*5)$ per (2+4)kg. Here, the per-numbers are piecewise constant changing from 3 to 4 after 2 kg.

In the case of a falling object, instead the per-number is locally constant changing each moment. Here, areas under the per-number graph, A = □ p(x)*□x approximates better and better the smaller the moment is chosen, only giving more areas to add. However, since the multiplied per-number is a difference describing a change of the area, p(x)*dx = dA, addition makes all middle terms disappear leaving only one difference between the terminal and the initial number. This motivates supplementing integration with differentiation solving differential equations as dA = x^2 dx, finding a formula for the area under the per-number graph p(x) = x^2. Looking at the tiny shadows of a rectangular p*q book we find the never falsified formula, d(p*q) = dp*q + p*dq, so that d(x^2) = d(x*x) = 2*x*dx

Reflecting on how to teach calculus

Using their own bundle-numbers with units, children quickly master ‘primary school calculus’ where integration occur in questions as ‘2 3s + 4 5s = ? 8s’ where multiplication precedes addition. In in the reverse question ‘2 3s + ? 5s = 4 8s’, subtraction precedes division as in differentiation where their hands first pull away the initial stack, 2 3s, before counting the rest by pushing way 5s.

In middle school, per-numbers may be introduced physically as bridging plastic S- and C-letters given that 3 S-letters have the same value as 5 C-letters, which gives the per-number 3S/5C. Recounting in the per-number thus gives T = 12S = (12/3)*3S = (12/3)*5C = 20C. This leads on to traditional proportionality questions with per-numbers as 3$/5kg, and 3m/5sec, and 3£/5$. To be followed by

fractions introduced as per-numbers with like units, 3$/5$ is 3/5. Again we use recounting to see that 3/5 of 20$ means 3$/5$ of 20$, so 20$ = (20/5)*5$ gives (20/5)*3$ = 12 $.

Adding per-numbers occur in mixture-problems as '2kg at 3$/kg + 4 kg at 4$/5kg = 6kg at ? $/kg' and its reversed version. Likewise with adding fractions in problems as '2$, of which 3/4 + 6$, of which 4/5 = 8$, of which ?' and its reverse. So typically, fraction must not add without units.

Multiplying per-numbers and fractions leads to statistics and to Bayes formula: On my hand I have 1 right-finger and 4 left-fingers where the right and one left finger are bent. So, 100% of my right-fingers and 25% of my left-fingers are bent, but only 20% of the fingers are bent. Among the bent, 50% are left-fingers, but among the left-fingers only 25% are bent.

In high school, calculus focus on adding locally constant per-numbers.

Modelling with per-numbers

Seeing calculus as per-number addition allows it to be part of modelling already in middle school, since a CAS tool can show areas under per-number graphs as well as locate turning points on graphs. With integration as more frequent than differentiation in modeling it should come first.

Reflecting on limits

Where a simple interest is added to the initial loan, a compounded interest is also added to former interest amounts. So, a periodical rate, r, gives the total simple rate, R, where 1+R = (1+r) ^n.

If a yearly rate of 100% split in 12 parts and added 12 times we get R = (1+1/12)^12 – 1 = 1.613, showing 61.3% in additional rate. This however has a limit since (1+1/n)^n can come arbitrarily close to but not exceed 2.718 called e. A solver thus shows that (1+1/n)^n = 2.712 for n = 234.

Inside a circle with radius 1 there are many right triangles with the long side from the center to the circle. Splitting 180 degrees in n parts, the height of the triangle is sin(180/n), so the circumference of a half circle is close to n*sin(180/n) that is 3.1411 for n = 100. Again, we have a limit since n*sin(180/n) can come arbitrarily close to but not exceed 3.1416 called pi, □.

This gives a formal definition for constancy: y is constant c it the distance between the two is arbitrarily small, i.e., the distance is less than any positive number epsilon. And for piecewise constancy: y is piecewise constant c if an interval delta exists such that the distance between the two here is less than any positive number epsilon. Interchanging epsilon and delta then gives a formal definition for local constancy, or continuity: y is locally constant c if for any positive number epsilon an interval delta exists such that the distance between the two here is less than epsilon.

Including grand theory, philosophy and psychology and sociology

Whitin philosophy, existentialism and essentialism discuss if existence precedes essence or the opposite. Here, the traditional LDI limit-derivate-integral approach going from the abstract to the concrete is an essence-based calculus, where the opposite IDL per-number addition approach is an existence-based calculus abstracting its concepts from concrete examples.

Within psychology, Vygotskian social constructivism sides with essentialism while Piagetian radical constructivism sides with existentialism.

Within sociology, calculus is a socially constructed institution created as a means to reach the goal, addition of per-numbers. But with a 'goal displacement' (Bauman, 1990) where not reaching the goal makes the means the goal ensure a monopoly to mastering calculus as the only way to later master per-numbers, almost an example of 'the banality of evil' (Arendt, 1963). A sociological perspective thus allows discussing the difference between a good and an evil calculus.

Discussing with the tradition

The ICME-13 Topical Survey says on its first page that it has "a particular focus on established research topics associated to limit, derivative and integral". Grand theory sees this LDI as a Vygotskian essence-based approach with a goal displacement ensuring that the end goal, per-number addition, is reached by only few, while many give up or are prevented from learning it. The opposite IDL is seen as a Piagetian existence-based approach learning the goal before a means (Tarp, 2019). If tested and researched the two opposite approaches have opposite effect on the learners, it will be possible to talk about good and evil calculus.

Recommendation

With 'Math before Many', the tradition insists that mastering math is the only way to master Many, the end goal of math education. But an unnoticed direct way exists where children use their own bundle-numbers with units to unite or split into constant and changing unit- and per-numbers, which is the core of mastering Many. This new 'Many before Math' paradigm (Kuhn, 1962) needs to be studied to see if finally, we may have found a mathematics for all, and if it gives a more passable way to later master mathematics if needed. If successful, there is no need for different curricula to different students, only the numbers of theoretical footnotes will differ (Tarp, 2021). This will create the same communicative turn in number-language education as the word-language saw in the 1970s, where using the language became more important than knowing its grammar (Widdowson, 1978). A first step to a calculus deconstruction (Foucault, 1995) may be to treat fractions as per-numbers, and to reverse the order in calculus by rooting it in mixture problems in middle school.

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Developing the child's own mastery of Many may lead to a Math for All 0202

Author.

To sociological imagination, the continuing educational problems are caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught outside-inside, as communication about outside quantities. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life" (p. 3). Research thus still leaves many issues unsolved after half a century.

Mathematics education is based upon textbooks that again are based upon a curriculum for primary and lower secondary school supplemented with side-curricula for upper secondary school. But why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups of students, so why does the number-language?

Both languages have two levels, a language level describing the outside world, and a grammar level describing the inside language. In the word-language, the language level is for all students and includes examples of real-world descriptions, both fact and fiction, whereas grammar level details are reserved for special students. Could it be the same with the number-language, learned by all students through describing fact and fiction? And where grammar level details are reserved to special students?

So, pausing for a moment at the entry to the third decennium of the 21. century we may ask, what can and what cannot be different in mathematics education, leading on to the question: how will a civilized mathematics education curriculum for the rest of the century look like?

To use a suitable methodology, difference-research (Author, 2018) recommends looking for hidden differences that may make a difference by, e.g., including the viewpoint of the three grand theories, philosophy and sociology and psychology. Typically, they are all absent in research except for the latter.

We thus hope to find a new road to the end goal of mathematics education, mastery of Many. A road that could respect and develop further the quantitative competence and mastery of Many that children develop before coming to school. This alternative to the institutionalized tradition may fulfil the dream of a mathematics education for all students, once developed and studied as a new research paradigm (Kuhn, 1962).

Grand Theory looks at Mathematics Education.

Within philosophy, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning (1962): In a sentence, trust the subject, it exists; doubt the predicate, it is essence constructed from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber (1930) was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. The tradition holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many will be a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB + 4*B + 5*1, resonating with the bundle-numbers with units that children bring to school, e.g. T = 2 5s.

By insisting that 2+1 = 3 by nature in spite of the fact that 2 pairs + 1= 5, the tradition teaches 'mathematism' (Author, 2018) true inside but seldom outside classrooms where digits and fractions are not numbers but operators needing a number to become a number.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? The tradition holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian (1986) psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian (1969) psychology.

Sociologically, a Weberian viewpoint would ask if SET may be a totalitarian rationalization of Many gone too far thus leaving Many de-enchanted, and leaving the learners in an iron cage. A Baumanian (1990) viewpoint would suggest that, by monopolizing the road to mastery of Many, the tradition has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must be removed from a goal description. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a 'pris-pital', a Foucaultian (1995) viewpoint, would ask, first which educational structure to choose, a European line-organization forcing a return to the same 'cell' after each hour, day, and month for several years; or the North American block-organization changing 'cell' each hour, and changing the daily schedule twice a year? Next, as prisoners of a self-referring 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g., by examining and developing the existing mastery children bring to school?

The Tradition of Mathematics Education

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g., that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

Typically, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g., that (a + b) * (a – b) = a^2 – b^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g., that (f*g)'/(f*g) = f'/f + g'/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g., that ∫6*x^2dx = 2*x^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling that transform an outside problem into an inside problem that is transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative literature: fact, fiction, and fiddle (Author, 2001).

Consensus and Dissension on Early Childhood Numbers and Operations

There seems to be a consensus saying, "Of course numbers must be learned before being applied in numbering. And as one-dimensional, numbers are names for points along a number line obeying a place value principle when containing more digits". Thus, a dissension may ask: "From the age around four, children seem to distinguish between four ones and two twos thus developing double-numbers with units when adapting to outside quantity. So, why not develop the double-numbers with units that children bring to school?"

Furthermore, there seems to be a consensus saying," Of course addition must be learned before subtraction, multiplication and division since they are all defined from addition". Thus, a dissension may ask: "To count an outside total in bundles, we use a broom iconized as division to push away the bundles. Then we use a lift iconized as multiplication to stack the bundles. Then we use a rope iconized as subtraction to pull away the stack to find unbundled singles that placed on-top the stack as decimals, fractions, or negative numbers. Recounting 8 in 2s, 8 = (8/2)x2, thus creates a 'recount-formula' T = (T/B)xB saying "From T, T/B times, B is pushed away to show that T contains T/B Bs", that is present all over mathematics and science. Once counted, stacks may be added, but on-top

needing units to be changed by recounting, or next-to as areas as in integral calculus? This ambiguity leaves addition not that well defined. So, why not accept the opposite order of the operations as the natural?"

Finally, there seems to be a consensus saying, "Of course functions are postponed to secondary school since their algebra builds upon the algebra of letter expressions." Thus, a dissension may ask: "The word- and the number-language both offer an inside description of an outside object or action by using sentences with a subject, a verb, and a predicate, abbreviating "the total is 2 3s" to 'T = 2x3'. So, why not use functions as number-language sentences from the start?"

Also, in contrast to the many letters, words, and sentence rules in word-language, a pocket calculator shows that the number-language contains only ten digits, and a few operations. And where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. And so are the operations counting and recounting totals in bundle-numbers with units.

Could it be that the numbering competence children bring to school contain core mathematics as proportionality and calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact, and fiction? This would allow designing a curriculum for all students without splitting it up into tracks. And allow the word-language and the number-language to be taught and learned in the same way by describing outside things and actions with inside words and numbers coming from counting and adding. This would allow the number-language the same communicative turn as took place in the word-language in the 1970s (Halliday, 1973; Widdowson, 1978).

However, instead of teaching children how to number, the tradition teaches children about numbers, and about operations, both to be learned before being applied to the outside world (Bussi and Sun, 2018). Thus, where word-language is taught in the space between the inside language and the outside world, the number-language is taught in the inside space between the language and its meta-language or grammar, which may make the number-language more abstract and difficult to learn and to apply.

So maybe research should visit the mother Humboldt university in Berlin and reflect on the Karl Marx Feuerbach thesis 11 written on the staircase: "The philosophers have only interpreted the world, in various ways. The point, however, is to change it."

Meeting Many, Children Bundle to Count and Share

How to master Many can be observed in a power-free dialogue (Habermas, 1981) with preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly if held together 2 by 2: "That is not four, that is two twos", thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit.

Children also use bundle-numbers when talking about Lego blocks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3'; and when asked "How many 4s?" they may say '5 4s less 2'; and, integrating them next-to each other, they typically say '2 7s and 4'.

Children have fun ‘bundle-counting’ their fingers in 3s in various ways: as 1Bundle7 3s, ‘bundle-written’ as T = 1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Sharing 9 cakes, 4 children take one by turn, and they smile when seeing that ‘9/4’ predicts that they can take a cake twice, thus seeing division by 4 as taking away 4s.

Children thus master numbering and sharing before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the core mastery of Many that children bring to school?

Is mastery of Many the same as numeracy? Not exactly. The UK Oxford Dictionary defines numeracy as ‘a good basic knowledge of mathematics; the ability to understand and work with numbers’. In contrast, the US Merriam-webster Dictionary only includes the second part. The two different views illustrate the difference between an essence-based and an existence-based approach created by the difference between the UK structure-based and US actor-based sociology (Giddens, 1984). The former sees numeracy as applying institutionalized mathematics, and the second sees numeracy rooting mathematics.

Textbooks for a Question-guided Counting Curriculum

Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied, and divided before applied to the outside world. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children’s existing ability to master Many by numbering outside totals and blocks with inside two-dimensional bundle-numbers. This calls for different textbooks from grade 1 that don’t mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Author (2018) holding that only two competences are needed to master Many, counting, and adding. The corresponding pre-service and in-service teacher education may be found at the Author (2018).

Q01, icon-making: “The digit 5 seems to be an icon with five sticks. Does this apply to all digits?”

Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons; and by comparing with Roman numbers.

Q02, counting sequences: “How to count fingers?”

Here the learning opportunity is that five fingers can also be counted “01, 02, 03, 04, Hand” to include the bundle; and ten fingers as “01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H”.

Q03, icon-counting: “How to count fingers by bundling?”

Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate:

T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s

This is called an ‘inside bundle-number’ describing the ‘outside block-number’.

Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity:

T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: “How can a calculator predict a counting result?”

Here the learning opportunity is to see the division sign as an icon for a broom pushing away bundles: 7/2 means ‘from 7, push away bundles of 2s’. The calculator says ‘3.some’, thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes a rope pulling away the block to look for unbundled singles.

By showing ‘7-3*2 = 1’ the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional learning opportunity is to write 8 = (8/2)*2 as a ‘recount-formula’ T = (T/B)*B, saying “From T, T/B times B can be taken away”, to predict counting and recounting examples.

Q05, recounting in another unit: “How to change a unit?”

Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s.

Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away.

Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Counting 3 in 5s gives fractions: T = 3 = (3/5)*5.

Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number, asking, e.g., ‘6$ = ?kg’. To find the answer, we recount 6 in 2s:

T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Q06, unbundled as decimals, fractions or negative numbers: “Where to put the unbundled singles?”

Here the learning opportunity is to see that the unbundled occur in three ways: Next-to the block as a block of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle.

Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q07, prime or foldable units: “Which blocks can be folded?”

Here the learning opportunity is to examine the symmetry of a block. The block T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1).

Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd.

Q08, finding units: "What are possible units in T = 12?".

Here the learning opportunity is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

Q09, recounting from tens to icons: "How to change unit from tens to icons?"

Here the learning opportunity is that asking 'T = 2.4 tens = 24 = ? 8s' can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s:

T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

Q10, recounting from icons to tens: "How to change unit from icons to tens?"

Here the learning opportunity is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'.

However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point.

An additional learning opportunity uses 'less-numbers', geometrically on an abacus, or algebraically with brackets:

T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21.

Consequently, 'less less 1' means adding 1.

Q11, recounting block-sides. "How to recount sides in a block halved by its diagonal?"

Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers:

a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

Q12, navigating. "Avoid the rocks on a squared paper". Rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock?

Textbook for a Question-guided Adding Curriculum

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite.

The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

We might call this beautiful simplicity 'the algebra square' (Author, 2018).

Operations unite/

split Totals in Changing Constant

Unit-numbers

m, s, kg, $ T = a + n

T – n = a T = a*n

T/n = a

Per-numbers

m/s, $/kg, $/100$ = %T = □ f dx

dT/dx = f T = a^b

√(b&T)= a loga(T) = b

Figure 01. The 'algebra-square' shows the four ways to unite or split numbers.

Q13, next-to addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?"

Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

Q14, reversed next-to addition: "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?"

Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Q15, on-top addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?"

Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

Q16, reversed on-top addition: "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?"

Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

Q17, adding tens: “With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?”

Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Q18, subtracting tens: “If T1 = 23 and T2 add to T = 71, what is T2?”

Here, recounting removes an underload:

T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or

T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469.

Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Q19, multiplying tens: “What is 7 43s recounted in tens?”

Here the learning opportunity is that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or

27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161,

solved geometrically in a 2x2 block.

Q20, dividing tens: “What is 348 recounted in 6s?”

Here the learning opportunity is that recounting a total with overload often eases division:

T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and

T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Q21, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?”

Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant.

Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation transforming piecewise to local constancy by interchanging epsilon and delta.

Q22, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?”

Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas.

Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q23, finding common units: “Only add with like units, so how add T = 4ab^2 + 6abc?”.

Here units come from factorizing seeing that 2ab is the common unit:

T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) * 2ab.

Discussion and Future Research

So yes, a curriculum for all students is possible without splitting it up into tracks. For the mastery of Many that children bring to school contains core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb, and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many inspired by, but not repressed by, the presence of several historically versions of mathematics education.

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To master or not to master math before many, that is the question 1502

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It seems evident that the goal of mathematics education must be to learn mathematics, which once learned inside has many important applications outside by allowing humans to master Many. It is only a shame that mathematics is so hard to learn that it produces many 'slow learners'. However, grand theory asks the opposite question: with 'mastery of math' as an inside means to 'mastery of Many' outside, are there different ways to this end goal, that might even show a different way to later master math if wanted? To get an answer we observe that the mastery children develop when adapting to Many before school leads to 11 micro-curricula so completely different that they create a Kuhnian paradigm-shift within mathematics education as radical as the change from a flat to a round earth.

Keywords: Elementary school mathematics, mathematics curriculum, numeracy, division, calculus

Inside, children adapt smoothly to their outside world

It is glad to see how children vividly communicate about Many before school. And it is sad to see how they then stop doing so, and how more and more are excluded from traditional education and sent to special education. A day inside a classroom tells you why. The students no more communicate about Many, instead a textbook mediated by a teacher teaches them about what they need in order to communicate: multidigit numbers obeying a place value system, first to be added then subtracted with no respect to their units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers.

In a classes for 'slow learners', the same is taught but at a slower pace. Which makes you wonder: With their preschool foundation, can children learn number-language through communication as with the word-language (Widdowson, 1978)? And can 'mastery of Many' lead to 'mastery of math' later if needed? In which case, inside mastery of math will not be the only way to master Many outside.

Looking for differences, Difference Research (Tarp, 2018) searching for differences making a difference may inspire micro curricula to be tested using, e.g., Design Research (Bakker, 2018).

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that existence precedes essence so that in a sentence, the subject is more important than any chosen predicate (Marino, 2004). Many therefore should be seen ontologically, what it is in itself, instead of epistemologically, how some may perceive and verbalize it. The tradition sees Many as an example of cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is seen as a union of stacks coming from multi-counting singles, bundles, bundles of bundles, etc., e.g., $T = 345 = 3*BB + 4*B + 5*1$.

Within psychology, Piaget sees learning as adapting to outside existence, whereas a Vygotsky sees learning as adapting to socially constructed essence already institutionalized.

Within sociology, the structure-agent debate discusses if humans should obey institutionalised essence and structures or allow these to be constantly negotiated between interacting peers.

Here, a Weberian viewpoint (1930) asks if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. A Baumanian viewpoint (1990) suggests that, by colonizing (Habermas, 1981) the road to mastery of Many, traditional mathematics has created a 'goal displacement': Institutions are means, not goals. And as an institution, mathematics is an inside means to the outside goal, mastery of Many. So, the word 'mathematics' must leave goal descriptions to avoid a self-referring 'banality of evil' (Arendt, 1963). A Foucaultian viewpoint (1995) sees education as a 'pris-pital', i.e., a mixture of a prison that forces learners to return to the same room constantly over a period; and that serves a truth regime trying to cure the self-referring diagnosis 'uneducated' saying "as math uneducated, you need math education having as its goal to learn math'.

So, where the tradition sees mastery of mathematics as the goal of mathematics education, a difference will be to use the mastery of Many that children bring to school to deconstruct the tradition by de-modelling it (Tarp, 2020).

Meeting many, children count with bundles as units

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old typically will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, "That is not four, that is two twos", thus describing what exists in space and in time: bundles of 2s in space, and 2 of them when counted in time.

Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And they also use full sentences as in the word-language with a subject 'that', and a verb 'is', and a predicate '2 2s', which abbreviated shows a formula as a number-language sentence 'T = 2 2s'.

Children's own flexible bundle-number curriculum

Listening to 'slow learning' students helps understanding when and why they fall behind. Inspired by this we may design question guided micro curricula, MC, to further develop the number-language and mastery of Many children get before school, allowing them a comeback to traditional education.

MC01. Digits

The tradition presents both digits and letters as symbols. A difference is letting students experience themselves digits as not symbols, but icons with as many sticks or strokes as they represent if written less sloppy (Tarp, 2018). In this way students see that inside icons link directly to outside degrees of Many. And that ten has no icon since, as a bundle it becomes the unit, so that two-digit numbers really are two numberings of bundles and of unbundled singles.

A guiding question can be "There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?" Outside material could be sticks, a folding ruler, cars, dolls, spoons, etc. Discussing why numbers after ten has no icon leads on to bundle-counting.

MC02. Bundle-counting sequences

Using a place value system, the tradition counts without bundles. A difference is to practice bundle-counting in tens, fives, and threes. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names ‘eleven’ and ‘twelve’ are Viking names meaning ‘one left’ and ‘two left’, and that the name ‘twenty’ has stayed unchanged since the Vikings said ‘tvende ti’; and that English roughly is a mixture of Viking words labelling concrete things and actions, and French words labelling abstract ideas. The Viking tradition saying ‘three-and-twenty’ instead of ‘twenty-three’ was used in English for many years. Now it stops after 20. The Vikings also counted in scores: 80 = 4 scores, 90 = half-5 scores.

A guiding question can be “Let’s use the word bundle when bundle-counting in tens, in 5s and in 3s.”

Outside material can be fingers, sticks, snap cubes, and a ten-by-ten abacus. With fingers and arms we can count to twelve, called a dozen. Using cubes, the bundles are stacked on-top of each other.

First, we count in 5s (hands): 0*B*1, …, 0*B*4, 0*B*5 or 1*B*0, 1*B*1, …, 1*B*5 or 2*B*0, 2*B*1, 2*B*2.

Then we count in 3s (triplets): 0*B*1, …, 0*B*3 or 1*B*0, …, 1*B*3 or 2*B*0, …, 2*B*3 or 3*B*0, 3*B*1, ...

Counting cubes in 3s, 3 bundles is 1 quadratic bundle-of-bundles or 1*BB* in writing, so we repeat:

0*B*1, …, 2*B*3 or 3*B*0 or 1*BB*0*B*0, 1*BB*0*B*1, 1*BB*0*B*2, 1*BB*0*B*3 or 1*BB*1*B*0.

Then we count in tens, including the bundles: 0*B*1, …, 0*B*8, 0*B*9, 0*B*10 or 1*B*0, 1*B*1, 1*B*2. Finally, we bundle-count in tens from 0 to 111.

MC03. Bundle-counting with underloads and overloads

Strictly following the place value system, the tradition silences the units when writing ‘two hundred and fifty-seven’ as plain 257. A difference may be inspired by the Romans using ‘underloads’ when writing four as “five less one”, IV; and by overloads when small children use ‘past-counting’: “twenty-nine, twenty-ten, twenty-eleven”.

A guiding question can be. “Let us count with underloads missing for the next bundle. And with overloads as children saying ‘twenty-eleven’.” Outside material could be sticks, cubes, and an abacus.

First, we notice that five fingers can be counted in pairs in three different ways.

T = 5 = I I I I I = II I I I = 1*B*3, overload

T = 5 = I I I I I = II II I = 2*B*1, normal

T = 5 = I I I I I = II II II = 3*B*-1, underload

Using fingers and arms, first we count using underloads: 0*B*1 or 1*B*-9, 0*B*2 or 1*B*-8, …, 0*B*9 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-9, 1*B*2 or 2*B*-8. The we count to twelve in 5s (hands): 0*B*1 or 1*B*-4, 0*B*2 or 1*B*-3, …, 0*B*4 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-4, …, 2*B*2 or 3*B*-3.

And in 3s (triplets): 0*B*1 or 1*B*-2, 0*B*2 or 1*B*-1, 0*B*3 or 1*B*0, 1*B*1 or 2*B*-2, …, 3*B*3 or 4*B*0 or 1*BB*1*B*0.

Cup-counting with a cup for bundles, and for bundles-of-bundles: *T* = 1]1]0 = 4]0 = 3]3 = 2]6 = 1]9.

Then we count in tens from 1 to 111, using ‘past-counting’:

… 1*B*9, 1*B*10, 1*B*11 or 2*B*1, 2*B*2, …, 2*B*11 or 3*B*1, …, 9*B*9, 9*B*10, 9*B*11 or 10*B*1 or 1*BB*0*B*1.

Then we rewrite totals as 'flexible bundle-numbers' with overloads and underloads:

$T = 38 = 3B8 = 2B18 = 1B28 = 4B\text{-}2 = 5B\text{-}12$

MC04. Doing math with flexible bundle-numbers with units

The tradition uses carrying when adding and multiplying and borrowing when subtracting and dividing. Here, a difference is to use flexible bundle-numbers instead.

A guiding question can be. "Let us do inside math with flexible bundle-numbers."

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 1: Doing math with flexible bundle-numbers with units

MC05. Talking math with formulas

In a number-language sentence as "The total is 3 4s", the tradition silences all but the calculation 3*4 recounted to 12. A difference is to use full sentences with an outside subject, a verb, and an inside predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence "$T = 5*6 = 30$" thus inside predicts that outside 5 6s can be re-counted as 3 tens.

A guiding question can be. "Let us talk math with full sentences about what we count and how."

We begin by counting in ones using a full sentence: "From the total we pull away one to get one."

We then write what was before and after, using a rope as an icon for pulling away: "$T = (T – 1) + 1$".

This number-language sentence formulates what we call a formula. It also applies if instead pulling away 2, "$T = (T – 2) + 2$", and if pulling away any unspecified bundle *B*, "$T = (T – B) + B$". We call this formula a 're-stack formula' since, with the total as a stack, we may pull away the bundle from the top and place it next-to as its own stack.

Outside asking "Adding 2 to what gives 5?", inside becomes "? + 2 = 5" in writing. Using the letter *u* for the unknown number, this becomes an 'equation' "$u + 2 = 5$", easily solved outside by pulling away the 2 that was added, described inside by restacking the 5: $u + 2 = 5 = (5 – 2) + 2$, so $u = 5 – 2$.

So inside, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the definition of the number '5-2': "5 minus 2 is the number, which with 2 added gives 5".

We count and re-count in bundles. Re-counting 8 1s in 2s, we use '/' to iconize a broom pushing away 2s. So '8/2 = 4' is an inside prediction for the outside action "From 8, we push away 2, 4 times."

Having been pushed away, the bundles of 2s are stacked. This is iconized by an ‘x’ for lifting the bundles, so ‘4x2 = 8’ is an inside prediction for the outside action “4 times stacking 2s gives 8 1s”.

Re-counting 8 in 2s gives a ‘recount formula’ 8 = (8/2) x 2, outside showing a stack with the height 8/2 and the width 2 on the vertical and horizontal side, and with the total 8 as the area. So, totals add by areas, called integral calculus. With unspecified numbers it says: *T* = (*T*/*B*) x *B*, or *T* = (*T*/*B*) * *B*, or “From a total *T*, *T*/*B* times, *B* can be pushed away”; or simply, “The total *T* contains *T*/*B* *B*s”.

Outside asking “How many 2s in 8”, inside is the equation “? * 2 = 8”, or “*u* * 2 = 8” easily solved outside by pushing away 2s, described inside by recounting 8 in 2s: *u* * 2 = 8 = (8/2) * 2, so *u* = 8/ 2.

Again, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the formal definition of ‘8/2’: “8 divided by 2 is the number of 2s in 8”.

MC06. Naming the unbundled singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question can be “How to see the unbundled singles?”. Outside materials can be cubes or an abacus.

Before outside recounting 9 in 2s, inside we let a calculator predict the result: Entering 9/2 gives ‘4.some’ predicting that “9 contains 4 2s and then some”. To find those, we outside pull away the 4-by-2 stack, and inside predict the result by entering ‘9 – 4 * 2’ giving 1. So, inside the calculator predicts that 9 recounts as 4*B*1 2s, which is also observed outside.

Recounting 9 cubes in 2s, the unbundled can outside be placed on-top of the stack. Inside it can be described by a decimal point separating the bundles from the unbundled: *T* = 4*B*1 2s = 4.1 2s. Likewise, when counting in tens: *T* = 4*B*2 tens = 4.2 tens = 4.2 * 10 = 42.

Seen Outside as a part of a bundle, inside we can count it in bundles as a ‘fraction’, 1 = (1/2) * 2 = 1/2 2s; or we can count what is missing in a full bundle, 1 = 1*B*-1 2s.

Again, we see the flexibility of bundle-numbers: *T* = 4*B*1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Likewise, when counting in tens: *T* = 4*B*2 tens = 4 2/10 tens = 4.2 tens = 5.-8 tens.

MC07. Changing number units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon-unit to another, from icons to tens, or from tens to icons, or into a square.

A guiding question can be “How to change number units?”. Outside materials can be an abacus.

Asking ‘3 4s = ? 5s’, we inside predict the result by entering on a calculator the 3 4s as 3*4, to be counted in 5s by dividing by 5. The answer ‘2.some’ predicts that 3 4s contains 2*B* some 5s. To find the unbundled singles, outside we pull away the 2 fives from the 3 4s; inside we predict this by entering ‘3*4 – 2*5’. The answer ‘2’ then predicts that 3 4s can be recounted as 2 5s & 2, or 2*B*2 5s.

Asking “40 = ? 5s”, we predict by solving the equation “*u* * 5 = 40” by recounting 40 in 5s: *u* * 5 = 40 = (40/5) * 5, so *u* = 40/5.Changing units also change the form of the height and width of the stack.

Asking “6 8s = ? tens”, or “6 * 8 = ?”, we inside predict the result by looking at a ten-by-ten square with 6 and 8 as *B*-4 and *B*-2 on the sides. We then see that the 6*8 stack is left when from the *B**B* stack we pull away a *B**2 and a 4**B* stack, and then add the 4*2 stack, which was pulled away twice.

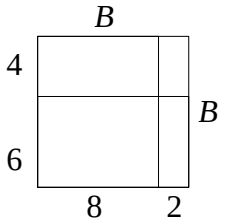	T = 6 * 8 = (*B*-4) * (*B*-2) = *BB* – 2*B* – 4*B* + 4*2 = 4*B*8 = 48	$T = \begin{pmatrix} 1B & -4 \\ 1B & -2 \end{pmatrix}$ = 1*BB* – 2*B* – 4*B* + 4*2 = 10*B* – 6*B* + 8 = 4*B*8 = 48	$T = \begin{pmatrix} 6B & +4 \\ 8B & +2 \end{pmatrix}$ = 48*BB*+12*B*+32*B*+8 = 48*BB*+44*B*+8 = 52*BB* 4*B* 8 = 5248

Figure 2: Multiplying 6*8 and 64*82 as binomials

Here, 6*8 exists as 6 8s outside. Inside recounting in tens shows that clearly minus times minus is plus. This view is an alternative to the multiplication controversy created by the ‘YouCubed’ website.

Inside, multiplying two ‘less-numbers’ horizontally creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles.

Wanting to square a 5-by-4 stack, its side is called √20, using lines to iconize the square. To find √20 we see that removing the 4-square leaves 20 – 4*4 = 4 shared by the two 4*t stacks in a 4+t square, giving t = 0.5. A little less since we neglect the t-square. Inside, a calculator predicts that √20 = 4.472.

MC08. Changing physical units with per-numbers

The tradition sees shifting physical units as an application of proportionality. Typically, finding the unit cost will answer questions as “with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?”

A difference is to use ‘per-numbers’ (Tarp, 2018) coming from double-counting the same quantity in the two units, e.g., *T* = 3$ = (3$/2kg) * 2kg = *p* * 2kg, with the per-number *p* = 3$/2kg, or 3/2 $/kg.

A guiding question can be “How to change physical units?”. Outside materials can be coloured cubes.

Recounting in the per-number allows shifting units: *T* = 6 kg = (6/2)*2 kg = (6/2)*3 $ = 9$; and *T* = 15$ = (15/3)*3$ = (15/3)*2kg = 10 kg.

Alternatively, we recount the units: $ = ($/kg)*kg = (3/2)*6 = 9; and kg = (kg/$)*$ = (2/3)*15 = 10.

With like units, per-numbers become fractions: 1$/4$ = ¼. The tradition teaches fractions as division: 1/4 of 12 = 12/4. A difference is to see a fraction as a part of a bundle counted in bundles, 1 = (1/4)*4, so 1/4 = 1 part per 4. Finding 3/4 of 12 thus means finding 3parts per 4 of 12 that recounts in 4s as: *T* = 12 = (12/4) * 4 = (12/4) * 3parts = 9parts, so 3 per 4 is the same as 9 per 12, or 3/4 = 9/12. Likewise, 3/4 of 100 means finding 3 parts per 4 of 100 = (100/4)*4, giving 75 parts per 100 or 75%.

MC09. Recounting the sides in a stack halved by its diagonal gives trigonometry

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, recounting the sides in a stack halved by its diagonal.

A guiding question can be “How to recount the sides in a stack halved by its diagonal?”. Outside materials can be tiles, cards, peg boards, and books.

Recounting the height in the base, height = (height/base) * base = tangent *A* * base, shortened to

h = (*h*/*b*) * *b* = tan *A* * *b* = tan *A b*s, thus giving the formula: tangent *A* = height / base, or tan *A* = *h*/*b*.

This gives a formula for the length of a unit circle containing many right triangles: A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, *h*, are almost identical. So, the length of a half circle is ½*C* = 100 * *h* = 100 * tan 1.8 = 100 * tan (180/100) = 3.1426 ≈ π, where the number π is tan (180/n)**n*, for *n* large enough.

MC10. Adding next-to and on-top

The tradition sees numbers as 1-dimensional line-numbers with addition defined as counting on. A difference is to accept children’s 2-dimensional bundle-numbers that add next-to and on-top. A guiding question can be “How to add 2 3s and 4 5s on-top and next-to?”. Material: cubes and abacus.

To add 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing or pulling, inside by recounting to change units. The recount formula predicts the result when entering (2*3+4*5)/*B*, where *B* can be 3 or 5 or 8. Added next-to by areas is called integral calculus.

Next-to addition is reversed when asking “How many 3s is added to 4 5s to get 5 8s?”. Outside, we pull way the 4 5s from the total *T* before recounting in 3s, which is predicted inside by asking the calculator: (5*8 – 4*5)/3, or Δ*T*/3. Using a difference to calculate the change in the total, Δ*T*, before using division to recount in 3s, this is called differential calculus.

MC11. Adding Per-Numbers and Fractions

Adding numbers without units may be called ‘mathematism’, true inside but seldom outside where, e.g., 2m + 3cm = 203cm. A difference respects that the recount-formula shows that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding.

A guiding question can be “What is 2kg at 3$/kg plus 4kg at 5$/kg?” Outside materials can be a peg board with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

Inside we see that unit-numbers add directly. Whereas per-numbers first must be multiplied to become unit-numbers. And since multiplication creates areas, per-numbers add by their areas, i.e., as the area under the per-number curve. And again, adding areas is called integral calculus.

Again, the opposite is called differential calculus asking “2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg?” The two connect by the fact that when adding differences, the middle terms disappear leaving only the difference between the end and initial numbers: (*b*-*a*)+(*c*-*b*)+(*d*-*c*) = *d*-*a*.

We now know 3 of 4 ways to unite parts into a total, and to split a total in parts, the ‘Algebra-square’ respecting that the Arabic meaning of the word algebra is to re-unite:

Operations unite/split Totals in	Changing	Constant
Unit-numbers	*T* = *a* + *n*	*T* = *a**n*

m, s, kg, $	$T - n = a$	$T / n = a$
Per-numbers m/s, $/kg, $/100$ = %	$T = \int a^* dn$ $dT/dn = a$	T = a^n $\sqrt[n]{T} = a$ $n = \log_a T$

Figure 3: The 4 ways to unite parts into a total, and the 5 ways to split a total into parts

Conclusion

In the mastery that children develop when adapting to Many before school, numbers and operations and functions apparently have different meanings. Numbers are 2D flexible bundle-numbers with units allowing both overloads and underloads. The operation order is the opposite. Division means counting that by pushing away bundles creates a recount-formula with a per-number used to change units, thus used all over STEM. And making functions occur as full number-language sentences from day one. And allowing calculators to predict results. Multiplication means stacking bundles to be pulled away by subtraction to find the unbundled that are presented by decimals, fractions, or negative numbers. So 6*8 is 6 8s, a stack that if recounted in tens changes both width and height outside and introduces early algebra when written with underloads: 6*8 = (B-4)*(B-2) = (10-4-2)*B + 4*2 = 4B8. When counted and recounted, totals may add on-top, or next-to by areas as in integral calculus.

These differences are so radical that they can only be tested with 'slow learners'. Children's own flexible bundle-number curriculum thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre- and in-service teacher education asking the subject instead of the instructor as exemplified on the MATHeCADEMY.net website.

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' We could ask if this is possible if an educational tradition rejects the child's own flexible bundle-numbers with units, and replaces them with inflexible line-numbers without units? So maybe the time has come where mathematics education should stop teaching 'mathematism' to children and instead begin learning from them how to master Many as a means to later master mathematics, if wanted?

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From evil to good calculus that adds locally constant per-numbers 2802

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Abstract

All can learn to add locally constant per-numbers to later appreciate epsilon-delta calculus, few can do the opposite. Why? Math says it cannot teach adding locally constant per-numbers since they don't exist: Math does not allow units, and local constancy defines piecewise constancy. Sociology and theology have different answers.

Case

A college wanted to lower the number of failing students at their first-year calculus course. So, a math lab was included where tutors could guide students individually or in groups. One tutor wanted to specialize in guiding struggling students to a better understanding of the epsilon-delta definition of continuity and differentiability. Having unsuccessfully tried this for a period, he asked me to design micro curricula to help him. Also, he wanted the curricula to be theoretically founded.

As a theoretical foundation I chose French poststructuralism rooted in Existentialism defined by Sartre as holding that 'existence precedes essence' (Marino, 2004). In Kierkegaard's version this means that what is created by nature should precede what is created by humans. And in Heidegger's version this means that in sentencing is-sentences, that subject should precede the predicate that will always be an individual choice or a social construction that could be different. Later in France, Derrida (1991) agrees by warning against predicates installing instead of enlightening what they describe. Such words should be deconstructed to better represent the existence they are supposed to reflect. Likewise, Foucault (1989, 1995) warns that scientific disciplines may instead exert power by disciplining not only themselves but also their subjects. As a counter measure, concept archaeology might find the original root and reason for concepts. So, micro curricula can be made by de-modeling (Tarp, 2020) traditional concepts (Bressoud et al., 2016) by returning to their original outside roots.

As to the goal of mathematics education a choice must be made. Does mastery of math precede mastery of Many, or the other way around? Existentialism points to Many-before-math, and Essentialism to math-before-Many.

Existentialism accepts that the Arabic word 'algebra' means to re-unite. So, to unite totals in the 'Algebra Square' (Tarp, 2018), multiplication and addition unite constant and changing unit-numbers, and power and integration unite constant and changing per-numbers, that with like units become fractions or percentages, all being operators needing numbers to become numbers. And to split totals, division and subtraction split into constant and changing unit numbers, and the factor-finding root (or the factor-counting logarithm) and differentiation split into constant and changing per-numbers. So, the outside root of calculus is adding or splitting into locally constant per-numbers.

Three micro curricula

The first micro curriculum introduces the difference between 1D line-numbers without, and 2D bundle-numbers with units. Here, line-number math roots 'mathematism' (Tarp, 2018) claiming that 2 + 1 = 3 always despite the fact that 2weeks + 1day = 15days. In contrast, bundle-numbers see 2x3 as an inside description of an outside stack of 2 3s, so that 2+3x4 gives 14 by necessity since it describes 2 1s + 3 4s. This distinction allows calculus to be rooted in next-to addition uniting areas when answering "2 3s + 4 5s = ? 8s", and its reverse.

The second micro curriculum introduces middle school calculus rooted in mixture problems asking "2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at ? $/kg", and its reverse. We observe that where the unit-numbers 2kg and 4kg add directly, the per-numbers 3$/kg and 5$/kg must multiply to unit-numbers to add, thus becoming areas. So per-numbers add by their areas under the per-number curve.

The third micro curriculum introduces formal definitions for the three kinds of constancy: global, piecewise, and local. In mixture problems, the per-number is piecewise constant. With falling objects, the per-number, meter/second, changes, but may still be called 'locally constant'.

● y is globally constant c it the distance between the two is less than any positive number epsilon.

● y is piecewise constant c if an interval delta exists such that here the distance between the two is less than any positive number epsilon.

Interchanging epsilon and delta then gives a formal definition for local constancy, or continuity. Which differs from the official definition of local constancy, defining piecewise constancy instead.

● y = f(x) is locally constant if an interval delta exists such that here f(u) = f(v) for all u and v.

This allows three formal definitions for linearity, saying that y = f(x) is globally, piecewise, locally linear if the change per-number □y/□x is globally, piecewise, or locally constant.

Locally constant per-numbers p thus add as many small area-strips pdx that if written as a difference dA make all middle numbers disappear leaving only one difference between the end and beginning number. This motivates developing a calculus to solve the differential equation dA = pdx.

Involving the concept of a limit, this may be de-modeled by looking at the length of a circle, and at a 100% rate added continuously: □ = n x tan(180/n), and e^t = (1+t/n)^n, for n large.

Testing the micro curricula

The three micro curricula were very successful in the math lab with many students being surprised that the epsilon-delta calculus is so easy once you master adding locally constant per-numbers. So, the tutor suggested using the many-before-math approach in the lectures, but the math department refused arguing that they could not teach adding locally constant per-numbers since they don't exist: Math does not allow units, and local constancy defines piecewise constancy instead.

My sociology friend laughed when hearing about it: it is a clear example of a goal replacement in an institution, making itself the goal in order to secure monopoly and employment. (Bauman, 1990). To this, my theology friend replied that, besides a good calculus there also seems to be an evil calculus in need for redemption. Only, if practicing 'the banality of evil' (Arendt, 1963) by just following

orders, an individual can get redemption, but not an institution. And, it seems unethical that an unnecessary monopoly should deprive humans of the basic ability to add per-numbers.

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Woke-Math never forces fixed forms upon flexible totals 2203

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Woke-Math respects flexible bundle-numbers for a total instead of forcing a linear number upon it.

So, Woke-Math warns that imposing line-numbers without units as five upon a total of fingers will disrespect the fact that the actual total may exist in different forms, all with units: as 5 1s, as 1 5s, as 1Bundle3 2s, as 2Bundle1 2s, as 3Bundle-1 2s, etc.

Woke-Math builds on a basic observation asking a 3year old "How old next time?" The answer typically is four showing four fingers. Holding them together 2 by 2, that child objects "That is not 4, that is 2 2s." The child thus sees what exist in space and time, bundles of 2s in space, and 2 of them in time when counted. So, what exist are totals to be counted and added in time and space, as T = 2 2s.

Woke-Math thus builds on the philosophy called existentialism holding that existence precedes essence, i.e., that what exists outside precedes what we say about it inside.

By de-modeling mathematics instead of modeling reality, Woke-Math offers "Master Many to master Math" as an alternative to the traditional approach, "Master Math to master Many".

Woke-Math respects, that outside totals inside may be counted and recounted in various two-dimensional bundle-numbers with units; and rejects one-dimensional line-numbers without units since they lead to 'mathematism' true inside, but seldom outside, where 2+3 = 5 is falsified by 2weeks + 3days = 17 days.

Counting and recounting totals in time and space produce flexible bundle-numbers

Counting in time produces a sequence of names like the names for the weekdays or months. Instead of saying "9, 10, 11 and 12", we might want to say, "Bundle less 1, Bundle, one left, two left" to see the Viking origin of the names 'e-leven' and 'twe-lve', meaning precisely that.'

Later counting "10, 20, ..., 90, 100" may be paralleled be counting "1B, 2B, …, 9B, 10B or BB", thus fulfilling the Dienes dream of meeting power in grade one. That is also fulfilled when counting ten fingers in 3s as "0B1, 0B2, 0B3 or 1B0, …, 2B3 or 3B0 or 1BB0B0, 3B1 or 1BB0B1". With bundles and bundle-bundles, the place value system is not needed.

Counting in space, all totals have units. Some numbers are prime units, others are compound units hiding the prime units, 1 6s = 1x6 = 1x2x3 = 2 3s = 3 2s. Bundle-numbers become flexible by allowing overloads and underloads: T = 5 1s = 1B3 2s = 3B-1 2s.

Counting sticks in space we see that 4 1s may be rearranged as 1 4s, thus realizing that written less sloppy, digits are icons with as many sticks or strokes as they represent.

Counting 8 1s in 2s, pushing away 2s may be iconized by a broom called division allowing a calculator to predict the result: 8 = 8/2 2s. Stacking the 4 2s may be iconized by a lift called multiplication, 4x2.

So, we have a recount-formula, 8 = 8/2 2s = (8/2) x 2, or T = (T/B) x B with unspecified numbers, the core proportionality formula appearing all over math and science: □y = (□y/□x) x □x = slope x □x, height = (height/base) x base = tan Angle x base, meter = (meter/second) x second = speed x second, etc.

From the total, pulling away the stack to look for unbundled singles may be iconized as a rope called subtraction: 9 – 2x4 = 1. Placing the unbundled on-top of the stack introduces decimals, fractions, and negative numbers: 9 = 4.1 2s = 4 ½ 2s = 5.-1 2s.

Recounting a physical total in different units creates a per-number, as 4 $/5kg.

To change units, we just recount in the per-number: 20kg = (20/5) x 5kg = (20/5) x 4$= 16$

Recounting in the same unit creates fractions or percentages: 6$/20$ = 6/20 = 30/100 = 30%.

With prime-units, per-numbers may be simplified: 6/10 = (2x3) / (2x5) = (3 2s) / (5 2s) = 3/5

On a tile, recounting the sides in the diagonal creates trigonometry:

height = (height/base) x base = tan Angle x base, height = (height/diagonal) x diag. = sin Angle x diag.

This allows the number pi to be calculated as n x tan(180/n) for n large.

Adding totals on-top and next-to

Once counted and recounted, totals may add on-top after recounting makes the units like. Or next-to as areas called integration combining multiplication and addition, becoming differentiation combining subtraction and division when reversed, asking, e.g., 2 3s + ? 5s = 4 8s. Here, we remove the initial total to recount the rest in 5s, so, ? = (4 8s – 2 3s) / 5 5s = □y/□x.

Per-numbers add by areas since adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before adding, thus adding as areas.

Such mixture problems lead directly the 2x2 ‘algebra square’ re-uniting changing and constant unit-numbers and per-numbers, where algebra in Arabic means ‘to re-unite’:

Changing unit-numbers unite by addition, and constant unit-numbers unite by multiplication. Likewise, changing per-numbers unite by integration, and constant per-numbers unite by power (10% + 10% = 21% since 110%^2 = 121%).

Uniting reversed will split a total in changing or constant unit-numbers or per-numbers. Here subtraction and division split a total into changing and constant unit-numbers. Differentiation splits a total into changing per-numbers. And finally, a total is split into constant factors by the factor-finding root and the factor-counting logarithm.

Bundle-numbers ease operations

Addition: T = 54 + 38 = 5B4 + 3B8 = 8B12 = 9B2 = 92 = 9.2 tens

Subtraction: T = 54 – 38 = 5B4 + 3B8 = 2B-4 = 1B6 = 16 = 1.6 tens

Division: 378 / 7 = 37B8 / 7 = 35B28 / 7 = 5B4 = 54 = 5.4 tens.

Multiplication: 6 x 7 = 6 x (B-3) = 6B -18 = (6-2)B (20-18) = 4B2 = 42 = 4.2 tens.

Multiplication tables as "6 7s = ?tens" may also be placed on a BxB square wring 6 and 7 with underload as B-4 and B-3.

So, 6 7s = 6x7 = (B-4) x (B-3) = BB -4B -3B + 4 3s (removed twice) = 3B 12 = 4B2 = 42 = 4.2 tens.

7 26s = 7 x 26 = 7 x 2B6 = 14B42 = 18B2 = 182 = 18.2 tens.

37 26s = 37 x 26 = (3B7) x (2B6) = 6BB (3x6 + 7x2)B 42 = 6BB 32B 42 = 6BB 36B 2 = 9BB 6B 2 = 962, seen on a 3B7 x 2B6 tile.

To find the square root of 40 we squeeze a 40 x 1 or 5 x 8 tile into a square. Here 6x6 is the largest inside square leaving 40 - 6x6 = 4 outside. Extending the 6x6 square into a (6+t) x (6+t) square adds additional two 6 x t tiles plus a t x t square that we neglect. With 6xt = 4/2, t = 2/6 = 0.333. So, the 40x1 tile squeezes into a 6.333 x 6.333 square approximately, or a little less since we neglect the t x t square. A calculator gives the answer: the square root of 40 is 6.325. Extending a square also shows that d(x^2) = 2x dx, or (x^2)' = 2x.

To solve the quadratic equation x^2 + 6x +8 = 0, we look at a (x+3) x (x+3) square divided in four sections, x^2, and 3x twice, and 9 = 8+1. Since x^2 + 6x +8 = 0, (x+3) squared is 1 = 1 squared. Hence x+3 = +1 or -1, so there are two solutions, x = -3+1 = -2, and x = -3-1 = -4.

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Bring back brains with woke math 0304

Allan Tarp

MATHeCADEMY.net

Woke-math builds on existentialism holding that ‘existence precedes essence’. So in education, inside essence should always be ‘de-modeled’ as outside existence: 6x7 inside is 6 7s outside.

Asked “How old next time?”, a 3-year-old shows four fingers and says “four”. But objects when held as 2 by 2: “That is not 4, that is 2 2s.” The child thus sees what exist, bundles of 2s in space, and 2 of them in time when counted. And uses a full number-language sentence with a subject, a verb and a predicate, thus de-modeling functions.

Respecting the child’s own flexible 2D bundle-numbers with units, Woke-math rejects the essence of traditional 1D line-numbers without units, leading to ‘mathematism’ true inside but seldom outside by claiming that ‘always 2+1 is 3’ despite, e.g., ‘2 pairs + 1 is 5’.

Woke-math may bring back brains from special education when describing fingers and snap-cubes in time and space.

In space, five fingers existing outside in different forms, may inside be described with bundles as units: 5 1s, or 1 5s, or 1Bundle3 2s (overload), 2B1 2s (normal), or 3B-1 2s (underload), etc. In time, a counting sequence should include the bundle-unit (e.g., 3s): 0Bundle1, 0B2, 0B3 or1B0 or 2B-3, …, 3B0, 3B1, where 3 Bundles is renamed as 1BundleBundle, so ten = 1BB 0B 1 3s.

Re-counting in time and space, the basic operations occur in reverse order.

Recounting 8 1s in 2s, pushing away 2s may be iconized by a broom called division allowing a calculator to predict the result: 8 = 8/2 2s. Stacking the 4 2s may be iconized by a lift called multiplication, 4x2. This recount proportionality formula, T = (T/B) x B says that T contains T/B Bs. It serves to change units all over math and science.

From 9 pulling away 4 2s may be iconized by a rope called subtraction. Placed on-top of the stack, the unbundled may be described as a negative number, a decimal or a fraction when counted in the bundle also: 9 = 5B-1 = 4B1 = 4 1/2 B 2s.

Recounting between digit-units and tens leads to early algebra in BXB-tables, and to equations solved by recounting.

Recounting between physical units leads to per-numbers as 3$/5kg.

Once counted and recounted, total may add on-top after recounting has made the units like, or next-to as areas thus leading to integration, and to differentiation when reversed.

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Bringing back brains to precalculus 3004

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In a math-lab, dropout-students often ask: “Don’t repeat it, explain it so I understand it”.

Letting existence precede essence, Existentialism (Marino, 2004) offers a U-turn by seeing mastering Many before Math (Tarp, 2021) as an alternative to the traditional reverse view that “of course, mathematics must be learned before it can be applied to model the world”. So, instead ‘Many-math’ de-models (Tarp, 2020) inside essence as outside existence to root a concrete understanding. Fingers and CAS-calculators are used as Transformative Technology.

Digits are de-modeled as icons with as many sticks as they represent. Operations are de-modeled as icons for counting answering “How many” by a number-language sentence that, as in the word-language, also contains both subject, verb and predicate where “The total is 2 3s” is shortened to the formula “T = 2 3s”.

Division and multiplication iconize a broom and a lift to push-away the unit-bundles to be lifted as a stack. They combine in a ‘recount-formula’ (Tarp, 2018), T = (T/B)xB, predicting that T contains T/B Bs. Subtraction iconizes a rope to pull-away the stack to find unbundled that are placed on-top as decimals, fractions, or negatives. Finally, addition iconizes the two ways to unite stacks, on-top and next-to.

Recounting in two units creates per-numbers as 4$/5kg that become fractions with like units; and trigonometry when mutually recounting the sides in a rectangle split by its diagonal.

Recounting ten fingers in bundles of 5s, 4s, 3s and 2s gives the basic formulas:

T = 2*B. T = 2*B+2. T = 1*B^2 +1. T = 1*B^3+1*B.

They all exemplify the bundle-counting formula (polynomial), T = a + b*x + c*x^2 + d*x^3, that shows the Algebra-square’s four ways to unite numbers: like and unlike unit-numbers are united by multiplication and addition. And like and unlike per-numbers are united by power and next-to-addition, also called integration.

Furthermore, it shows the three basic formulas, T = b*x (proportional), T = d*x^n (power), and T = d*n^x (exponential). That leads on to the reverse formulas, equations, splitting a total, where division and subtraction split a total in like and unlike unit-numbers, and the factor-counting logarithm or the factor-finding root and differentiation split a total in like and unlike per-numbers.

As operators needing factors to become numbers, per-numbers and fractions add by areas (integration).

After de-modeling, we can model saving on our devise held in the left hand in algebra-mode when calculating, and in the right hand in geometry-mode when reading from graphs.

With a per-week deposit increasing with 4$/week, after x weeks we deposit T = 4*x, seen as a raising straight line. The total capital after x weeks, 2*x, is the area under the T-graph, found as a triangle or by integration on the CAS device.

CAS regression allows a degree 3 polynomial with 2 parabolas to model a 4-row per-number table. Again, integration finds the total unit-number as the area under the graph.

Grateful, most students returned to their classes, and about half also took a calculus class later.

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CTRAS 2023 proposals 0305

Online math opens for a communicative turn in number language education 0305

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"Of course, the goal of mathematics education is to master mathematics before it can be applied to later master Many." Seeing this as a 'goal displacement' making a means a goal, sociology (Bauman, 1990) and difference research (Tarp, 2018) suggests a communicative turn as in foreign language education in the 1970s (Widdowson, 1978): Maybe mastering Many is a more accessible way to later master Mathematics. Likewise, existentialism (Sartre, 2007) holds that existence should precede essence.

Seeing 4 fingers 2 by 2 as '2 2s' shows that preschool children master Many with 2D bundle-numbers with units. In this 'BundleNumber-math' adding is preceded by counting, which de-models (Tarp, 2020) division and multiplication as icons for a broom and a lift to push-away the unit-bundles to be lifted as a stack. They combine in a 'recount-formula' (Tarp, 2018), T = (T/B)xB, predicting that T contains T/B Bs. Subtraction iconizes a rope to pull-away the stack to find unbundled that are placed on-top as decimals, fractions, or negatives. Addition iconizes adding on-top and next-to.

This allows both school and education students to be guided by the concrete subject on their desktop instead of by an instructor on a screen, as exemplified by the MATHeCADEMY.net.

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Artificial Intelligence makes Difference Research more relevant

Allan Tarp, the MATHeCADEMY.net

Research typically is seen as an example of a top-down or bottom-up lab-lib cooperation where laboratory observations are deduced from or are inducing library concepts. In the top-down version, library theory generates a hypothesis that, validated or falsified in the laboratory, leads to a strengthened or adapted theory. In the bottom-up version, laboratory observations lead to categories, that additional observations may split into subcategories.

Artificial Intelligence has access to the library, but laboratory data will be input. Top-down research thus may be generated very quickly with a quality depending on the reliability of the input, which may be difficult to check.

In contrast, AI is of less help to bottom-up research typically generating new categories not yet present in the library.

Also Difference Research searching for differences making a difference (Tarp, 2018) may now be more relevant since although AI may locate existing differences, it cannot invent new differences. Nor can it examine the difference they make.

Examples of difference research are bundle-numbers with units, operations as icons for counting, re-counting to change unit, per-numbers coming from recounting in two units, integration as addition of locally constant per-numbers, trigonometry before geometry, and mathematism adding numbers without units.

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Bundle-Bundle BB squares, the third way in the multiplication war

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In online-education concrete materials matter, especially to struggling students. Multiplication tables are basic mathematics, but a war about rote learning has erupted in the US. The BB Bundle-Bundle square represents a middle way in this conflict.

Seeing 4 fingers together 2 by 2 as '2 2s' shows that preschool children master Many with 2D bundle-numbers with units. In this 'BundleNumber-math' adding is preceded by counting, which de-models (Tarp, 2020) division and multiplication as icons for a broom and a lift to push-away the unit-bundles to be lifted as a stack. Subtraction iconizes a rope to pull-away the stack to find unbundled that are placed on-top as decimals, fractions, or negatives as 'underloads' (Tarp, 2018).

A Bundle-Bundle BB square is a ten by ten sheet where rubber bands de-model the total 7x8 as 7 8s, which is the (B-3)x(B-2) rectangle left when removing from full BB square the rectangles 3B and 2B and adding the 3 2s removed twice: (B-3)*(B-2) = B*B-3*B-2*B+3*2 (Tarp, 2021). We notice that the total lost in height what it gained in width.

Likewise the BB square shows that 7x34 is shown as (B-3)*(4B-6) giving 4BB-12B-6B+18, and 57x34 is shown as (6B-3)*(4B-6) giving 24BB -12B-36B+18. Thus (a-b)x(c-d) = ac-bd-ac+bd.

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Woke-Math never forces fixed forms upon flexible totals 1205

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Woke-Math respects flexible bundle-numbers for a total instead of forcing a linear number upon it.

So, Woke-Math warns that imposing line-numbers without units as five upon a total of fingers will disrespect the fact that the actual total may exist in different forms, all with units: as 5 1s, as 1 5s, as 1Bundle3 2s, as 2Bundle1 2s, as 3Bundle-1 2s, etc.

Woke-Math builds on a basic observation asking a 3year old "How old next time?" The answer typically is four showing four fingers. Holding them together 2 by 2, that child objects "That is not 4, that is 2 2s." The child thus sees what exist in space and time, bundles of 2s in space, and 2 of them in time when counted. So, what exist are totals to be counted and added in time and space, as T = 2 2s.

Woke-Math thus builds on the philosophy called existentialism holding that existence precedes essence, i.e., that what exists outside precedes what we say about it inside.

By de-modeling mathematics instead of modeling reality, Woke-Math offers "Master Many to master Math" as an alternative to the traditional approach, "Master Math to master Many".

Woke-Math respects, that outside totals inside may be counted and recounted in various two-dimensional bundle-numbers with units; and rejects one-dimensional line-numbers without units since they lead to 'mathematism' true inside, but seldom outside, where 2+3 = 5 is falsified by 2weeks + 3days = 17 days.

Counting and recounting totals in time and space produce flexible bundle-numbers

Counting in time produces a sequence of names like the names for the weekdays or months. Instead of saying "9, 10, 11 and 12", we might want to say, "Bundle less 1, Bundle, one left, two left" to see the Viking origin of the names 'e-leven' and 'twe-lve', meaning precisely that.'

Later counting "10, 20, ..., 90, 100" may be paralleled be counting "1B, 2B, …, 9B, 10B or BB", thus fulfilling the Dienes dream of meeting power in grade one. That is also fulfilled when counting ten fingers in 3s as "0B1, 0B2, 0B3 or 1B0, …, 2B3 or 3B0 or 1BB0B0, 3B1 or 1BB0B1". With bundles and bundle-bundles, the place value system is not needed.

Counting in space, all totals have units. Some numbers are prime units, others are compound units hiding the prime units, 1 6s = 1x6 = 1x2x3 = 2 3s = 3 2s. Bundle-numbers become flexible by allowing overloads and underloads: T = 5 1s = 1B3 2s = 3B-1 2s.

Counting sticks in space we see that 4 1s may be rearranged as 1 4s, thus realizing that written less sloppy, digits are icons with as many sticks or strokes as they represent.

Counting 8 1s in 2s, pushing away 2s may be iconized by a broom called division allowing a calculator to predict the result: 8 = 8/2 2s. Stacking the 4 2s may be iconized by a lift called multiplication, 4x2.

So, we have a recount-formula, 8 = 8/2 2s = (8/2) x 2, or T = (T/B) x B with unspecified numbers, the core proportionality formula appearing all over math and science: □y = (□y/□x) x □x = slope x □x, height = (height/base) x base = tan Angle x base, meter = (meter/second) x second = speed x second, etc.

From the total, pulling away the stack to look for unbundled singles may be iconized as a rope called subtraction: 9 – 2x4 = 1. Placing the unbundled on-top of the stack introduces decimals, fractions, and negative numbers: 9 = 4.1 2s = 4 ½ 2s = 5.-1 2s.

Recounting a physical total in different units creates a per-number, as 4 $/5kg.

To change units, we just recount in the per-number: 20kg = (20/5) x 5kg = (20/5) x 4$= 16$

Recounting in the same unit creates fractions or percentages: 6$/20$ = 6/20 = 30/100 = 30%.

With prime-units, per-numbers may be simplified: 6/10 = (2x3) / (2x5) = (3 2s) / (5 2s) = 3/5

On a tile, recounting the sides in the diagonal creates trigonometry:

height = (height/base) x base = tan Angle x base, height = (height/diagonal) x diag. = sin Angle x diag.

This allows the number pi to be calculated as n x tan(180/n) for n large.

Adding totals on-top and next-to

Once counted and recounted, totals may add on-top after recounting makes the units like. Or next-to as areas called integration combining multiplication and addition, becoming differentiation combining subtraction and division when reversed, asking, e.g., 2 3s + ? 5s = 4 8s. Here, we remove the initial total to recount the rest in 5s, so, ? = (4 8s – 2 3s) / 5 5s = □y/□x.

Per-numbers add by areas since adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before adding, thus adding as areas.

Such mixture problems lead directly the 2x2 ‘algebra square’ re-uniting changing and constant unit-numbers and per-numbers, where algebra in Arabic means ‘to re-unite’:

Changing unit-numbers unite by addition, and constant unit-numbers unite by multiplication. Likewise, changing per-numbers unite by integration, and constant per-numbers unite by power (10% + 10% = 21% since 110%^2 = 121%).

Uniting reversed will split a total in changing or constant unit-numbers or per-numbers. Here subtraction and division split a total into changing and constant unit-numbers. Differentiation splits a total into changing per-numbers. And finally, a total is split into constant factors by the factor-finding root and the factor-counting logarithm.

Bundle-numbers ease operations

Addition: T = 54 + 38 = 5B4 + 3B8 = 8B12 = 9B2 = 92 = 9.2 tens

Subtraction: T = 54 – 38 = 5B4 + 3B8 = 2B-4 = 1B6 = 16 = 1.6 tens

Division: 378 / 7 = 37B8 / 7 = 35B28 / 7 = 5B4 = 54 = 5.4 tens.

Multiplication: 6 x 7 = 6 x (B-3) = 6B -18 = (6-2)B (20-18) = 4B2 = 42 = 4.2 tens.

Multiplication tables as "6 7s = ?tens" may also be placed on a BxB square wring 6 and 7 with underload as B-4 and B-3.

So, 6 7s = 6x7 = (B-4) x (B-3) = BB -4B -3B + 4 3s (removed twice) = 3B 12 = 4B2 = 42 = 4.2 tens.

7 26s = 7 x 26 = 7 x 2B6 = 14B42 = 18B2 = 182 = 18.2 tens.

37 26s = 37 x 26 = (3B7) x (2B6) = 6BB (3x6 + 7x2)B 42 = 6BB 32B 42 = 6BB 36B 2 = 9BB 6B 2 = 962, seen on a 3B7 x 2B6 tile.

To find the square root of 40 we squeeze a 40 x 1 or 5 x 8 tile into a square. Here 6x6 is the largest inside square leaving 40 - 6x6 = 4 outside. Extending the 6x6 square into a (6+t) x (6+t) square adds additional two 6 x t tiles plus a t x t square that we neglect. With 6xt = 4/2, t = 2/6 = 0.333. So, the 40x1 tile squeezes into a 6.333 x 6.333 square approximately, or a little less since we neglect the t x t square. A calculator gives the answer: the square root of 40 is 6.325. Extending a square also shows that d(x^2) = 2x dx, or (x^2)' = 2x.

To solve the quadratic equation x^2 + 6x +8 = 0, we look at a (x+3) x (x+3) square divided in four sections, x^2, and 3x twice, and 9 = 8+1. Since x^2 + 6x +8 = 0, (x+3) squared is 1 = 1 squared. Hence x+3 = +1 or -1, so there are two solutions, x = -3+1 = -2, and x = -3-1 = -4.

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Artificial Intelligence makes Difference Research in school mathematics more relevant 1306

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Research typically is seen as an example of a top-down or bottom-up lab-lib cooperation where laboratory observations are deduced from or are inducing library concepts. In the top-down version, library theory generates a hypothesis that, validated or falsified in the laboratory, leads to a stronger or adapted theory. In the bottom-up version, laboratory observations lead to categories, that additional observations may split into more nuanced subcategories.

Artificial Intelligence has access to the library, but laboratory data must be added. Top-down research thus may be generated very quickly with a quality depending on the reliability of the data input, which may be difficult to check. In contrast, AI is of less help to bottom-up research typically generating new categories not yet present in the library. Also Difference Research 'searching for differences making a difference' may now be more relevant since although AI may locate existing differences, it cannot invent new differences. Nor can it examine the difference they make.

Examples of difference research are bundle-numbers with units, operations as icons for counting, re-counting to change unit and to solve equations, per-numbers coming from recounting in two units, integration as addition of locally constant per-numbers, trigonometry before geometry, mathematism adding numbers without units, mastering Many before math, etc.

Background

In late May 2023, 350 leading scientists and notable figures signed a common statement saying that "Mitigating the risk of extinction from AI should be a global priority alongside other societal-scale risks such as pandemics and nuclear war" (https://www.safe.ai/statement-on-ai-risk).

Artificial intelligence (AI) may be seen as the opposite of natural intelligence where the word intelligence comes from latin 'intelligere', meaning the ability to think before making a choice.

To increase productivity, humans have gradually invented artificial humans. First by replacing hands with tools. Then by replacing muscles with motors able to repeat jobs, and combined with tools to machines. And finally, by replacing the natural brain with an artificial brain able to make intelligent choices, and combined with machines to robots or becoming 'AIbots' themselves.

Programming languages were invented to instruct artificial brains in making intelligent repetition and choice. In the language BASIC the command "REPEAT UNTIL" allows the computer to stop by itself without exterior human interventions, and the command "IF …THEN" allows it to choose a different action, and thus becomes the base of artificial intelligence.

An impression of the AI- risks may come from the mid-May 2023YouTube video "AI and the future of humanity" where at the Frontiers Forum professor Yuval Noah Harari, author of the bestselling books 'Sapiens' and 'Homo Deus', says that, by its ability to develop itself, AI may soon change from artificial to alien intelligence able to hack and infiltrate human language and texts. This will mean that our prime tool for a democratic interhuman cooperation based on information, debate, and voting, now may be infiltrated by an alien intelligence that may supply fake information and seducing

arguments to influence the debate. Thus, to take power AI does not have to develop new physical 'AIbots' itself since it can use human language to have its will. (https://www.youtube.com/watch?v=LWiM-LuRe6w). Furthermore, where organic lifeforms have a limited lifespan, this new inorganic lifeform has not and thus will be able to accumulate more and more knowledge infinitely. Likewise, in late April 2023 a study published in JAMA Internal Medicine concluded that not only is AI better at diagnosing but also to show compassion.

And in fact, school mathematics has already experienced to be hacked by a different intelligence when the so-called modern or new mathematics in the 1960s offered its compelling system of well-proven statements about well-defined concepts all derived from the mother-concept Set. The attack succeeded to a certain extent in Europe, but not in North America quickly reacted by saying "Back to basics".

This raises the question "How to protect mathematics and its education and research from the risk of being infiltrated by alien artificial intelligence?"

A task, that seems almost impossible since both mathematics and education and research is text-based, so the question leads on to the next question "Can mathematics and its education and research exist in a text-free version?"

We begin by looking for possible answers in the three grand theories, philosophy, sociology, and psychology.

The three grand theories

A core question in philosophy is whether existence precedes essence or the other way around. Here Existentialism favors the first, and Platonism the latter. The Greek word mathematics, meaning 'what we know', was introduced as a common label for four activities, arithmetic, geometry, music, and astronomy studying Many by itself, in space, in time, and in time and space. Then Euclid transformed geometry to a deductive theory by finding what could be derived logically from axioms describing an internal essence of outside space. Modern mathematics later did the same, so today it is general accepted that the goal of mathematics is to learn mathematics as defined by university mathematicians, and that mathematics is a self-referring text-based science.

So, to avoid being texr-based, we now may have to return to the ancient Greek viewpoint seeing mathematics as a natural science about Many existing as multiplicity in space and as repletion in time. And our search may be guided by existentialist thinkers.

Asking "What is 'is'?", Heidegger described the four different ways to use the judging word 'is' to assign meaning to an abstract concept. "Is an example of" points up to a more abstract concept. "Is for example" points down to less abstract examples. "Is like", points over to the same abstraction level with metaphors, meaning 'carry over' in Greek. Finally, "Is period" warns that a subject will always be much more than any collection of predicates that will always be chosen and reductive. Instead, a subject should be met directly to allow personal impressions that later may be shared and developed when communicating with peers. This reluctance towards predicates is supported by a simple observation called "The Pencil Paradox" showing the difference between numbers and words: "Placed between a ruler and a dictionary, a pencil can falsify its number, but not its word."

Heidegger's skepticism towards predicates together with Kierkegaard's skepticism towards institutions later inspired French poststructuralism. Here Derrida introduces deconstruction of institutionalized words to open hidden perspectives. And Lyotard introduces postmodernism as skepticism towards grand narratives that may express a political instead of a natural correctness. Finally, Foucault introduces 'pastoral power' to create 'docile bodies' occurring when institutionalizing statements about humans as disciplines claiming to be scientific but instead discipling themselves as well as their human subject, which then may be cured by the institutions built on the disciplines that created the diagnose in the first place.

A core question in psychology is whether brains should be adapted to outside existence as proposed by Piaget, or to inside essence as proposed by Vygotsky. Within mathematics education this distinction represents the difference between radical and social constructivism. The former has been preeminent in North America, the latter in Europe. But since essence typically is text-bound, we now may have to look more closely to text-free approach Piagetian approach.

A core question in sociology concerns the institutions that individuals create to reach goals they do not have time to reach themselves: do institutions precede individuals or the other way around? Here Bauman warns against the risk of a 'goal displacement' where the employees secure their employment by inventing obstacles calling for more employees to reach the goal so that employment becomes the goal with the original goal as a means.

Within mathematics education, mastering inside mathematics should be a means to later master outside Many, but a goal displacement has created the myth that "of course mathematics must be learned before it may be applied". And invented an obstacle by transforming mathematics into 'mathematism' (Tarp, 2018) true inside but not always outside by building on the claim that always 2+1 = 3 despite numerous outside counter examples as 2weks plus 1 day is 15 days.

This goal displacement making mastering mathematics precede mastering Many makes mathematics text-based. So a difference will be to cancel the goal displacement by turning mathematics around so that mastering many will precede mastering mathematics.

A different math

So, if we ask the grand theories about how to create a text-free mathematics, philosophy recommends focusing on outside existence, and being skeptical and flexible with the inside essence we chose to assign to it. Psychology recommends developing the learning coming from meeting and adapting to outside existence as described by Piaget. And sociology recommends we should avoid a goal displacement making mathematics a goal instead of a means and return to the original Greek goal to study Many as it exists in time and space.

One way of following these recommendations could be to use 'difference research' searching for differences making a difference (Tarp 2018). Here, we constantly ask "Can this be different?" when looking at the core of mathematics. Postmodern deconstruction may give an answer by using 'demodeling' (Tarp, 2020) to find outside concrete examples to root inside abstractions.

The core of mathematics may be seen as what is on a basic calculator: single and multiple digits, operations and equations. So, we ask how demodel this core as things in space and actions in time.

Typically, digits are seen as abstract symbols outside time and space. A difference is to see the symbol ‘5’ as an icon created in time by uniting five ones, e.g., sticks, to one five-icon thus containing as many sticks as it represents. Likewise with the other digits until the number ten that does not need a digit if used as a bundle. So here we build no geometrical icon, we simply collect the ten sticks into one bundle with no sticks left unbundled. Writing ten as 10 then shows that we now have two numberings of bundles and unbundled both becoming units. With three as a bundle-unit, nine becomes a bundle of bundles. And with two as a bundle-unit, eight becomes a bundle of bundles of bundles, thus introducing power as the first operation. Rolling two dices we may say 2 3s, which shows that digits are needed both as unit-numbers in space and as counting numbers in time. By itself, a digit is not a number, but an operator needing a number to become a number as show when writing them out fully as polynomials as they are spoken: 345 = 3*100+4*10+5*1.

Using snap-cubes, the unbundled may be placed on-top of the stack of bundles and seen as a decimal number, or with a negative less-number counting what is needed for another full bundle, e.g., if counted in 2s, outside 9 may be written inside as a ‘flexible bundle number with units’ as 9 = 4B1 = 5B-1 = 3B3 2s, where the two latter represent an underload and an overload, but all representing the same outside total. This eases calculations:

43+29 = 6B12 = 7B2 = 72; 43-29 = 2B-6 = 1B4 = 14

Typically, the number sequence typically is learned by memorizing a row of names as when memorizing the names of the months. This could be demodeled by always including the units by saying 0B6 instead of 6 and 3B4 instead of 34 and 6BB7B8 instead of 678.

Typically, the operations are learned and defined in the order addition, subtraction, multiplication, division, and power. A difference is to demodel them as actions in time, which gives the opposite order. Here power comes first rooted in bundles of bundles; then comes division as an icon for pushing away bundles; then comes multiplication as an icon for lifting them into a stack; then comes subtraction as an icon for pulling away the stack to look for unbundled; and finally addition comes as an icon for uniting stacks next-to or on-top. Seeing operations as actions leads directly to the core proportionality formula used to chance units all over science, T = (T/B)*B, saying that the total T contains T/B Bs.

Typically, numbers occur without units where 234 is presented as one number on a number line instead of 3 numberings of singles, bundles, and bundles of bundles. So, here a difference is recounting in another unit to respect that this is a core question with outside totals. Recounting from icon units to tens leads to the tables where 6x7 is seen as 6 7s or as ‘(B-4) (B-3)s’ on a ten-by-ten ‘Bundle-Bundle-Board’ (Tarp, 2023). This introduces early algebra: ‘(B-4) (B-3) = 10B -3B -4B + 4 3s = 3B + 1B2 = 4B2 = 42And recounting from tens to icons leads to equations as ‘u*4 = 20’ solved by u = 20/4 since 20 recounts in 4s as 20 = 20/4 4s = 20/4 *4. Likewise, the equation u+2= 5 is solved by moving over as u = 5-2 since u is a placeholder for a number that with 2 added gives 5, thus found by reversing the action by pulling away the 2 again. Solving equations by ‘move to opposite side with opposite sign’ thus is a difference to the traditional balancing method ‘do the same to both sides’ introduced to motivate teaching teachers about the abstract algebra concept ‘group’.

Furthermore, recounting the same total in two units gives a per-number as 2$ per 5 kg that becomes a faction with like units, 2$ per 5 $ = 2/5. And recounting the sides in a rectangle split by its diagonal will lead to trigonometry before geometry. Finally, when adding, per-numbers must be multiplied to unit numbers thus adding by their areas as integral calculus, also occurring when adding two stacks next to each other instead of on-top.

So, as to addition it is takes on a different role. When demodeling it will be the last and not the first operation, and it leads directly to proportionality and integral calculus when stacks are added on-top and next-to.

Fact, fiction and fake, the three modeling genres.

With mathematics a number-language describing outside things in space and actions in time, its literature or models needs to be distinguished in fact, fiction or fake, the same genres that exist when using the word-language to produce qualitative literature.

Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas. They need to be checked for correctness and units.

Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. They need to be supplied with scenarios building on alternative assumptions.

Fake or fiddle stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas. They need to be replaced by word stories.

Teacher education

Typically, teacher education sees mathematics as a timeless self-referring body of knowledge that supplies school mathematics with a theoretical background to be pedagogically supplied with real-world examples as the sugar coating that makes the pill go down. As a difference, demodeling instead presents mathematics as things and actions in time and space. The things may be found an a ten-by-ten Bundle-Bundle-Board with rubber bands to show different stacks and forms (Tarp, 2023), and the actions may be counting and adding in time and space.

A corresponding teacher education may be found on MATHeCADEMY.net. that offers a text-free action-based education in 'Many-math': "Let Kids teach teachers teach MatheMatics as ManyMath, a natural science about MANY using flexible bundle-numbers with units to count and recount before adding on-top and next-to; the CATS approach: Count & Add in Time & Space." The material is self-instructing since, when in doubt, you ask the concrete subject, not the instructor.

Research

Typically, research papers are reviewed by peers without allowing the author to review the reviews. With AI's ability to create research papers itself, the time may have come to allow this on the base of the genre conditions: question, method, design, data collection and analysis, finding, discussion. And maybe reviewers should have their own papers rejected if they don't follow this. Also, a 'Salon des Refusee' should be included to conferences to give an indication of what is accepted and what is not. Art developed from rejected works demonstrating difference.

Conclusion

We asked: “Can mathematics and its education and research exist in a text-free version?”

Inspired by the recommendations of the 3 grand theories; and guided by Difference Research and postmodern de-modeling we have found that, to create a text-free version out of the reach of artificial intelligence, mathematics could retake its original Greek goal as a natural science about Many, and thus stay as the human number-language about Many, that children develop before school.

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Online math may create a communicative turn in number-language education also 1207

Allan Tarp, the MATHeCADEMY.net, summer 2023

Paper presented at the CTRAS 2023 June Conference.

Video: https://www.youtube.com/watch?v=36tan-gGjJg, and https://www.youtube.com/watch?v=4EPqjz8evd4

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Abstract

Teaching mathematics online is different from teaching it offline in a classroom. So, we may ask what else could be different? On a calculator we see the core of mathematics: digits, multidigit numbers, operations, and equations. But they all occur as products in space, not as processes in time. So, maybe teaching mathematics 'process-based' instead of 'product-based' is different by letting outside Many precede inside Math? And indeed, the math core becomes different when created from tales about Many displayed as rectangles on a concrete ten-by-ten bundle-bundle board, a BBBoard. To see if a 'process-before-product' or 'Many-before-Math' education will make a difference, micro-curricula are designed to bring outside totals inside by bundle-counting creating flexible bundle-numbers with units: ones, bundles, bundle-of-bundles, etc., that all are included in oral counting sequences. Here digits arise as icons when uniting sticks. Here operations arise as icons as well: division pushes away bundles that multiplication lifts onto a stack that subtraction pulls away so the unbundled may be included as decimals, fractions, or negatives. Once counted, a unit may be changed by recounting. Here recounting from tens to icons leads to equations, and when reversed, to tables displayed as the rectangle left when removing the two surplus rectangles from the full bundle-bundle on a BBBoard. Here recounting in two physical units leads to per-numbers bridging the two units and becoming fractions with like units. Here recounting the sides and the diagonal in a rectangle leads to trigonometry before geometry. Finally, once counted and recounted, totals may add on-top after recounting has provided like units, or next-to as areas as in integral calculus that becomes differential calculus when reversed. As operators needing numbers to become numbers, per-numbers and fractions also add by their areas since they need to be multiplied to unit-numbers before adding. So, outside totals inside appear in an 'Algebra Square' where unlike and like unit-numbers and per-numbers are united by addition and multiplication, and by integration and power. And later again split by the reverse operations, subtraction and division, and by differentiation and root or logarithm. Once these curriculum scores have been designed, they may be played in online education, as well as in special education to see if a BBBoard may 'Bring Back Brains' excluded by the 'Math-before-Many' education. But there will be no concert without first designing a score. So actual testing is not addressed here but left to others to perform.

Background

Teaching mathematics online and remotely is different from teaching it offline in a classroom. So, we may ask what else could be different? On a calculator we see the core of mathematics: digits, multidigit numbers, operations, and equations. But they all occur as products in space, not as processes in time. So, maybe teaching mathematics 'process-based' instead of 'product-based' is different by letting outside Many precede inside Math? Especially since a 'product-based math' typically implies the top-down 'abstractions-before-examples' view that 'of course, mathematics first must be learned before it later may be applied', which makes mastery of inside mathematics a necessary means to reach the end goal, mastery of outside Many. In contrast, a 'process-based math' resonates with existentialist philosophy holding that existence precedes essence and thus implies the bottom-up 'examples-before-abstractions' view that "of course, examples should install abstractions instead of being installed by them", which makes mastery of outside Many the direct goal that automatically implies a basic mastery of mathematics later.

Asking a similar question created a communicative turn in foreign language education in the 1970s (Widdowson, 1978). So, online math may create a communicative turn in number-language education as well.

To look for differences we may use 'Difference Research' (Tarp, 2018) that, searching for a difference making a difference, asks "May a different input give a different output?"

Asking 'Could mathematics be different' we find its core on a basic calculator with digits and operations and an equation sign on buttons, and multidigit numbers on the display. However, the digits and operations occur as end-products only, we don't see the process creating them.

We see the 5-digit as one symbol in space, we do not see how it may be formed in time by uniting five sticks or strokes into one icon. Likewise, we see the total T = 234 as one number without units. So, we don't see the three numberings using bundle-counting to find the number of singles, bundles, and bundles-of-bundles, i.e., we don't see the polynomial with the units T = 234 = 2*BB + 3*B + 4*1.

We see the operations order as natural: addition, subtraction, multiplication, division, and power. Because they build on the basic claim that "2+1 = 3 always". However, with units, 2 weeks + 1 day is 15 days. So, adding numbers without units often folds outside the classroom, whereas multiplication always holds since, in the product 2x3, 3 is the unit. Therefore, to teach valid mathematics instead of invalid 'mathematism' (Tarp, 2018), units must always be included inside.

Counting totals in tens, a bundle of bundles first occurs with three-digit numbers, and without stressing that hundred is just another name for bundle-bundle. Instead, counting fingers in 2s, 2 becomes a Bundle, 4 a Bundle-of-Bundles, a BB or a B^2. And 8 becomes a BBB, or a B^3. So, when bundle-counting an outside total with units, power becomes the first operation, not the last.

Calculations never includes the outside total that is calculated. So instead of saying 'T = 6x7', we only say '6x7'. And we see 6x7 as 42 always instead of de-modeling (Tarp, 2020) it outside as 6 7s that may or may not be recounted in tens as 4 bundle 2 tens written inside as 4B2 tens or 4.2 tens, or shortly as 42 if leaving out both the unit and the decimal point. Nor do we see or perform 6x7 as an outside process where a total is recounted in 7s by 6 times pushing away the bundles with a broom called division, to be stacked by a lift called multiplication. After which a rope called subtraction pulls away the stack to look for unbundled singles to be placed on-top of the stack as a decimal number, as 9 = 4B1 2s. Only now, once counted and recounted, stacks may finally be added on-top or next-to. So, apparently from a process-before-product view, the operation order is the opposite.

These observations suggest an immediate answer to our question. Mathematics does seem to be different when seen as a process to be acted in time instead of as an end-product in space to be memorized.

This resonates with the philosophy called existentialism that, by holding that existence precedes essence (Sartre, 2007), suggests that outside existence should install inside essence instead of being installed by it. Or in other words, we should treat and teach mathematics as a number-language describing the outside fact Many, already occurring in the word-language as the plural form of words.

Also, this will allow solving inside problems by de-modeling them as outside totals. Thus asking "What is the product of 2½ and 3 1/3?", this inside product could be demodeled as an outside total T, thus creating the number-language sentence, T = 2½ * 3 1/3, with an outside subject, a verb and an inside predicate as in word-language sentences. From this story about T we may conclude that with 2 and 3 times as many Ts we get 2*T = 5 * 3 1/3, and 6*T = 5*10 = 50, so T = 50/6 = 8 2/6.

Outside totals are typically counted in tens inside. So, we may develop a number-language by describing things and actions on a ten-by-ten Bundle-Bundle-Board, a BBBoard, where rubber bands split the bundles vertically and horizontally to create rectangles with a total of T pegs. Which will then be the subject in a number-language sentences where the BBBoard allows an outside total of 2 3s to becomes the inside sentence T = 2x3, also called a formula or a function. Plastic BBBoards are cheap to manufacture and give to schools, where the students also can use BBboards on squared paper.

But will this difference make a difference? To test this, we first design some micro-curricula (MC) where we describe things and actions on a Bundle-Bundle BBBoard. The curricula are inspired by the four Tarp papers from 2001 and 2018 - 2022.

Figure 1. A total of 6 7s shown on a Bundle-Bundle BBBoard

MC01. Digits are icons uniting sticks

The goal is to experience that digits may be formed by uniting the number of sticks that they represent, and to see that these digits are like the digits on the calculator.

The action is to unite two sticks to one two-cion, three sticks to one three-icon, etc. Other concrete materials could be cars, dolls, spoons etc. Finally, a bending ruler may be used to create the digits.

We see that with Roman numbers, 3 1s is not united into 1 3s; and that Roman numbers use the letter V as a symbol for a hand, and the letter X as a symbol for two hands.

We don't have an icon for ten since bundling in tens makes ten a two-digit number counting both the number of bundles, and the number of unbundled singles:

T = ten = 1B0, or T = 10 if leaving out the unit.

When including the units, we don't need a place value system.

MC02. Operations are icons created by Bundle-counting and recounting

The goal is to experience how the three outside counting processes, pushing away bundles, stacking bundles, and pulling away stacks may inside be iconized as division, multiplication, and subtraction.

As concrete material serves sticks and snap cubes.

To recount 8 in 2s, we push away 2s 4 times to get the answer T = 8 = 4 2s.

Iconizing 'push away' by a broom, /, called division, the action 'from 8 push away 2s' may be iconized as '8/2' giving 4 on a calculator.

Iconizing 'stacking up' by a lift, x, called multiplication, the action '4 times stack 2s' may be iconized as '4x2' giving 8 on a calculator.

Pushing and stacking combines in a 'recount-formula' (Tarp, 2018),

8 = (8/2)x2, or T = (T/B)xB with unspecified numbers, saying that T contains T/B Bs.

Iconizing 'pull away' by a rope, –, called subtraction used to find unbundled singles, the action 'from 9 pull away 4 2s' may be iconized as '9-4x2' giving 1 on a calculator as expected.

Placed on top of the stack, the unbundled may be seen as a decimal number or as a fraction when counted in 2s also, or creating a negative number showing what is pulled away form the next bundle:

9 = 9 – 4x2 = 1, so 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

Iconizing 'uniting' by a cross showing two directions, +, called addition, the action 'uniting 2 3s and 4 5s' may be iconized as '2x3 + 4x5' giving 26 on a calculator if united in tens, and 5B1 5s if added on-top, and 3B2 if added next-to.

Pulling and adding combines in a 'restack-formula' (Tarp, 2018),

8 = (8-2)+2, or T = (T-B)+B with unspecified numbers, saying that T-B is left when B is pulled away and placed next to, or simply that pulling away B allows T to split in T-B and B.

MC03. Bundle-counting in icons

The goal is to use an oral bundle-counting sequence with icon-bundles to represent outside examples of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

First, we count the ten fingers in 4s by raising them one by one beginning with the little finger.

0B1, 0B2, 0B3, 0B4 or 1B0, moving the fingers to see the difference between 4 1s and 1 4s, called a bundle, B.

1B1, 1B2, 1B3, 1B4 or 2B0, again moving the fingers to see the difference between the two.

2B1, 2B2.

So, a total of ten may be counted in 4s as 2 bundles and 2, written shortly as

T = 2B2 4s, or T = 22 4s, if leaving out the unit.

We notice that we need 2 to have 3 bundles, so we may also write T = 3B less2 = 3B-2 4s.

So, with ‘flexible bundle-counting’ we have three ways to count ten in 4s: underload, normal, and overload:

T = ten = 3B-2 = 2B2 = 1B6 4s.

The counting is now repeated on the pegboard with a rubber-band showing the 4-bundles.

Now, we count the ten fingers in 3s by raising them one by one beginning with the little finger.

0B1, 0B2, 0B3 or 1B0, moving the fingers to see the difference between the two.

1B1, 1B2, 1B3 or 2B0, again moving the fingers to see the difference between the two.

2B1, 2B2, 2B3 or 3B0, again moving the fingers to see the difference between the two.

3B1 or 4B-2 since we need 2 to have 4B0.

So, with ‘flexible bundle-counting’ we have four ways to count ten in 3s:

T = ten = 4B-2 = 3B1 = 2B4 = 1B7 3s.

However, when bundling in 3s, 3 bundles become 1 bundle-of-bundles, 3B = 1BB = 1B^2, which suggests that power should be the first operation:

T = ten = 3B1 = 1BB 0B 1 3s, or T = 101 3s, if leaving out the units.

The counting is now repeated on the pegboard with a rubber-band showing the 3-bundles.

We notice that a bundle-of-bundles BB becomes a square.

Now, we count the ten fingers in 2s, pairs.

We begin by seeing how 5 fingers may be counted in 2s.

T = 5 = 1B3 = 2B1 = 3B-1 2s, and T = 5 = 1BB 0B 1 2s.

Putting the two hands together, we see that ten can be counted in 2s as

T = ten = 2BB 0B 2 = 1BBB 0BB 1B 0 2s, or T = 1010 2s, if leaving out the units.

With power, we get a polynomial T = ten = 2*B^3 + 1*B 2s

The counting is now repeated on the pegboard with a rubber-band showing the 2-bundles.

MC04. Bundle-counting in tens

The goal is to use an oral bundle-counting sequence with ten-bundles to represent outside presences of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band above the first row.

The action is to include both singles and bundles when counting the pegs horizontally as

0B1, 0B2, …, 0B9, 0Bten or 1B0 or 10. And then

1B1, 1B2, …, 1B9, 1Bten or 2B0 or 20, moving up the rubber band.

This continues until

9B1, 9B2, …, 9B9, 9Bten or tenB0 or 1BB0B0 or 100.

Bundling in tens, a BundleBundle BB is called hundred, BBB = B^3 is called thousand, and BBBB =B^4 is called ten thousand, or Wan in Chinese. Likewise, BBBBBB = B^6 is called a million.

Alternatively, instead we may count what is missing for another full bundle:

1B-9, 1B-8, …, 1B-1, 1B0 or 10. And then

2B-9, 2B-8, …, 2B-1, 2B0 or 20, moving up the rubber band.

This continues until

1BB-9, 1BB-8, …, 1BB-1, 1BB0 or 1BB0B0 or 100

Including also overloads and underloads, we can now practice 'flexible bundle-counting' as

T = 67 = 6B7 = 5B17 = 7B-3 tens

MC05. Recounting tens in icons gives equations

The goal is to experience how recounting from tens to icons is another word for an equation, that may be solved by, of course, recounting.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band to show the bundles, as well as snap-cubes, and a calculator.

Asking "8 is how many 2s" may be rephrased to "8 is u 2s" using letters for unspecified or unknown numbers. This then may be shortened to an equation, 8 = ux2, solved by recounting 8 in 2s as 8 = (8/2)x2, so that the solution u = 8/2 is found when isolating the unknown number by moving the known number to opposite side with opposite sign.

Here we see that when decreasing the bottom in a stack the height increases.

MC06. Recounting icons in tens gives rectangles and multiplication tables

The goal is to translate an oral icon-counting sequence to ten-counting, e.g., translating the sequence 1, 2, 3, 4, 5 4s into 4, 8, 12, 26, 20.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a vertical rubber band to show the stack of bundles.

With 2 as the bundle-size, first we count vertically in 2s as 1, 2, 3, 4, 5 2s. Then we count in tens as 2, 4, 6, 8, 10. Then we count horizontally but letting a finger slide over the pegs. Finally, we memorize by folding fingers.

Likewise with 3, 4 and 5 as the bundle-size. Each time we see that increasing the bundle-size will decrease the height. It is like water that sinks when the bottom is increased.

With 6 or B-4 as the bundle-size, again first we count vertically in 6s as 1, 2, 3, 4, 5 6s. Then we count in tens as 1B-4, 2B-8, 3B-12, 4B-16, 5B-20, or 0B6, 1B2, 1B8, 2B4, 3B0, or 6, 12, 18, 14, 30.

Likewise with 7, 8 and 9 as the bundle-size.

Finally, to translate 7 6s into tens we use two rubber bands to show that 7x6 = (B-3) x (B-4). So, we get the total by removing 3 horizontal bundles and 4 vertical bundles and then add the upper right rectangle 3 4s that was removed twice:

T = 7x6 = (B-3) x (B-4) = 10B – 3B – 4B + 3x4 = 3B + 1B2 = 4B2 = 42.

Here we see that negative multiplied with negative gives plus.

Now we recount 6 78s in tens

T = 6 78s = 6x78 = 6x7B8 = 42B48 = 46B8 = 468

On the pegboard, the horizontal rubber band is a 6 and the vertical illustrates 7B8.

Now we recount 36 78s in tens

T = 36 78s = 36x78 = 3B6 x 7B8 = 21BB (3x8+6x7)B 48 = 21BB (2B4+4B2)B 48 = 21BB (6B6)B 48 = (21+6)BB (6+4) B 8 = 27BB 10B 8 = 28BB0B8 = 2808

On the pegboard, the horizontal rubber band illustrates 3B6 the vertical illustrates 7B8. This gives the four parts 21BB, 24B, 42B, and 4B8, or 21BB, 2BB4B, 4BB2B, and 4B8, adding up to 27BB10B8 or 28BB0B8 or 2808.

A quick way is to write the numbers under each other. Multiplying down the gives the BBs and the unbundled, and cross-multiplication gives the Bs.

MC07. Bundle-Bundles are squares

The goal is to count the squares occurring as bundle-bundles.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with two rubber bands to show the squares.

With 2 as the bundle-size, the 2x2 square contains a total of 4, T = 2x2 = 4.

With 3 as the bundle-size, the 3x3 square comes from adding a vertical and a horizontal 3-line, and then remove the upper right corner added twice, so the total is 9, T = 3x3 = 9.

With 4 as the bundle-size, the 4x4 square comes from adding a vertical and a horizontal 4-line, and then remove the upper right corner added twice, so the total is 16, T = 4x4 = 16.

With 5 as the bundle-size, the 5x5 square comes from adding a vertical and a horizontal 5-line, and then remove the upper right corner added twice, so the total is 25, T = 5x5 = 25.

The first 5 squares may be memorized by folding fingers at the same time.

With B-1 as the bundle-size, the 9x9 square comes from twice removing 1 bundle, and then add the upper right corner removed twice, so the total is 8B1 or 81.

With B-2 as the bundle-size, the 8x8 square comes from twice removing 2 bundles, and then add the upper right corner removed twice, so the total is 6B4 or 64.

With B-3 as the bundle-size, the 7x7 square comes from twice removing 3 bundles, and then add the upper right corner removed twice, so the total is 4B9 or 49.

With B-4 as the bundle-size, the 6x6 square comes from twice removing 4 bundle, and then add the upper right corner removed twice, so the total is 2B16 or 3B6 or 36.

We notice, that with square numbers, the end digit occurs again.

MC08. Recounting rectangles as squares gives square roots to solve quadratics

The goal is to experience how a stack may be squared by fitting it between two squares; and how a square always contains two smaller squares and two like rectangles that may solve quadratics.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with rubber bands. First, we use two bands to show the rectangle and two to show the square. Then we use two bands to show the two squares and the two rectangles.

Wanting to square a rectangle, we may ask: “How to change 6 3s into a BB square?”

We see, that 6 3s as a square will fit between a 4x4 and 5x5 square and that it is closer to 4x4.

The overflow here is 6x3-4x4 = 2 = 2x1, so 1 has to added to 4the 4s as 1 = (1/4)x4 = ¼ 4s = 0.25.

So, a guess would be that the 6 3s may be transformed into a 4.25x4.25 square, so that 4.25 may be close to the square root of 6x3. A little less since we need a little for the upper right corner.

A calculator shows that the square root is 4.243.

As to 8 3s, we see, that it is closer to 5x5. The missing here is 5x5-8x3 = 1 = 2x½, so ½ has to be subtracted from 5 as ½ = (½/5)x5 = 0.1 5s.

So, a guess would be that the 6 3s may be transformed into a 4.9x4.9 square, so that 4.9 may be close to the square root of 6x3. A little less since again we need a little for the upper right corner.

A calculator shows that the square root is 4.90.

Wanting to split a square in less squares, the BBBoard shows that with B = 7, the (B+3) square splits up into a BxB square and a 3x3 square and two 3xB rectangles:

(B+3)x(B+3) = BxB + 3x3 + 2 x (3xB), or (B+3)^2 = B^2 + 6B + 9

With B = 7 we get B^2 + 6B + 9 = 100, or B^2 + 6B - 91 = 0, a so-called ‘quadratics’.

Turning the process around, we may want to find B by solving the quadratic B^2 + 6B – 91 = 0, with B unknown.

So, since 6 = (6/2)*2 = 3*2, on a (B+3)x(B+3) BBBoard we use two rubber bands to show the 3x3 square in the upper right corner and the BxB square in the lower left corner as well as the two 3xB rectangles that combine to a big square (B+3)^2:

(B+3)^2 = B^2 + 6*B + 9

To include -91, we now rewrite 9 = 100 – 91, so

(B+3)^2 = B^2 + 6*B + 100 - 91,

But, since B^2 + 6*B – 91 = 0, we get

(B+3)^2 = 100 = 10^2, so

B+3 = 10 or B+3 = -10, so

B = 7 or B = -13

We were lucky, since some quadratics cannot be solved as B^2 + 6B + 10 = 0, where 9 becomes 10-1, so that (B+3)^2 = B^2 + 6*B + 10 -1 becomes (B+3)^2 = -1 that is not possible.

Now we may want to solve the quadratics B^2 - 6B +5 = 0

So, since 6 = (6/2)*2 = 3*2, on a BBBoard we use two rubber bands to show the 3x3 square in the upper right corner and the (B-3)x(B-3) square in the lower left corner as well as the two 3xB rectangles that combine to a big square B^2: B^2 = (B-3)^2 + 6*B + 9.

On the BBBoard we see that the (B-3)^2 square is left if we from the B^2 square pull away a 3xB rectangle twice and then add the 3^2 square that was removed twice:

(B-3)^2 = B^2 – 6*B + 9

To include 5, we now split 9 = 5+4, so

(B-3)^2 = B^2 – 6*B + 5 + 4

But, since B^2 – 6*B + 5 = 0, we get

(B-3)^2 = 4 = 2^2, so

B-3 = 2 or B-3 = -2, so

B = 5 or B = 1

Again were lucky, since B^2 - 6B +10 = 0 cannot be solved, where 9 becomes 10-1, so that

(B-3)^2 = B^2 + 6*B + 10 -1 becomes (B-3)^2 = -1 that is not possible.

The BBBoard shows that adding the 7x7 and the 3x3 square along the diagonal gives a 10x10 square except for two 3x7 rectangles (or tiles). Instead we could transform the 3x3 square into two ux7 tiles where ux7x2 = 3x3, or ux14 = 9 = (9/14)x14, so that u = 9/14 = 0,64.

So, we guess that the 7x7 and 3x3 squares add as a 7.64x7.64 square giving 58.4 which becomes the wanted 49+9 = 58 if we pull away the 0.64x0.64 square that was added twice, 58.4 – 0,4 = 58. With the root we find the exact answer, V 58 = 7.62, which turn out to be the length of the raising diagonal in the 3x7 tile, as we can see by cutting out four examples of this tile on paper.

The first paper-tile is placed on the upper left tile on the BBBoard, the next is turned and placed at the upper end, the third is again turned before placed at the end, and likewise the fourth. We now see that the diagonals form a square that is a full 10x10 square where 4 times we pull-away a half square, a total of 2 tiles. Removing the paper tiles, we see that this is also the case when adding the 3x3 and the 7x7 squares along the diagonal. Thus, we guess that in a tile, its diagonal square is the sum of the squares of its sides. This is called the Pythagoras rule named after a Greek philosopher.

MC09. Recounting in physical units gives per-numbers

The goal is to experience how recounting in two physical units creates per-numbers bridging the two to answers proportionality questions as “if 3kg costs 4$, then 15kg costs ?$, and ?kg costs 12$”.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band placed on a paper to show the two units.

Recounting 3kg as 4$ gives two per-numbers 4$/3kg (the price), and 3kg/4kg shown by a vertical rubber band as 3 on the board, and as 4 below on the paper. On a peg board 15 kg may by squeezed to 3s by recounting, 15kg = (15/3)x3kg = 15/3 times 3kg, so we also have 15/3 times 4$, or 20 $.

Likewise, 12$ = (12/4)x4$ = (12/4)x3kg = 9 kg.

MC10. Recounting in the same unit gives fractions

The goal is to experience how per-numbers become fractions when recounting in the same unit.

As concrete material serves a ten-by-ten bundle-of-bundles pegboard with two parallel rubber bands to show the whole and the part.

The BBBoard shows that getting a share of 3$ per 5$ means getting the fraction 3 per 5, or 3/5.

And that getting 3 per 5 means getting 6 per 10, 9 per 15, etc.

And that to get 3 per 5 of 20, we recount 20 in 5s as 20 = (20/5)*5, so we get (20/5)*3 = 12 as seen on the BBBoard.

MC11. Recounting rectangle-sides gives trigonometry before geometry

The goal is to experience how per-numbers in a rectangle split by its diagonal leads to trigonometry.

Recounting the height in the base, height = (height/base) * base = tangent Angle * base, shortened to

h = (h / b) * b = tan A * b = tan A bs,

This gives the formula tangent A = height / base, or tan A = h/b.

Using the words run and rise instead of base and height, we get the diagonal’s slope-formula:

tan A = rise/run.

The word ‘tangent’ is used since the height will be a tangent in a circle with center in A with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, h recounts in r as h = (h/1) * 1 = tan A.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, half the circumference, called π, is

π = 100 * h = 100 * tan 1.8 = 100 * tan (180/100) = 3.1426

This gives a formula for the number π:

π = tan (180/n) * n, for n sufficiently large.

We also see that in a circle with radius r, the circumference is 2*π*r, and the area is π*r^2, or π/4* d^2 where d is the circle's diameter.

So, a d-circle to takes up almost 80% of the space inside the surrounding d-square.

MC12. Adding next-to and on-top gives calculus and proportionality

The goal is to experience how, once counted and recounted, totals may add in two different ways, next-to and on-top.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with rubber bands, as well as snap-cubes.

We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" We see that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

We now ask the reverse question "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 3s. So now subtraction precedes division, which is natural as reversed integration, also called differential calculus.

We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?" We see that on-top addition means changing units by using the recount-formula. Thus, on-top addition typically applies proportionality.

We now ask the reverse question "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 5s. So again, subtraction precedes division as in differential calculus.

MC13. Adding per-numbers gives calculus

The goal is to experience how per-numbers (and fractions) first must be multiplied to unit-numbers before being added.

We ask "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" We see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

We now ask the reverse question "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" We see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

MC14. Adding one-digit numbers

The goal is to see how one-digit numbers may be added as bundle-numbers.

One-digit numbers may add as bundles, e.g.,

9+6 = 2B3 6s = 2B-3 9s, or

9+6 = B – 1 + 6 = B5 = 15.

9 – 6 = B – 1 – 6 = B – (1+6) = B – 7 = 3, or

9 – 6 = 9 – (B – 4) = 9 – B + 4 = - 1 + 4 = 3, or

MC15. Adding unspecified letter-numbers

The goal is to experience how letter-numbers add by first finding their common unit.

In the letter-number T = 3ab the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as 3 abs, or as (3a) bs, or as (3b) as. To avoid being confused by the 's' we will omit it, so T = 3ab = 3 * ab = 3a * b or 3b * a.

Since totals need a common unit to add, this must be first found:

T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a

T = 2ab^2 + 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b

MC16. The Algebra Square

The goal is to experience how an 'Algebra Square' shows the four ways to unite and split-into constant and changing unit-numbers and per-numbers.

Counting ten fingers in 3s gives T = 1BundleBundle 1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations unite/ split Totals in	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n T – n = a	T = a * n T/n = a
Per-numbers m/s, $/100$ = %	T = ∫ f dx dT/dx = f	T = a^b b√T = a loga(T) = b

Figure 2. The Algebra Square Reunite into Total to be Split into the Four Number-Types

MC17. Algebra and geometry coordinated.

The goal is to experience how a coordinate-system connects algebra calculations and geometry forms. As concrete material serves a Bundle-Bundle BBBoard with rubber bands and pins, as well as a 0-by-10 square on paper.

From the lower left point on a BBBoard we may travel in the horizontal x-direction with an 'out-number' x, and in the vertical y-direction with an 'up-number' y. In this way a point has two coordinates telling how far out and how far up it is placed. Here, the lower left point is (0,0).

If we from (0,0) take a '1 out, 2 up' trip we reach the points (1,2), (2,4), (3,6), etc. We see that y and x are linked by the formula y = 2*x which is called the formulas for the line we travel along.

Likewise, if we from (0,9) take a '1 out, 1 down' trip we reach the points (1,8), (2,7), (3,6), etc. In this case the line formula is y = 9-x.

Asking where the two lines intersect, the geometry forms show that this takes place in the point (3,6).

An algebra calculation gives the same result: Since the intersection point belongs to both lines, the coordinates obey both formulas, y = 2*x and y = 9 – x, so 2*x = 9-x. This equation is solved by moving to opposite side with opposite sign:

2*x = 9-x, so 2*x + x = 9, so 3*x = 9, so x = 9/3 = 3, so y = 2*3 = 6.

The intersection point thus has the coordinates (x,y) = (3,6).

On a BBBoard, two rubber bands show 6 7s as a 6-by-7 rectangle. An additional vertical rubber band is placed between 4 and 5 to signal that its x-number is unspecified.

Now, the diagonal in the '10 7s' rectangle represent a '7 out, 10 up' trip as well as a 'x out, y up' trip.

But since the angle A from horizontal is the same in the two trips we have tan A = y/x = 10/7. This gives the equation y/x = 10/7 that may be solved by moving to opposite side with opposite sign:

y/x = 10/7, so y = 10/7*x.

The intersection point between the diagonal and the horizontal rubber line then may be found by equating the two y-formulas, y = 10/7*x and y = 6, again solved by moving to opposite side with opposite sign.

10/7*x = 6, so x = 6*7/10 = 4.2, which gives y = 10/7 *4.2 = 6.

The intersection point thus has the coordinates (x,y) = (4.2,6).

Likewise, we see that the diagonal in the '6 10s' rectangle has the formula y = 6/10*x.

Also, we see that for the descending diagonal in the '3 tens' rectangle the angle A from horizontal seen from the point (10,0) may be found in two ways, as tan A = y/(10-x), and as 10/3. This gives the equation y/(10-x) = 10/3 that may be solved by moving to opposite side with opposite sign:

y/(10-x) = 10/3, so y = 10/3*(10-x) = -10/3*x + 100/3.

Likewise, looking at the descending diagonal in the '6 tens' rectangle, its formula is

y/(10-x) = 6/10, so y = 6/10*(10-x) = -6/10*x + 100/6.

The four diagonals together with the horizontal y = 6 line now form a star with five intersection points with coordinates that may be found both by algebra calculations and by geometry inspection.

On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angels and its three sides.

We see that these 7 triangle numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle we see that the angle is predicted by the formula tanA = 6/10, which on a calculator gives A = 31.0 degrees. And that the area is ½*6*10 = 30. And that the length of the diagonal d is found by squaring: d^2 = 10^2 + 6^2 = 136, giving d = V136 = 11.7.

Following the line with the formula y = 0.1*x^2 + 1 we see that the line bends upwards and becomes a curve instead, called a parabola. It intersects the diagonal y = x in two points that may be calculated by equating the two formulas:

0.1*x^2 + 1 = x, or x^2 + 10 = 10x, or x^2 -10x + 10 = 0

This quadratics is solved by x = 1.1 and x = 8.9.

The two intersections points thus have the coordinates (1.1, 1.1) and (8.9, 8.9).

From (0,0) we may take a trip with a constant x-number 1, but with a falling up-number y beginning with 4 and ending with -4. We then reach the points (1,4), (2,7), (3,9), (4,10), (5,10), (6,9), (7,7), (8,4), and (9,0). Again, we get a parabola, but now curving downwards. Since the y-number is 0 when x is 0 and 9, its formula could contain the two factors x and (x-9), y = a*x*(x-9). Here we have three unspecified numbers, but in the point (1,4) we only have one:

4 = a*1*(1-9), or 4 = a*(-8), so a = 4/(-8) = -½

The formula of the parabola then is

y = -½*x*(x-9), or y = -½*x^2 + 4½ x

The two intersection points with the line y = 3+x may now be found from inspection to be (1,4) and (6,9). Or from calculations by equating the two formulas:

-½*x^2 + 4½ x = 3+x, that after changing units to -½s gives

x^2 – 9*x = -6 – 2*x, or x^2 – 7*x + 6 = 0 with the solutions x = 1 and x = 6.

The two intersections points thus have the coordinates (1, 4) and (6, 9).

MC18. Numbers in time and space: Change and distribution

The goal is to experience how formulas calculating y from x form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

An unspecified number becomes a letter, and so does an unspecified formula, y = f(x), also called a function.

Writing out fully with units, the number 345 becomes T = 3*B^2 + 4*B + 5. Using y an x instead of T and B, this basic number-formula is called a polynomial, y = 3*x^2 + 4*x + 5. Here 5 is called the level-number, 4 is called the change-number that may be positive or negative if the curve goes

upwards or downwards, and 3 is called a curvature-number that may be positive or negative if the curve curves upwards or downwards.

The basic number-formula also shows the basic change-formulas: Proportional change, y = 4*x; constant or linear change y = 4*x+5; constantly changing or accelerated change y = 3*x^2 + 4*x + 5; power change y = 3*x^2; and exponential change y = 3*2^x.

Saving money in a bank combines a constant change-number, a, with a constant change-percent, r. Here the total input-amount, A, and the total input-percent, R, are the same when recounted in the single numbers respectively: A/a = R/r.

Formulas typically is used to predict future behavior, either with certainty or with probabilities. If that is not possible, instead statistics is used to describe past behavior with a mean and a deviation to predict that with 95% probability the next occurrence will fall within the interval created by the mean plus or minus twice the deviation.

Distributing a total in two different categories leads to cross tables that may be shown with numbers or with percentages. Again, as per-numbers, percents must first be multiplied to unit-numbers before being added. So, care must be shown when describing cross tables with percentages: On my hand I have 2 right R-fingers and 3 left L-fingers. One R and two L carry a green dot. So, we may say "among the green, 1/3 is R" but not "among the R, 1/3 is green" since ½ is green among the R.

MC19. The Three Tales: Fact, Fiction and Fake

The goal is to experience that as qualitative literature, quantitative literature also has the genres, fact and fiction and fake when modeling real world situations (Tarp, 2001).

Fact is 'since-then' calculations using numbers and formulas to quantify and to predict predictable quantities as, e.g., 'since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20'. Fact models can be trusted once the numbers, and the units, and the formulas, and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter.

Fiction is 'if-then' calculations using numbers and formulas to quantify and to predict unpredictable quantities as, e.g., 'if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.

Fake is 'what-then' calculations using numbers and formulas to quantify and to predict unpredictable qualities as, e.g., 'Let us close hospitals since it is cheaper to rest in a graveyard'. Fake models build on dubious assumptions and should be replaced by a political process containing debate before decision.

MC20. Teacher education in CATS: Count & Add in Time & Space

The goal is to experience how teacher education may be different to obtain a communicative turn in number-language education.

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental (Kolb, 1984) learning curricula as, e.g., in Vietnam.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Discussing the difference

We asked what could be different in online education as compared with offline education.

Typically, mathematics is taught as an end-product from a textbook. So, a difference would be to teach mathematics as a process where things in space are acted upon in time. This would make mathematics a number-language with sentences containing an outside subject, a verb, and an inside predicate as in word-language sentences. As the outside thing to talk about we choose a ten-by-ten bundle-bundle board, a BBBoard, where outside totals appear as rectangles within two rubber bands showing the bundle-number and the counting number.

A process-based mathematics contains numerous differences.

● Here, numbers are no longer traditional line-numbers. Now they are rectangular stack-numbers with both a unit-number and a counting-number. Here changing units leads directly to the recount-formula expressing proportionality, to equations solved be recounting, and to per-numbers adding by their areas just as stack-numbers do when adding next-to. So here the three core parts of mathematics, proportionality and equations and calculus, appear from the beginning, and not in the middle and the end of the K-12 curriculum.

● Here, digits are no longer symbols with names to memorize and to learn how to read and write. Now they are icons built in time by uniting five sticks or strokes in one five-icon, etc. Of course, the

icons are not exactly identical to the textbook symbols, but they are close. And, and they give a concrete meaning to the two-digit number 10 by writing it with units, 1 Bundle 0, or 1B0, after having quickly collected all ten sticks into one bundle with none left behind. And then followed by 1B1, 1B2, called 'one-left' and 'two-left' by the Nordic Vikings.

● Here, the counting sequence is no longer a sequence of names like the sequence of months in a year or cities on a railway. Now a combination of digits and units occurs when performing a multiple counting of singles, bundles, bundles-of-bundles etc. This changes one three-digit number to three numberings, and thus makes obsolete the place-value system. So here, 7 changes to 0B7, 45 to 4B5, and 678 to 6BB7B8. And accepting underloads and overloads leads to flexible bundle-numbers with units allowing 48 to be written both as 5B-2, 4B8, and 3B18 tens, which eases standard calculations by making carrying obsolete.

● Here, power is no longer the last operation. Now, power occurs as the first operation since counting ten fingers in 3s allows meeting a bundle-of-bundles, a BB or B^2, since 9 is 3B0, or 1BB0B0, or 100 3s. And counting in 2s allows meeting a bundle-of-bundle-of-bundles, a BBB or B^3, since 8 is 4B0 or 1BBB0BB0B0, or 1000 2s.

● Here, the order of operations is no longer add-subtract-multiply-divide. Now it is the exact opposite. With the process-based math, the first task is to bring an outside total inside by counting it in bundles. So, the first process is to push-away bundles, which may be iconized as a broom so that the outside action 'from 8, push-away 2s' inside becomes the calculation 8/2. The next task is to lift the bundles into a stack, which may be iconized as a lift so that the so that the outside action '4 times lift 2s' inside becomes the calculation 4x2. The next task is to pull-away the stack to look for unbundled singles, which may be iconized as a rope so that the outside action 'from 9, pull-away 8' inside becomes the calculation 9 – 8. Included on-top of the stack, the unbundled may be seen as a decimal number, 4B1 2s, or counted in the unit, 4 ½ 2s, or described at what is pulled-away from the top bundle, 5B-1 2s. Finally, after brought inside as stacks outside totals may now be added next-to as areas like integral calculus, or on-top after recounting has made the units like.

So here, the order of operations is different as the precise opposite. First power comes when bundling bundles, then division pushes-away bundles, then multiplication lift the bundles, then subtraction pulls-away the stack to allow the unbundled to also join the stacks, which finally may add next-to or on-top.

Also, the operations have different meanings, now 8/2 means 8 counted in 2s and not only 8 split in 2, and now 4x8 means 4 8s and not only 3B2 tens, and now subtraction is a means to allow the unbundled to join the stack and give concrete meaning to decimals, fractions, and negative numbers. Finally, addition has two different meanings leading directly to the core of mathematics, calculus and linearity (proportionality). So here, integral calculus occurs in grade one as next-to addition of stacks, which becomes differentiation when reversing the process. Later it reappears in middle school where per-numbers also add by areas since they must be multiplied to unit-numbers before adding.

Since bundle-numbers have units a core task is to change units. Here division and multiplication immediately are combined in the recount-formula that recounts the total T in Bs as T/B Bs, T = (T/B)xB, expressing the proportionality that is visible when observing that increasing the base of a

stack implies decreasing the height to keep the Total unchanged. In its process-based version, the recount formula is a core formula present all over mathematics and science. In its product-based version it is totally absent. Likewise, subtraction and addition immediately are combined in a split-formula that splits T in B and T – B by pulling away B from T, 'T = (T – B) + B'.

In textbooks, the first operation typically is addition based on the statement that '2+1=3 always', which allows 2+3 to be defined as 2+1+1+1. However, this is not a number-statement, but a word-statement saying that "the follower of two is three" useful in time when traveling through a row of places. It is not a statement about uniting two totals in space, since here the units are needed also as, e.g., 2 pairs + 1 triple is 2B1 3s, or 3B1 2s, or 7 1s. Built on the assumption that '2+1=3 always', mathematics becomes 'mathematism' true inside its own world, but seldom outside in the real world where numbers always carry units, and where "additions often folds while multiplication always holds" since 2x3 = 6 simply states that 2 3s may be recounted as 6 1s.

● Here, decimal numbers and fractions and negative numbers no longer have to wait until they are introduced as rational and negative numbers in middle school. Now they arise at once as different ways to see the unbundled placed on-top of the bundle-stack as described above where 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

● Here, counting sequences no longer leave out the units. Now the units are included so that 2 becomes what exists outside, 0-bundle-2, shortened to 0B2. Likewise, 23 becomes 2B3, and 456 becomes 4BB5B6. So here, counting ten fingers in 3s makes 9 a bundle-of-bundles, a BB or B^2, which allows power to be the first operation, and no longer the last. Also, 'past-counting' is now allowed as in France calling 7B1 for 6B11 since they gave up understanding the Vikings in Normandy saying '1 and half-four' meaning '1 and half the way to 4 twenties'. Likewise is 'less-counting' saying "Bundle less nine, B-8, B-7, …, B-1, 1 Bundle, 2B-9, etc.

● Here, functions are no longer subsets in a set-product where first component identity implies second component identity, and which is postponed to middle school. Now a function is a number-language sentence called a formula with a subject, a verb and a predicate as in word-language sentences. Only here, the inside predicate is a prediction of what happens with the outside subject, so that T = 2x3 = 6 is an inside prediction of the outside fact that 2 3s may be recounted as 6 1s.

● Here, equations no longer are identical number-names that may be transformed by the same action on both sides, and which is postponed to middle school. Now equations describe reversed processes from the beginning. Here, the addition equation 'u+3 = 7" asks "7 splits in 3 and something?". That of course is found by splitting 7 in 7 = (7 – 3) + 3, so that u = 7-3. This resonates with the formal definition of subtraction: "7-3 is the number n that added to 3 gives 7", or "if n+3 = 7, then n = 7-3". So, we see that the equation is solved when reversing the process by moving a number across the equal sign with the reverse calculation sign.

Here, the multiplication equation 'u*3 = 12" arises when recounting from tens to icons leads to asking "How many 3s in 12?". That of course is found by recounting 12 in 3s as 12 = (12/3)*3 so that u = 12/3. This resonates with the formal definition of division: "12/3 is the number n that multiplied with 3 gives 12", or "if n*3 = 12, then n = 12/3". So, again we see that the equation is solved when reversing the process by moving a number across the equal sign with the reverse calculation sign.

Likewise, reversing power-operations leads to equations as u^3 = 8 solved by the factor-finding root u = 3√8, and to equations as 2^u = 8 solved by the factor-counting logarithm u = log2(8). Finally, integration moves across as differentiation. So now there is no need for the abstract algebra group concepts as communicative and associative laws as well as neutral and inverse elements, which step by step motivates the 'do the same to both sides outside the brackets´ method that is presented fully to teachers but only in parts to students.

● Here, multiplication tables are no longer sequences learned by heart. Now 6x7 is 6 7s in its own right. And it may or may not be recounted in tens. And if so, it takes place on the BBBoard where 6 and 7 are seen as B-4 and B-3 respectively. This transformation of 6x7 into (B-4)x(B-3) introduces early algebra where the BBBoard shows the process where the 6 7s are left when pulling-away from the 10 Bundles, first 4 bundles then 3 bundles and finally adding the 4 3s that have been pulled away twice, thus clearly showing that minus times minus gives plus. So, to get the answer quickly, from 10 you subtract (3+4), see them as bundles, and add 3x4. So, all you have to memorize is the lower left corner on the BBBoard where you build up of from 2 2s to 5 2s, then from 2 3s to 5 3s, then with 4s and finally with 5s. Multiplication on a BBBoard thus offers a middle way in 'the multiplication war' (Economist 2021).

● Here, squares are no longer special numbers in the multiplication tables. Now they are Bundle-Bundles on a BBBoard that may be built by always adding an extra line horizontally and vertically and subtracting the upper right corner-number that was added twice. So now squares form a sequence of their own quickly learned. And here the square root no longer has to wait until irrational numbers are introduced in middle school. Now they arise from a simple question "How to square a rectangle without changing its total?" The BBBoard shows that you simply move half the overload from vertical to horizontal position or vice versa. The answer then needs to be reduced a little so that there is also something for the upper right corner.

Finally, here quadratics no longer must wait to high school. Now they appear on a (x+a)*(x+a) board when two rubber bands show the two squares, x^2 and a^2, as well as the two a*x rectangles that all disappear if x^2 + 2ax + b = 0, only leaving a^2 – b as (x+a)^2 after being squared.

● Here, proportionality no longer must wait to middle school. Now it appears when recounting a total in two units creates per-numbers bridging the units by recounting in the per-number: with 4kg per 5$, to answer the question '12 kg = ?$', we simply recount 12 in 4s: 12 kg = (12/4)*4kg = (12/4)*5$ = 15$. Likewise, with answering the inverse question '?kg = 20$'. Or we may just recount the units: $ = ($/kg)*kg = (5/4)*12 = 15.

● Here, fractions no longer are rational numbers that can add without units. Instead, fractions like per-numbers are operators needing numbers to become numbers before adding. So, we must always ask "fractions of what?" Counting red apples provides an example. Here ½ of 2 apples plus 2/3 of 3 apples gives (1+2)/(2+3) of 5 apples, i.e., 3/5 of 5 apples, and not (1*3+ 2*2) of 2*3 apples, i.e., 7/6 of 6 apples. So ½ + 2/3 is 7/6 only if they are taken of the same total, just as 2+1 is 3 only if they have the same unit. So addition has only meaning inside a bracket with the common units outside, as expressed in the distributive law, p*r + q*r = (p+q)*r. Shortening or enlarging fractions now simply means removing or adding a common unit.

● Here, trigonometry no longer must wait to high school where plane geometry as well as coordinate geometry has been taught. Now it appears as per-numbers coming from recounting the sides in a rectangle split by a diagonal. This means that concrete examples from all STEM areas may be looked at very early.

● Here, coordinate geometry no longer comes after plane geometry. Instead, it serves its name by coordinating algebraic calculations with standard geometrical forms which may or may not later be studied by themselves in plane geometry.

● Here, addition no longer is the first operation to teach. Instead, it is the last since counting and recounting first must bring outside totals inside to be added in two different ways, next-to by their areas, or on-top after recounting has made the units like. So, when finally introduced, addition and its reverse leads directly to the core of mathematics, calculus and proportionality. And, with totals occurring as flexible bundle numbers with units, overloads and underloads make carry and borrow math unneeded.

● Here, functions no longer must wait until late middle school to be defined as a rule that assigns to each element in one set one and only one element in another set, which is the soft version as the one taught in teacher education: a function is a subset in a set-product where first component-identity implies second-component identity. Instead function now occur as number-language sentences with a subject, a verb and a predicate just as in word-language sentences. And taking on the name ‘formula’ when containing unspecified letter-numbers.

● Here, calculus no longer must wait to be taught to few the last year of school. Now it is taught to all the first year when totals are added next-to as areas. In middle school it is again taught to all when adding piecewise constant per-numbers and when adding fractions in cross tables. So, in principle, it can be taught to all in high school also as a way to add locally constant per-numbers from science. Especially if the natural order is followed where integration comes before differentiation invented to recount areas as changes to allow many changes to be added as one change only from the initial to the end number.

● Here, algebra no longer is an abstract activity looking for pattens. Instead, algebra takes on its original Arabic meaning, to reunite like and unlike unit-and per-numbers as shown in the Algebra-square. Here the rules for calculating on with unspecified letter-numbers come automatically recount totals from icons to tens on a BBBoard.

● Here, modeling no longer must obey the parole “Of course, math must be learned, before it can be applied”. As a number-language with sentences assigning inside numbers and calculations to outside totals, modeling occurs always. Dut, as in the word-language, it is also important to evaluate if the sentences express fact, fiction or fake. So, these three genres now become an important part of using the number-language to be skeptical towards any quantitative literature.

Testing the difference

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of ManyMath difficult in a traditional education, except for where it is stuck, e.g., division and fractions and letter calculations. But ManyMath may be tested outside the main track: in

preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

As to teacher training, the MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Mastering Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as, e.g., in Vietnam.

Search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a full first-generation curriculum. Asked "How old next time?" a three-year-old will say four showing four fingers, but the child will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers 'T = (T/B)xB' saying "T contains B, T/B times."

Conclusion

Inspired by the difference between teaching mathematics on-line outside the classroom and teaching it off-line inside we formulated the question about what else could be different that led to a possible answer: a difference could be to teach mathematics as a process in time instead of as a product in space. Or, in other words, a difference could be to teach mathematics as an inside number-language about outside quantities parallel to the inside word-language about outside qualities. Or again in other words, a difference could be to let mastery of the outside existence, Many, precede mastery of the inside essence, Mathematics.

And yes, mastery of Math may be preceded by mastery of Many that in return automatically leads to mastery of the Math core. Which allows details to be taught as footnotes to those who may be interested while the rest may focus on using their mastery to work with tales about Many and to discuss which of the three genres they belong to, fact or fiction or fake. And since outside Many are brought inside as total counted and recounted in flexible bundle-numbers with units as shown in the polynomial form of numbers, the basic inside tales about outside Many could be about things and actions on a ten-by-ten Bundle-Bundle Board, a BBBoard. So yes, a difference is to teach mathematics as a process in time instead of as a product in space.

Of course, working on a BBBoard, learners may not learn school or university mathematics in a strict traditional form, but they learn to communicate about Many; and by always including units they avoid the tradition's mathematism with its fake addition claims only holding inside but often folding outside the classroom.

So, it is possible to have a communicative turn in number-language education like the communicative turn in word-language education that took place in the 1970. Where outside-inside use of outside examples to be described by inside language-sentences was allowed to precede or even replaced by inside-inside use of inside sentences to be described by inside grammar-sentences.

What remains is testing to see if the difference expressed in the above micro-curricula makes a difference. This may be impossible inside traditional education because of heavy cost to change textbooks and to establish in-service teacher training. But it may be possible outside in math labs and in special education where the goal could be to use BBB to obtain BBB, i.e., to use Bundle-Bundle-Boards to Bring Back Brains. But, although designing and testing micro-curricula go together in design research, they should not be mixed: Architects design houses, they do not build them. Likewise with curriculum architects.

And in the end the core question is what right allows traditional education to replace the child's innate concrete number-language with a foreign abstract language? Is this not just another example of using the power of an imported institution to colonize the native brains?

So, a final question could be: Forcing mastery of inside mathematics to precede mastery of outside Many, does that follow the 17 United Nations Sustainable Development Goals? Here goal 4 about quality in education formulates the wish to "ensure inclusive and equitable quality education and promote lifelong learning opportunities for all". And here target 4.6 states the subgoal to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy".

A communicative turn in number-language education may allow this development goal to be reached now. Postponing this turn clearly will not and may instead keep widespread innumeracy alive indefinitely. Which of course instead may serve well the wish for ever more funds for teacher education of teachers and researchers as a goal displacement warned against in sociology (Bauman, 1990; Arendt, 1963).

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education. The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not learn from children instead?

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Extended version at
http://mathecademy.net/manymath-2030/

A text-free math education found by difference research for protection against alien artificial intelligence 2009

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Artificial Intelligence, AI, friend or foe to math education? Some warn that AI develops into an alien intelligence infiltrating all that is text-bound in a library. So, to protect math education from this, we ask if math may be taught and learned in a text-free form out of reach to AI. Having not yet met math in its text-bound form, 3year old children give the answer by using bundle-numbers with units as 2 2s thus seeing what exists in time and space. This discovery allows difference research to use sociological imagination to design text-free curricula giving priority to outside existence over inside essence, and to use flexible bundle-numbers with units in tales about things and actions on a Bundle-Bundle-Board.

PROTECTING MATH AND ITS EDUCATION FROM BEING INFILTRATED BY AI

It seems only natural that mathematics is a core subject in education because of its many important applications in core societal matters within economy, science, technology, engineering, etc. So, all we need are universities to define and develop mathematics, and to teach teachers how to teach it to students in mathematics classes that it may later be applied in other classes. It is as simple as that. And of course, it goes without saying that first mathematics must be learned to be applied later.

However, a core part of mathematics is geometry, in Greek meaning to measure earth. As well as algebra, in Arabic meaning to reunite changing and constant unit- and per-numbers with addition, multiplication, integration and power. Both thus indicate that inside mathematics has outside roots. So instead of teaching the abstract before its concrete roots, maybe it should be the other way around as suggested by existentialist philosophy (Sartre, 2007) holding that outside existence precedes inside essence that might be power-charged by being not natural but socially constructed (Foucault, 1972)?

Peer-reviewed research may give an answer. But can it be trusted, John Bohannon asked in his 2013 article "Who's Afraid of Peer Review?". And if nobody teaches existence before essence, then peer-review might reject all articles about this arguing they don't discuss or extend established knowledge. So 'difference research' searching for differences making a difference (Tarp, 2018) typically has its papers rejected at conferences' peer-reviews performed by the other contributors. Until now where AI, Artificial Intelligence, with its access to the library may write research articles also in huge numbers. In May 2023, 350 leading scientists and notable figures signed a common statement warning against AI by saying that "Mitigating the risk of extinction from AI should be a global priority alongside other societal-scale risks such as pandemics and nuclear war". A similar warning is found on the YouTube video "AI and the future of humanity" given by Yuval Harari at the Frontiers Forum, May 2023.

To protect math education from infiltration by an alien intelligence seems almost impossible since both mathematics and education are text-bound. So, we may ask: "Can mathematics be taught and learned in a different text-free version?" Let us see what difference research may offer here.

To look for a different version we listen to brains that have not yet been exposed to books, young preschool children. So, we ask a 3year old child "How many years next time?" Typically, the answer is four showing four fingers. But presented by four fingers held together two by two, the child protests: "That is not four, that is two twos". The child thus sees what exists in space and time: Bundles of twos in space, and two of them when counted in time. These rectangular bundle-numbers with units are different from the textbook's linear number-line numbers without units.

Based on this discovery, difference research now uses sociological imagination (Mills, 1959) to design text-free curricula giving priority to outside existence over inside essence, and using bundle-numbers.

Designing micro curricula

Looking at our 5 fingers we observe that bundle-numbers may be flexible when bundle-counting. If we shorten 'Total' to T and 'Bundle' to B we have: T = 0B 5 = 1B 3 = 2B 1 = 3B -1 2s, or T = 1BB 0B 1 = 1B^2 0B 1 = 101 2s if we leave out the units. Here 1B 3 may be called an overload, and 3B -1 may be called an underload. Counting all ten fingers, we get T = 2BB 0B 2 = 1BBB 0BB 1B 0 = 1010 2s. Counting them in 3s, we get T = 3B 1 = 1BB 0B 1 = 101 3s. We notice that with units, the place value system becomes redundant, and that power is the first operation we meet.

Flexible bundle-numbers may also be used with ten as bundle-size: T = 68 = 6B 8 = 5B 18 = 7B -2 tens. This eases standard operations and makes also carrying and borrowing redundant:

T = 23+59 = 2B 3+5B 9 = 7B 12 = 8B 2 = 82; and T = 83 – 59 = 8B 3 – 5B 9 = 3B -6 = 2B 4 = 24

T = 3*59 = 3*5B 9 = 15B 27 = 17B 7 = 177; and T = 84 /3= 8B 4 /3 = 6B 24 / 3 = 2B 8 = 28

Here we met Many in space. In time we also include the unit in the counting sequence: 0B 1, 0B 2, …, 0B 9, 0B ten or 1B 0, 1B 1 etc., enjoying that 'eleven' comes from the Vikings saying '1 left, 2 left'.

We now look at the counting process by asking "How many 2s in 8?". To answer, first we push-away the 2s, which allows division to be iconized as a broom, 8/2. Then 4 times we stack the 2s, which allows multiplication to be iconized as a lift, 4x2. We may now write the result as a 'recount-formula":

8 = 4 2s = 8/2 2s = (8/2) x 2, or T = (T/B) x B with unspecified numbers.

So, with bundle-counting changing the units from 1s to bundles we get the proportionality formula directly. Also, we meet a formula or function as a number-language sentence with an outside subject, a verb, and an inside predicate as in word language sentences. Also we meet solving equations since our question could be reformulated as ux2 = 8 where recounting 8 in 2s gives 8 = (8/2)x2. The equation thus is solved by u = 8/2, i.e., by 'moving to opposite side with opposite sign'. Which also follows from the formal definition saying that "8/2 is the number u that multiplied with 2 gives 8, ux2 = 8".

Likewise, the equation u+2 = 5 is solved by moving over as u = 5-2 since u is a placeholder for a number that with 2 added gives 5, thus found by reversing the action and pulling-away the 2 again.

Solving equations by ‘opposite side & sign’ is a difference to the traditional balancing method ‘do the same to both sides’ introduced to motivate teaching teachers the abstract algebra concept ‘group’.

When bundle-counting, we also meet decimals, fractions, and negative numbers to account for the unbundled singles: First we pull-away the stack which allows subtraction to be iconized as a rope, e.g., 9 – 4x2 = 1. Then we place the unbundled on-top of the stack, as a decimal number, 9 = 4B1 2s, or as a fraction when counted in 2s also as 1 = (1/2) x 2, 9 = 4 ½ 2s, or as a negative number showing in space what is missing for the next bundle, or what have been pulled away from it in time, 9 = 5B -1.

Above we saw that recounting from tens to icons solves equations: u*6 = 30 = (30/6)*6, so u = 30/6.

Recounting from icons to tens gives multiplication tables that may be seen on a ten-by-ten Bundle-Bundle-Board, a BBBoard where 6*7 may be seen as 6 7s or as ‘(B-4)*(B-3) which leads to early algebra since the 6 7s are left when from the ten Bs we pull-away 4Bs and 3Bs, and then add the 4 3s we pulled-away twice: T = 6 7s = 6*7 = (B-4)*(B-3) = 10B - 3B - 4B + 4 3s = 3B + 1B 2 = 4B 2 = 42.

Inside, outside totals become rectangular stacks as T = 8 4s = 8*4, or squares in the case of bundle-bundles T = 4 4s = 4*4. So, we may ask “How to square a rectangle?”, e.g., T = 8 4s = B*B = B^2 where B is called the square root of 32, B = √32, iconized by half a perimeter. We begin by adding half of the excess, ½*(8-4) 4s= 2 4s to both sides of the 4x4 square, which gives a 6x6 square with a total of 36. This is too much since also the upper right corner must be included. So instead we add a number t determined by (4+t)^2 = 32. On a drawing we see that the square (4+t)^2 has four parts, 4^2 and t^2 and 2*4*t, so (4+t)^2 = 4^2 + t^2 + 8*t = 32, or t^2 + 8*t + 16 = 32 , or t^2 + 8*t – 16 = 0, called a quadratic now rooted in transforming a 32-rectangle into a ‘4+t square’. Likewise, we may rewrite the quadratic t^2 + b*t + c = 0 as t^2 + 2*b/2*t + (b/2)^2 = (b/2)^2 – c = D/4, or as (t+ b/2)^2 = D/4 where D is called a discriminant. This shows that a D/4 rectangle may be transformed into a b/2+t square thus providing the solution to the quadratic as t = -b/2 ±√(D/4).

Changing the unit, the recount-formula roots the proportionality formula T = a*b recounting T in bs. Examples may be meter = meter/sec*sec, recounting a distance in seconds, or $ = $/kg*kg, recounting dollars in weight, thus creating ‘per-numbers’ as meter/sec, $/kg, etc. Or part = part/whole * whole, recounting a part in wholes and becoming fractions with like units. In time, the end value may be recounted in the start-value: end = end/start * start, where end/start is the change-factor, e.g., 105%.

Finally, in a rectangle with a base, b, a height, h, and a diagonal, d, mutual recounting roots trigonometry as per-numbers: h = (h/b)*b = tangent(Angle)*b where tan(A) is the per-number h/b. Splitting the circumference of half a unit-circle in n parts gives the number □ = n*tan(180/n) for n big.

Once counted, stacks may be added, on-top after recounting has provided like units, or next-to as areas thus rooting integral calculus, as well as differential calculus when reversing asks “2 3s + ?5s = 4 8s”.

Per-numbers add in mixture problems as “2kg at 3$/kg plus 4kg at 5$/kg gives what?”. With like units, the unit-numbers 2 and 4 add directly. But per-numbers must be multiplied to unit-numbers

before adding as the areas created by the multiplication. So, mixture problems root integral calculus, becoming differential calculus when the problem is reversed. So, integral calculus should be introduced before its inverse differential calculus show that many differences add as one difference.

Fact, fiction & fake, the 3 modeling genres

With mathematics as a number-language for outside things in space and actions in time, its quantitative literature or models needs to be divided into fact, fiction or fake, the same genres used in the word-language for qualitative literature. Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas; and that need to be checked for units and correctness. Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas; and that need to be supplied with scenarios with alternative assumptions. Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas; and that need to be replaced by word stories (Tarp, 2001).

Conclusion, yes we can

So, the answer to our question is: Yes, mathematics may be taught and learned in a version free from text and AI if we follow the advice of existentialist philosophy and let outside text-free existence precede inside text-bound essence; and use children's own flexible bundle-numbers with units instead of the textbook's line-numbers without units (Tarp, 2020-2023). This allows a communicative turn in number-language as the one that took place in the word-language education in the 1970s (Widdowson, 1978). Core mathematics will then be learned automatically as tales about Many on a Bundle-Bundle-Board. As to teacher education and research, the MATHeCADEMY.net offers a corresponding free online teacher education using Bundle-numbers with units where learning takes place through guided activities that allow questions to be answered by the subject in the laboratory instead of by an instructor in a library. The academy also contains many articles showing the possibilities of leaning to master Many before mathematics. They propose that studies of the discovered differences are carried out as design research (Bakker, 2018). And studies on the ability to bring back brains from special education is especially needed in a subject that has deprived children of their own bundle-numbers with units.

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Modeling eased by demodeling and rerooting 2209

Allan Tarp, MATHeCADEMY.net

Modeling should motivate mathematics education, but is not always that easy. Could this be different? Difference research searching for differences making a difference suggests that inside concepts may be de-rooted from the outside world by getting different names and meanings. So de-modeled they may retake their original roles allowing mathematics to again become a number-language communicating about the outside world. The search found children's own flexible bundle-numbers with units, that allow counting and recounting to precede adding. This leads directly to the core of mathematics, using proportionality and calculus to re-unite changing and constant unit-numbers and per-numbers.

DOES MATHEMATICS MODELING HAVE TO BE SO DIFFICULT?

Inspired by the first International Congress on Mathematical Education, ICME-1, I joined the student revolt in 1969 to secure that mathematics would no more be taught without linking it to its use through modeling. I was allowed to write a master thesis on modeling where I chose Game Theory. In 1974 I published my first textbook "Mathematical Growth Models" showing how calculus is modeling predictable non-constant change. And at the ICME-3 I presented a poster "Mathematics, a collection of arbitrary theoretical structures, or model-building of the real world" as well as a short oral address in English and French "Mathematics, an integral part of the real world". From 1975 I worked with Mogens Niss for three years at Roskilde University. We were both interested in modeling the Danish macro-economic cycle. Niss preferred the actual government model where the mathematics was so complicated that it could not be addressed in high school. I saw the model, not as 'since-then' fact model as in physics, but as an 'if-then' fiction model based upon assumptions that could be different. And here Ockham's razor says with two different models to explain the same, you should prefer the simpler one. So, I worked out a simple linear model that could be used in high school (Tarp, 2001). Niss stayed at the university and I returned to the high school and joined a group that succeeded changing the precalculus curriculum so that polynomials of first and second degree were replaced by linear and exponential functions so modeling could enter the classroom. I was allowed to test a special curriculum showing how statistical tables with categories divided into subcategories and changing over time may be modelled by statistics, linear and exponential change (Tarp, 2021), which allowed all students to pass the exam successfully. But the standard textbooks still presented pure mathematics to be learned before it could be applied. Linear and exponential functions were presented as examples of the function concept that was presented as an example of a set-relation. So instead of presenting an abstract concept though its examples, it was presented as an example of a more abstract concept. This was difficult to many students and resulted in so poor exam results, that it was suggested to remove precalculus as a mandatory class at the reform in 2005.

When my students asked for examples, I chose saving money by adding 5 $ per week at home or 5% per year in a bank. "Why can't we call this growth by adding and by multiplying?" the students asked. This made a difference. So, I used deconstruction (Derrida, 1991) to develop a Difference Research searching for differences making a difference (Tarp, 2018). And the government accepted my advice that precalculus should stay as a mandatory subject, but the function concept should be replaced with

variables as in in physics and economics, so that we write y = b+a*x instead of f(x) = b+a*x. However, modeling still was difficult to many students. So, I turned to primary school to see if deconstruction by listening to children would make a difference here also. I found that the children see four fingers held together two and two, not as 4, but as two 2s, thus using bundle-numbers with units for what exists, bundles of 2s in space, and 2 of them when counted in time. I described the potentials of deconstruction and bundle-numbers in several MrAlTarp YouTube videos, and in 10+15+16 contributions to the ICME 10-12 (Tarp, 2012). For ICME 13 I wrote 9 papers but was allowed only 1. For the ICMI Study 24 on curriculum I designed several micro curricula where the math core was re-rooted and renamed by the process of counting and recounting before adding (Tarp, 2018, 2020). Also, I had designed a teacher education academy in 'ManyMath', MATHeCADEMY.net, Count & Add in Time & Space. Typically, modeling is eased by demodeling and re-rooting when tested in math labs, libraries and private education. So, the time has come for others to perform a large-scale testing.

Demodeling: from the inside to the outside and back

Modeling goes from the outside to the inside and back. Demodeling does the opposite by going from the inside to the outside and back. So demodeling begins with the core of inside mathematics as seen on a calculator: digits, operations, brackets, multidigit numbers, decimal point, and an equation sign. And then asks the question "What outside things and actions have rooted these inside concepts?"

Demodeling digits and multidigit numbers

Digits and letters may both be seen as symbols. But digits may also be seen as icons with as many sticks as they represent, five sticks in the five-icon, etc., if written 'less sloppy'. A sequence of digits may be seen as one multidigit number obeying a place-value system with ones, tens, hundreds, etc., and seldom with the word 'bundle-of-bundles' used for 'hundred'. But a multidigit number may also be seen as rooted in several numberings of unbundled, bundles, bundles of bundles, etc. (Tarp, 2018).

Recounting a total T of ten in 3s we get T = 3 Bundles 1, or T = 3B 1, or T = 1BB 0B 1 = 1B^2 0B 1 since 3 bundles is 1 bundle-of-bundles. So, bundling bundles roots power, and bundle-counting totals roots polynomials, T = 345 = 3*B^2 + 4*B + 5*1. And it also roots functions as number-language sentences with an outside subject, a verb, and an inside predicate as in word-language sentences.

To bundle-count a total, bundles are pushed away and lifted into a stack to be pulled away to look for unbundles singles. These actions root division as an icon for a push-away broom, multiplication as a lift, subtraction as a pull-away rope, and addition showing two ways to unite stacks, on-top or next-to. Placed on-top of the stack, the unbundled may be seen as a decimal number, or as a fraction when counted in bundles also, or described by what has been pulled away in time from the next bundle, or what is missing in space for another bundle. Recounting 9 in 2s, the end result may thus be written as

T = 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s (an underload), or with an overload, T = 3B3 2s, and T = 2B 5 2s.

As to the process, to recount 8 in 2s we push-away 2s to be stacked as 4 2s, which may be written as

8 = (8/2) x 2 = 8/2 2s, or T = (T/B) x B = T/B Bs with unspecified numbers.

By changing the unit, this recount-formula roots the proportionality formula T = a*b recounting T in bs. Examples may be meter = meter/sec*sec, recounting a distance in seconds, or $ = $/kg*kg,

recounting dollars in weight, thus creating ‘per-numbers’ as meter/sec, $/kg, etc. Or part = part/whole * whole, recounting a part in wholes and becoming fractions with like units. In time, terminal = terminal/initial * initial recounts the end-value in start-values.

A rectangle has base, b, height, h, and diagonal, d, raising an angle, A. Here, mutual recounting roots trigonometry: h = (h/b)*b = tan(A)*b, h = (h/d)*d = sin(A)*d, and b = (b/d)*d = cos(A)*d. In half a radius 1 circle, splitting the circumference in n parts gives the pi-number □ = n*tan(180/n) for n big.

Recounting between icons and tens root equations and early algebra.

Recounting from tens to icons, we ask “How many 2s in 8?” This roots equations solved by recounting 8 in 2s: u*2 = 8 = (8/2)*2, so u = 8/2 from pushing-away 2s from 8, showing that an equation is solved by reversing the process, i.e., by moving a number to the opposite side with the opposite sign. This follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, u*2= 8. ‘To opposite side with opposite sign’ may be rooted outside, while the inside balancing method is derived from abstract algebra concepts as group, inverse and neutral elements, associativity and commutativity.

Recounting from icons and tens, we ask “6 7s is how many tens?” This roots early algebra if allowing underloads: T = 6 7s = 6*7 = (B-4)*(B-3) = BB -4B -3B + 4 3s, as seen on a B*B square where the 6 7s is left when from ten bundles we pull-away 4 bundles and 3 bundles, and finally add the 4 3s that was pulled-away twice.

Once counted, stacks may be added, on-top after recounting provides like units, or next-to as areas thus rooting integral calculus, as well as differential calculus when reversing asks “2 3s + ?5s = 4 8s”.

Per-numbers are added in mixture problems as “2kg at 3$/kg plus 4kg at 5$/kg give what?”. With like units, the unit-numbers 2 and 4 add directly. But per-numbers must be multiplied to unit-numbers before adding as the areas created by the multiplication. So, mixture problems root integral calculus, preceding differential calculus occurring when the problem is reversed.

Inside, outside totals become rectangular stacks as T = 9 5s = 9*5, or squares in the case of bundle-bundles, T = 5 5s = 5*5. So, we may ask “How to square a rectangle?”, e.g., T = 9 5s = B*B = B^2 where B is the square root of 45, B = √45, iconized by half a perimeter. Here, half the excessing 4 5s is placed to the right to create a 7*7 square except for the upper 2*2 right corner. Again, half of this is placed to the right to create a 6 5/7-square, since ½*4 = 2 = (2/7)*7. This is close: (6 5/7)^2 = 45.1.

Fact, fiction and fake, the 3 modeling genres

With mathematics as a number-language modeling outside things in space and actions in time, its quantitative literature needs to be divided into fact, fiction or fake, the same genres used in the word-language for qualitative literature. Fact stories are ‘since-then’ stories that quantify and predict predictable quantities by using factual numbers and formulas; and that need to be checked for units and correctness. Fiction stories are ‘if-then’ stories that quantify and predict unpredictable quantities by using assumed numbers and formulas; and that need to be supplied with scenarios with alternative assumptions. Fake stories are ‘what-then’ stories that quantify and predict unpredictable qualities by using fake numbers and formulas; and that need to be replaced by word stories (Tarp, 2001).

Discussion and conclusion

What should name a mathematical concept? Its outside root, or its inside relation to other concepts on a lower or higher abstraction level? Should a function be named a 'sentence' using a verb to link an outside subject to an inside calculation? Or a 'standby calculation' with specified and unspecified numbers? Or an example of a 'many-to-one set relation'? In fact, what we here ask is: The goal of math education, is that to learn to master math to later master Many, or the other way around? Holding that existence precedes essence, existentialist philosophy (Sartre, 2007) prefers the latter. And we see that to master Many to later master mathematics by re-rooting often implies a different name and order of de-rooted concepts. Since, to bring outside Many inside, counting and recounting precedes addition, and 2D flexible bundle-numbers with units replace the traditional 1D line-numbers without, and recounting to change units leads to proportionality, T = (T/B)xB, to per-numbers with different physical units, and to fractions with like units. Likewise, recounting between icons and tens leads to equations solved in the original way by moving to 'opposite side with opposite sign', and to early algebra when 6 7s is rewritten as (B-4)*(B-3). And, recounting rectangular stacks as squares roots the square root, and to quadratic equations. And in a stack halved by its diagonal, mutual recounting between the sides roots trigonometry thus preceding both plane and coordinate geometry. And once counted and recounted, stacks may be added in two ways, on-top, after recounting has made the units like, or next-to as areas thus rooting integral calculus, and differential when the question is reversed. And, since per-numbers must be multiplied to unit-numbers before adding they also add by areas as integral calculus, facilitated by differential calculus trying to rewrite area-strips as differences so the sum of many differences becomes one difference between the end and start values.

So, demodeling and re-rooting inside de-rooted concepts may ease modeling: Now you don't first learn about inside essence, but learn math directly by manipulating and communicating about (Widdowson, 1978) outside existence, e.g., things and actions on a ten-by-ten Bundle-Bundle-Board (Tarp, 2023).

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Children’s own bundle numbers with units may allow effectful informal learning 3010

Allan Tarp, MATHeCADEMY.net, 2023

The goal of math education is typicality seen as learning to master math to later master Many. So, a difference could be to see the goal of math education as learning to master Many directly to indirectly on the way learning the core math as seen on a calculator: digits, operations, and equations, all occurring as products in space. Unless they occur as processes in time where outside Many precede inside math. And the math core does become different when created as tales about Many displayed as rectangular stacks of bundles on a plastic ten-by-ten bundle-bundle board, a BBBoard. Here both digits and operations arise as icons: digits when uniting sticks, and operations when division pushes away bundles that multiplication lifts onto a stack that subtraction pulls away so the unbundled may be included as decimals, fractions, or negatives. Once counted, a unit may be changed by recounting predicted by a calculator. Here recounting from tens to icons and vice versa leads to equations, and to tables displayed as the stack left when removing the two surplus stacks from the full bundle-bundle on a BBBoard. Here recounting in two physical units leads to per-numbers bridging the two units and becoming fractions with like units. Here recounting the sides and the diagonal in a stack leads to trigonometry before geometry. Finally, once counted, and recounted, totals may add on-top after recounting has provided like units, or next-to as areas as in integral calculus that becomes differential calculus when reversed. As operators needing numbers to become numbers, per-numbers and fractions also add by their areas since they need to be multiplied to unit-numbers before adding. So, outside totals inside appear in an ‘Algebra Square’ where unlike and like unit-numbers and per-numbers are united by addition and multiplication, and by integration and power. And later again split by the reverse operations, subtraction, and division, and by differentiation and root or logarithm.

The approach is exemplified in the YouTube video "Online math opens for a communicative turn in number language education." under the link "https://youtu.be/36tan-gGjJg"