Math Ed & Research 2021

No Power Point Presentations

- From loser to user, from special to general education, learning inside mathematics through outside action
- Teaching mathematics as communication, trigonometry comes before geometry, and probably makes every other boy an excited engineer
- Towards truth, beauty and goodness in mathematics education
- Replacing academic number sense with children's own many sense

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PREFACE

FROM LOSER TO USER, FROM SPECIAL TO GENERAL EDUCATION, LEARNING INSIDE MATHEMATICS THROUGH OUTSIDE ACTIONS

Although eager to begin school, some children soon fall behind and are sent to special education teaching the same at a slower pace. Wanting mathematics education to be for all leads to the question: Is this so by nature or by choice? Can it be otherwise? Observing how children communicate about Many before school, this paper asks what kind of mathematics can be learned if accepting the bundle-numbers as 2 3s that children bring to school. Using Difference Research, it turns out that accepting numbers with units means that counting, recounting and solving equations come before adding on-top or next-to introduce integral and differential calculus as well as proportionality in early childhood education. So, it is possible to institute an ethical mathematics education that transforms loser to users returning to general education as stars teaching fellow students and teachers how to master Many.

INSIDE, CHILDREN ADAPT SMOOTHLY TO THEIR OUTSIDE WORLD

It is glad to see how children vividly communicate about Many before school. And it is sad to see how they stop after beginning school, and how more and more are excluded from general education and sent to special education. A day inside a classroom tells you why. The students no more communicate about Many, instead a textbook mediated by a teacher teaches them about what they need in order to communicate: multidigit numbers obeying a place value system, first to be added then subtracted with no respect to their units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers.

In a special education class, the same is taught but at a slower pace. Which makes you wonder: With their preschool foundation, can children learn number-language through communication as with the word-language (Widdowson, 1978)? And, can this mastery of Many lead to mastery of mathematics later, if needed? In which case, mastery of mathematics wouldn't be the only way to master Many.

Looking for differences, Difference Research (Tarp, 2018) searching for differences that might make a difference may inspire micro curricula to be tested using, e.g., Design Research (Bakker, 2018).

MEETING MANY, CHILDREN COUNT WITH BUNDLES AS UNITS

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, "That is not four, that is two twos", thus describing what exists: bundles of 2s, and 2 of them.

Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And they also use full sentences as in the word-language with a subject 'that', and a verb 'is', and a predicate '2 2s', which abbreviated shows a formula as a number-language sentence 'T = 2 2s'.

A BUNDLE-NUMBER CURRICULUM FOR SPECIAL EDUCATION

Listening to special education students helps understanding when and why they fall behind. Inspired by this we may design question guided micro curricula, MC, to further develop the number-language and mastery of Many they acquired before school, allowing them an comeback to general education.

MC01. Digits

The tradition presents both digits and letters as symbols. A difference is letting students experience themselves digits as not symbols, but icons with as many sticks or strokes as they represent if written less sloppy (Tarp, 2018). In this way students see inside icons as linking directly to outside degrees of Many. And that ten has no icon since, as a bundle it becomes the unit, so that two-digit numbers really are two countings of bundles and unbundled singles.

A guiding question can be "There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?" Outside material could be sticks, a folding ruler, cars, dolls, spoons, etc.

Discussing why numbers after ten has no icon leads on to bundle-counting.

MC02. Bundle-Counting Sequences

Using a place value system, the tradition counts without bundles. A difference is to practice bundle-counting in tens, fives, and threes. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names 'eleven' and 'twelve' are Viking names meaning 'one left' and 'two left', and that the name 'twenty' has stayed unchanged since the Vikings said 'tvende ti'; and that English roughly is a mixture of Viking words labeling concrete things and actions, and French words labeling abstract ideas. The Viking tradition saying 'three-and-twenty' instead of 'twenty-three' was used in English for many years (see, e.g., Jane Austen). Now it stops after 20. The Vikings also counted in scores: 80 = 4 scores, 90 = half-5 scores.

A guiding question can be "Let's use the word bundle when bundle-counting in tens, in 5s and in 3s."

Outside material could be fingers, sticks, cubes, and a ten-by-ten abacus. Using fingers and arms we can count to twelve, also called a dozen. Using cubes, the bundles are stacked on-top of each other.

First, we count in 5s (hands): 0*B*1, …, 0*B*4, 0*B*5 or 1*B*0, 1*B*1, …, 1*B*5 or 2*B*0, 2*B*1, 2*B*2.

Then we count in 3s (triplets): 0*B*1, …, 0*B*3 or 1*B*0, …, 1*B*3 or 2*B*0, …, 2*B*3 or 3*B*0, 3*B*1, ...

Counting cubes in 3s, 3 bundles is 1 bundle-of-bundles or 1*BB* in writing, so we repeat: 0*B*1, …, 2*B*3 or 3*B*0 or 1*BB*0*B*0, 1*BB*0*B*1, 1*BB*0*B*2, 1*BB*0*B*3 or 1*BB*1*B*0.

Then we count in tens, including the bundles: 0*B*1, …, 0*B*8, 0*B*9, 0*B*10 or 1*B*0, 1*B*1, 1*B*2.

Finally, we bundle-count in tens from 0 to 111.

MC03. Bundle-Counting with Underloads and Overloads

Strictly following the place value system, the tradition silences the units when writing 'two hundred and fifty-seven' as plain 257. A difference may be inspired by the Romans using 'underloads' when writing four as "five less one", IV; and by overloads when small children use 'past-counting': "twenty-nine, twenty-ten, twenty-eleven".

A guiding question can be. "Let us count with underloads missing for the next bundle. And with overloads as children saying 'twenty-eleven'. Outside material could be sticks, cubes, and an abacus.

First, we notice that five fingers can be counted in pairs in three different ways

T = 5 = I I I I I = ~~II~~ I I I = 1*B*3, overload

T = 5 = I I I I I = 𝍸 𝍸 I = 2*B*1, normal

T = 5 = I I I I I = 𝍸 𝍸 𝍸 = 3*B*-1, underload

Using fingers and arms, first we count using underloads: 0*B*1 or 1*B*-9, 0*B*2 or 1*B*-8, …, 0*B*9 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-9, 1*B*2 or 2*B*-8. The we count to twelve in 5s (hands): 0*B*1 or 1*B*-4, 0*B*2 or 1*B*-3, …, 0*B*4 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-4, …, 2*B*2 or 3*B*-3.

And in 3s (triplets): 0*B*1 or 1*B*-2, 0*B*2 or 1*B*-1, 0*B*3 or 1*B*0, 1*B*1 or 2*B*-2, …, 3*B*3 or 4*B*0 or 1*BB*1*B*0. Cup-counting with a cup for bundles, and for bundles-of-bundles: T = 1]1]0 = 4]0 = 3]3 = 2]6 = 1]9.

Then we count in tens from 1 to 111, using past-counting: … 1*B*9, 1*B*10, 1*B*11 or 2*B*1, 2*B*2, …, 2*B*11 or 3*B*1, …, 9*B*9, 9*B*10, 9*B*11 or 10*B*1 or 1*BB*0*B*1.

Then we rewrite totals as 'flexible bundle-numbers' with overloads and underloads:

T = 38 = 3*B*8 = 2*B*18 = 1*B*28 = 4*B*-2 = 5*B*-12

MC04. Doing Math with Flexible Bundle-Numbers

The tradition uses carrying when adding and multiplying, and borrowing when subtracting and dividing. Here, a difference is to instead use flexible bundle-numbers.

A guiding question can be. "Let us do inside arithmetic with flexible bundle-numbers."

Overload	Underload	Overload	Overload
65 + 27	65 - 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 - 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 1: Doing Arithmetic with Flexible Bundle-Numbers

MC05. Talking Math with Formulas

In a number-language sentence as "The total is 3 4s", the tradition silences all but the calculation 3*4. A difference is to use full sentences with an outside subject, a verb and an inside predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence "*T* = 5*6 = 30" thus inside predicts that outside 5 6s can be re-counted as 3 tens.

A guiding question can be. "Let us talk math with full sentences about what we calculate and how."

We begin by counting in ones using a full sentence: "From the total we pull away one to get one."

We then write what was before and after, using a rope as an icon for pulling away: "*T* = (*T* – 1) + 1".

This number-language sentence formulates what we call a formula. It also applies if instead pulling away 2, "*T* = (*T* – 2) + 2", and if pulling away any unspecified bundle *B*, "*T* = (*T* – *B*) + *B*". We call this formula a 're-stack formula' since, with the total as a stack, we pull away the bundle from the top and place it next-to as its own stack.

Outside asking “Adding what to 2 gives 5?”, inside becomes “? + 2 = 5” in writing. Using the letter *u* for the unknown number, this becomes an ‘equation’ “*u* + 2 = 5”, easily solved outside by pulling away the 2 that was added, described inside by restacking the 5: *u* + 2 = 5 = (5 – 2) + 2, so *u* = 5 – 2.

So, inside an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the definition of the number ‘5-2’: “5 minus 2 is the number *u* that added to 2 gives 5”.

We now use full sentences when counting in bundles, e.g., re-counting 8 1s in 2s. We use ‘/’ to iconize a broom brushing away 2s. So ‘8/2 = 4’ is an inside prediction for the outside action “From 8, brush 2 away, 4 times.”

Having been brushed away, the bundles of 2s are stacked. This is iconized by an ‘x’ for lifting the bundles, so ‘4x2 = 8’ is an inside prediction for the outside action “4 times stacking 2s gives 8 1s”.

Re-counting 8 in 2s thus gives a ‘re-count formula’ 8 = (8/2) x 2, outside showing a box with the counter 8/2 and the unit 2 on the vertical and horizontal side, and with the total 8 as the area. So, totals add by areas, called integral calculus. With unspecified numbers it says: *T* = (*T*/*B*) x *B*, or *T* = (*T*/*B*) * *B*, or “From a total *T*, *T*/*B* times, *B* can be brushed away”.

Outside asking “How many 2s in 8”, inside is the equation “? * 2 = 8”, or “*u* * 2 = 8” easily solved outside by brushing away 2s, described inside by recounting 8 in 2s: *u* * 2 = 8 = (8/2) * 2, so *u* = 8/ 2.

Again, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the formal definition of ‘8/2’: “8 divided by 2 is the number *u* that multiplied to 2 gives 8”.

MC06. Naming the Unbundled Singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question can be “How to see the unbundled singles?”. Outside materials can be cubes or an abacus

Before outside recounting 9 in 2s, inside we let a calculator predict the result: Entering 9/2 gives ‘4.some’ predicting that “from 9, brush away 2s can be done 4 times”. To find unbundled singles, outside we pull away the 4-by-2 stack, inside predicted by entering ‘9 – 4 * 2’ giving 1. So, inside the calculator predicts that 9 recounts as 4*B*1 2s, which is also observed outside.

Recounting 9 cubes in 2s, the unbundled can be placed on-top of the stack. Here it can be described inside by a decimal point separating the bundles from the unbundled: T = 4*B*1 2s = 4.1 2s. Likewise, when counting in tens: T = 4*B*2 tens = 4.2 tens = 4.2 * 10 = 42.

Seen as part of a bundle, inside we can count it in bundles as a ‘fraction’, 1 = (1/2) * 2 = 1/2 2s; or we can count what is missing in a full bundle, 1 = 1*B*-1 2s.

Again, we see the flexibility of bundle-numbers: T = 4*B*1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Likewise, when counting in tens: T = 4*B*2 tens = 4 2/10 tens = 4.2 tens = 5.-8 tens.

MC07. Changing Number Units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon to another, from icons to tens, or from tens to icons, or into a square.

A guiding question can be “How to change number units?”. Outside materials can be an abacus.

Asking ‘3 4s = ? 5s’, we inside predict the result by entering on a calculator the 3 4s as 3*4, to be counted in 5s by dividing by 5. The answer ‘2.some’ predicts that from 3 4s, 2 times, 5 can be brushed away. To find unbundled singles, outside we pull away the 2 fives from the 3 4s; inside we predict this by entering ‘3*4 – 2*5’. The answer ‘2’ predicts that 3 4s can be recounted as 2 5s & 2, or 2*B*2 5s.

Asking “40 = ? 5s”, we predict the result by solving the equation “u * 5 = 40” by recounting 40 in 5s:

u * 5 = 40 = (40/5) * 5, so u = 40/5.

Asking “6 8s = ? tens”, or “6 * 8 = ?”, we inside predict the result by looking at a ten-by-ten square with 6 and 8 as *B*-4 and *B*-2 on the sides. We then see that the 6*8 box is left when from the *B**B* box we pull away a 4**B* and a 2**B* box and add the 4*2 box pulled away twice.

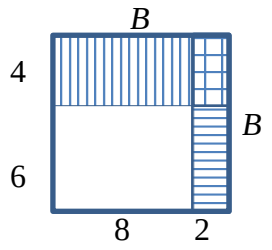	T = 6 * 8 = (*B*-4) * (*B*-2) = *BB* – 2*B* – 4*B* + 8 = 4*B*8 = 48	$T = \begin{pmatrix} 1B-4 \\ 1B-2 \end{pmatrix}$ = 1*BB* – 2*B* – 4*B* + 8 = 10*B* – 6*B* + 8 = 4*B*8 = 48	$T = \begin{pmatrix} 2x-3 \\ 4x-5 \end{pmatrix}$ = $8x^2$ – 10x – 12x + 15 = $8x^2$ – 22x + 15

Figure 2: Multiplying Numbers as Binomials

Inside, multiplying two ‘less-numbers’ horizontally creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles.

Wanting to square a 5-by-4 box, its side is called $\sqrt{20}$, using lines to iconize the square. To find $\sqrt{20}$ we see that removing the 4-square leaves 20 – 4*4 = 4 shared by the two 4**t* boxes in a 4+*t* square, giving *t* = 0.5. A little less since we neglect the *t*-square. Inside, a calculator predicts that $\sqrt{20}$ = 4.472.

Intersection points between lines and circles leads to quadratic equations as x^2+ 6*x* + 8 = 0, easily solved when rotating the upper of two *x* by *x*+3 playing cards to create a square with sides *x*+3, and with zero area except for the 1 left in the upper 3-by-3 square after 8 is removed.

In total, $(x+3)^2$ = 0 + 1 = 1 = $\sqrt{1}^2$, so *x* + 3 = ± 1, giving *x* = - 2 and *x* = - 4.

MC08. Changing Physical Units with Per-Numbers

The tradition sees shifting physical units as an application of proportionality. Typically finding the unit cost will answer questions as “with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?”

A difference is to use ‘per-numbers’ (Tarp, 2018) coming from double-counting the same quantity in the two units, e.g., *T* = 3$ = (3$/2kg) * 2kg = *p* * 2kg, with the per-number *p* = 3$/2kg, or 3/2 $/kg.

A guiding question can be “How to change physical units?”. Outside materials can be colored cubes.

Recounting in the per-number allows shifting units:

T = 6 kg = (6/2)*2 kg = (6/2)*3 $ = 9$; and *T* = 15$ = (15/3)*3$ = (15/3)*2kg = 10 kg.

Alternatively, we recount the units: $ = ($/kg)*kg = (3/2)*6 = 9; and kg = (kg/$)*$ = (2/3)*15 = 10.

With like units, per-numbers become fractions: 1$/4$ = ¼. The tradition teaches fractions as division: 1/4 of 12 = 12/4. A difference is to see a fraction as a part of a bundle counted in bundles, 1 = (1/4)*4, so 1/4 = 1 part per 4. Finding 3/4 of 12 thus means finding 3parts per 4 of 12 that recounts in 4s as:

T = 12 = (12/4) * 4 = (12/4) * 3parts = 9parts, so 3 per 4 is the same as 9 per 12, or 3/4 = 9/12.

So, finding 3/4 of 100 means finding 3 parts per 4 of 100 = (100/4)*4, giving 75 parts per 100 or 75%.

MC09. Recounting the Sides in a Box Halved by its Diagonal Gives Trigonometry

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, mutually recounting the sides in a box halved by its diagonal.

A guiding question can be "How to recount the sides in a box halved by its diagonal?". Outside materials can be tiles, cards, peg boards, and books.

Recounting the height in the base, height = (height/base) * base = tangent A * base, shortened to

h = (*h* / *b*) * *b* = tan *A* * *b* = tan *A b*s, thus giving the formula tangent *A* = height / base, or tan *A* = *h*/*b*.

Using the words 'run' and 'rise' for 'base' and 'height', we get the formula: tan *A* = rise/run, giving the steepness or slope of the diagonal. The word 'tangent' is used since the height will be a tangent in a circle with centre in *A* with the base as its radius.

This gives a formula for the circumference since a circle contains many right triangles: In an *h*-by-*r* half-box, *h* recounts in *r* as *h* = (*h*/*r*) * *r* = tan *A* * *r*.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With *A* as 1.8 degrees, the circle and the tangent, *h*, are almost identical. So, half the circumference is

½*C* = 100 * *h* = 100 * tan 1.8 * *r* = 100 * tan (180/100) * *r* = 3.1426 * *r*

Calling the circumference for 2 * π * *r*, we get a formula for the number π:

π = tan (180/*n*) * *n*, for *n* sufficiently large.

MC10. Adding Next-To and On-Top

The tradition sees numbers as one-dimensional cardinality with addition defined as counting on. A difference is to accept childrens' bundle-numbers as 2-dimensional boxes that add next-to and on-top. A guiding question can be "How to add 2 3s and 4 5s on-top and next-to?". Material: cubes and abacus.

To add 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing one or both; inside recounting changes units. Or to use the recount formula to predict the result by entering (2*3+4*5)/*B*, where *B* can be 3 or 5 or 8. Added next-to by areas is called integral calculus.

Adding 20% to 30$, we have two units with the per-number 30$ per 100%. Adding 20% to 100% gives 120%, recounting in 100s as 120% = (120/100)*100% = (120/100)*30 $ = 120%*30$. So, we add 20% by multiplying with 120%, also called to multiply with the index-number 120.

Reversing adding next-to and on-top, a guiding question can be "How many 3s to add to 4 5s to get a total of 6 5s or 5 8s?". Outside materials can be cubes and an abacus.

To find the answer outside, we pull way the 4 5s from the total *T* before recounting in 3s, which is predicted inside by asking the calculator: (6*5 – 4*5)/3, or (5*8 – 4*5)/3, i.e., as Δ*T*/3. Using a difference to calculate the change in the total, Δ*T*, before recounting, this is called differential calculus.

MC11. Adding Per-Numbers and Fractions

Seeing fractions as numbers that add without units, the tradition teaches 'mathematism', true inside but seldom outside classrooms where, e.g., 2m + 3cm = 203cm. A difference is respecting that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding.

A guiding question can be "What is 2kg at 3$/kg plus 4kg at 5$/kg?" Outside materials can be a peg board with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

Inside we see that unit-numbers add directly. Whereas, per-numbers first must be multiplied to become unit-numbers. And since multiplication creates a box with an area, per-numbers add by their areas, i.e., as the area under the per-number curve. And again, adding areas is called integral calculus.

And again, the opposite is called differential calculus asking "2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg." The two connect by the fact that adding serial differences, the middle terms disappear leaving only the difference between the end and initial numbers: (*b*-*a*) + (*c*-*b*) + (*d*-*c*) = *d*-*a*.

MC12. Change by Adding or by Multiplying

The tradition taches adding arithmetic and geometric sequences as series. A difference is adding constant unit-numbers and per-numbers. A guiding question can be "What is 2$ plus 3$/day?" and "What is 2$ plus 3%/day?" Outside materials can be a peg board and an abacus.

Inside we see that adding 3$/day to 2$ gives a total of $T = 2 + 3*n$ after *n* days. This is called change by adding, or linear change with the general formula $T = b + a*n$. And that adding 3%/day to 2$ gives a total of $T = 2*103\%\text{^}n$ after *n* days since adding 3% means multiplying with 103%. This is called change by multiplying or exponential change having the general formula $T = b * a\text{^}n = b*(1+r)\text{^}n$.

Reversing change by adding means facing an equation as $100 = 20 + 5*u$, easily solved by restacking and recounting: 100 = (100-20) + 20 = 80 + 20, so $u*5 = 80 = (80/5)*5$, so $u = 80/5 = 16$.

Reversing change by multiplying gives two equations. In the equation $20 = u\text{^}5$, we want to find the factor *u* of which 5 gives 20, predicted by the factor-finding root $\sqrt[5]{20} = 1.82$. In $20 = 5\text{^}u$ we want to find the number *u* of 5-factors that give 20, predicted by the factor-counting logarithm $\log_5(20) = 1.86$. We now know all the ways to unite parts into a total, and to split a total in parts, the 'Algebra-square':

Operations **unite**/split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a*n}$ $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int a*\mathrm{d}n}$ $\frac{\mathrm{d}T}{\mathrm{d}n} = a$	$\boldsymbol{T = a\text{^}n}$ $\sqrt[n]{T} = a \qquad n = \log_a T$

Figure 3: The 4 Ways to Unite Parts into a Total, and the 5 Ways to Split a Total into Parts.

OBSERVATIONS

The special education students were asked to write or phone short messages to a friend about how they experienced the twelve micro curricula. Typical answers expressed positive attitudes towards learning that digits are icons with sticks, that hundred is a bundle-of-bundles, that with bundles you don't need the place value system, that over- and underloads are allowed, that recounting is predicted by a recount-formula that also solves equations, that negative numbers and decimals and fractions simply tell how to see the unbundled, that the multiplication tables come when recounting from icons to tens, that boxes can be squeezed to change units or to become squares, that physical units are changed by recounting in the per-numbers, that recounting a box halved by its diagonal introduces trigonometry and a formula for pi, that number-boxes can be added on-top, but also next-to as integration adding areas, also occurring when adding per-numbers. And they proudly talk about returning to general education and becoming stars when teaching fellow students and the teacher new ways to do math.

CONCLUSION

The ancient Greek civilization talked about common ethos, individual ethics and collective moral. As to a school, we would expect a civilized ethos of this institution to be its original meaning, a timeout to reflect ongoing activity, as when called by a coach in a match. Foucault (1972) thus warns against a school becoming a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cells daily, and accept the diagnose 'un-educated' to be cured by, of course, education as defined by a self-referring 'truth regime'. To avoid this, mathematics education should be a timeout where the ethics of civilized educators would be that of foster-parents guiding their foster-children to better master and communicate about Many. And as to moral, we would expect a civilized mathematics education curriculum to support the foster-parents when helping children improve their mastery of outside existence instead of forcing them to first master an inside constructed essence. That, as an institutionalized means, may be tempted by a goal displacement (Bauman, 1990, p. 84): By making itself so difficult that only few will arrive at the outside end-goal of mastering Many, inside mathematics could increase its power and funding in order to finally, if ever, reach its outside goal.

Thus, to the benefit of all students, a moral mathematics education should use guidance to develop the mastery and communication that children build up when adapting to Many before school. And it would be immoral and unethical to force upon them the necessity of a construct called one dimensional place-value line-numbers adding without units in order to exclude some to special education just repeating the same at a slower pace. This paper shows that this immoral mathematics education is not there by necessity, but by choice. And that a moral version comes from not teaching but learning from children.

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TEACHING MATHEMATICS AS COMMUNICATION, TRIGONOMETRY COMES BEFORE GEOMETRY, AND PROBABLY MAKES EVERY OTHER BOY AN EXCITED ENGINEER

ABSTRACT

Before 1970 both foreign language and mathematics were hard to learn because the two taught grammar before language. Then a turn took place in foreign language education allowing students to learn it through communication. Mathematics education never had a similar turn, so it is still hard for many. Therefore, this paper asks if it is possible to learn mathematics as communication. We see that three different kinds of mathematics are taught, pre-setcentric, setcentric and post-setcentric. And that the three grand theories disagree as to which to recommend. Being inspired by the fact that children communicate about the physical fact Many with two-dimensional box- and bundle-numbers with units, a curriculum is designed where trigonometry is rooted in a mutual recounting of the three sides in a box halved by its diagonal. So, the answer is: Yes, core mathematics can be learned as communication about boxes since it is directly connected to counting and recounting Many in boxes and bundles.

Keywords: Engineer, Geometry, Math, STEM, Trigonometry.

1. MATHEMATICS, EASY OR HARD OR NECESSARY?

Mathematics is easy, or is it? Well, not according to numerous PISA test results. So apparently mathematics is hard. But is mathematics then hard by nature or by choice? The ancient Greek sophists warned that in social matters we should tell nature from choice to avoid being patronized by choice masked as nature.

So, let us look closer at the typical rationality behind mathematics education: "Of course, mathematics is a core subject in education because of its many important social applications. And, of course, you cannot apply mathematics before you have learned it. So, once inside you master mathematics, you will be able to master the outside fact Many!"

But if I don't master mathematics, will I then have to give up on mastering Many? Or, since mastering Many seems to be the end goal with mathematics as a means, we could ask the following question: Is it possible to master Many before mastering what the school call mathematics? If so, then afterwards, those finding it important may try to master mathematics itself, while others focus on the STEM subjects probably wanting to become technicians or engineers.

So, to answer our research question "Can also mathematics be learned as communication?" we begin by looking at the background of mathematics education.

2. BACKGROUND

We, the people, have holes in the head as have other animals. The holes serve to meet our basic needs for food for our stomachs, oxygen for our lungs, and information for our brain. To share what we need, we form societies where human societies are especially efficient since, when standing up, our forelegs became arms with hands to grasp while exchanging sounds for what is grasped and shared, thus allowing us to share information also through communication in a common language that we speak, and maybe also read and write. Using a keyboard, we observe that it contains letters, digits and operations allowing us to communicate in our two languages, the word-language where letters unite

to words that unite to sentences typically containing a subject, a verb, and a predicate as ‘these are apples’; and a number-language where digits unite to numbers that together with operations unite to formulas or number-language sentences, also containing a subject, a verb, and a predicate as ‘the total is four apples’ shortened to ‘T = 4’.

Adapting to our surroundings we learn to speak our word-language before school that then teaches us to read and write. Later it teaches grammar, the meta-language that describes the language that describes the world.

We also learn some number-language before school, but here the school starts teaching it from the beginning as a subject called mathematics, which is the grammar of the number-language. Mathematics thus teaches grammar before language as did foreign language education until around 1970 when it took a turn and began teaching communication before grammar [1]. Number-language education never made the same turn, so let us try to see if that is possible: In mathematics education, can we teach language before grammar? Can mathematics be learned as communication?

3. HOW WELL-DEFINED IS MATHEMATICS?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy [2], seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric [3].

The Arabic conquest of the silver mines in Spain made Europe descent into a dark middle age with mathematics frozen because of the rigidity of Euclidean geometry and the inability of Roman numbers to perform multiplication [4]. Later, silver found in German Harz financed the Renaissance reopening the trade with silk and pepper from India through Arab middle men bringing Hindu-Arabic bundle-bundle numbers, algebra, and trigonometry to Europe. Later again, the search for a trade route on open sea around Africa let Newton to invent calculus as a means to add locally constant per-numbers.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for calculus, trigonometry as well as the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. Algebra thus contains the four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1, i.e. 3 bundles of bundles and 4 bundles and 2 unbundled ones, i.e. 3 boxes. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation: subtraction, division, root and logarithm, and differentiation [5].

Operations unite/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a*n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\,dx$ $dT/dx = f$	$T = a\text{^}b$ $b\sqrt{T} = a \quad loga(T) = b$

So, as a label, mathematics has no existence itself, only its content has; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek ‘many-math’ rooted in Many was replaced

by the ‘New Math’. Here the invention of the concept Set created a Setcentric [6] ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts.

However, ‘well-defined’ meant defining by self-reference, i.e., defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

The Diversity of Numerical Communication Shown by Six Ways to Solve Proportionality Questions

The need to change units has made the two proportionality questions the core outside asked questions. With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g.

Q1: “7 meters need how many seconds?”, and Q2: “How many meters is covered in 12 seconds?”

● Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: ‘2m takes 5s, 7m takes ?s’ to get to the answer (7*5/2)s = 17.5s.

Then we ask, Q2: ‘5s gives 2m, 12s gives ?m’ to get to the answer (12*2)/5s = 4.8m.

2 new methods came, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

● Q1: 1m takes 5/2s, so 7m takes 7*(5/2) = 17.5s.

Q2: 1s gives 2/5m, so 12s gives 12*(2/5) = 4.8m.

● Q1: 2/5 = 7/x, so 2*x = 7*5, x = (7*5)/2 = 17.5.

Q2: 2/5 = x/12, so 5*x = 12*2, x = (12*2)/5 = 4.8.

● Alternatively, we may recount in the ‘per-number’ 2m/5s coming from ‘double-counting’ the total T.

Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s;

Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

Or, we may simply recount the units:

● sec = (sec/m)*m = 5/2*7 = 17.5

m = (m/sec)*sec = 2/5*12 = 4.8

● Set introduced modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function f(x) = c*x from the set of m-numbers to the set of s-numbers,

having as domain DM = {x∈R ⏐ x>0}. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: c*2 = 5, (c*2)*½ = 5*½, c*(2*½) = 5/2, c*1 = 5/2, c = 5/2. With f(x) = 5/2*x, the inverse function is f-1(x) = 2/5*x. So with 7m, f(7) = 5/2*7 = 17.5s; and with 12s, f-1(12) = 2/5*12 = 4.8m.

4. GRAND THEORY LOOKS AT MATH EDUCATION

With mathematics education as a human activity, we might want to see how the three basic human sciences look at it, sociology focusing on 'they', philosophy focusing on 'it', and psychology focusing on 'I'.

Sociology

Two kinds of sociology exist with different views on the role of institutions as means to reach a common goal. Seeing a great potential in institutions to create human welfare, Europe has developed a structural sociology [7]. Seeing institutions as constraining human behavior, North America has developed an agency-based sociology called pragmatism [8].

Sociologically, mathematics education is a double institution containing two social constructions: a text communicating something about something, and a mediation of this text. However, as a rational institution with a goal, it shares the danger of becoming an 'iron cage' [9] with a 'goal displacement' [10] seeing its own survival and growth as the real goal, and the original goal as a means that as long as it is not reached, will secure the continuing existence and possibly growth of the institution. In short, as long as the number-language children develop before school is neglected, and as long as mathematics is considered a hard subject, mathematics education will be needed, and so will research in mathematics education.

As to mathematics education, Europe favors the setcentric New Math, rejected by North America going back to the pre-setcentric version of traditional school mathematics. Here mathematics as communication may offer a post-setcentric alternative that may be attractive to North America as a completely new paradigm [11].

Philosophy

Two kinds of philosophy exist with different views on what is more important, outside ontological existence or its inside epistemological constructed representations. European rationalism favors the platonic view that what exists physical is but examples of metaphysical forms only accessible to philosophers educated at the Plato academy. Hence epistemology precedes ontology. Anglo-Saxon empiricism favors the view of existentialism that existence precedes essence [12].

As to mathematics education, rationalism favors setcentric mathematics, whereas empiricism favors the pre-setcentric version or the existentialist post-setcentric version [5].

Psychology

Psychology focuses on how the human brain adapt to the outside world. Two kinds of psychology exist with different views on what is more important to adapt to: what exits in the outside world, or what is inside constructed as representations. Here Europe favors Vygotsky [13] arguing that the goal of education is that learners adapt to institutionalized knowledge mediated by teachers with respect to

what learners already knows, their individual zones of proximal developments, which puts a big responsibility on the teacher to be able to practice a differentiated teaching.

In contrast, North America favors Piaget [14] arguing that the goal of education is that learners adapt to the outside world by creating individual schemata used to assimilate the world to, or to be accommodated when meeting resistance from the world or from peers.

As to mathematics education, psychology has created two forms of constructivism, a Vygotsky-based social constructivism favored in Europe; and a Piaget-based radical constructivism favored in North America.

5. A CONTEMPORARY MATH CURRICULUM

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g., that 2+3 = 5. This offers a 'natural' curriculum with one-dimensional linear multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

Reverse operations may create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions, decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as $\sqrt{}2$ and log 3 where $\sqrt{}2*\sqrt{}3 = \sqrt{}6$, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g., that (a+b) * (a–b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles, and areas. Later, geometry and algebra are coordinated in coordinate geometry. To be followed by a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = $\sqrt{}3/2$.

In a formula, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified formula. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g., that (f*g)'/(f*g) = f'/f + g'/g. And with a reverse operation, integration, again creating additional education about results of operating on calculations, e.g., that $\int 6*x^2dx = 2*x^3 + c$, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as has the qualitative: fact, fiction and fiddle [15].

How Children Communicate About Many

How to master Many may be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them. Children thus use two-dimensional bundle-numbers inside as representation for outside boxes, which

resonates nicely with the box-structure of Hindu-Arabic numbers, bout not with the one-dimensional cardinal numbers that schools teach.

Children also use box-numbers when talking about snap cubes as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 2x2s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as T = 7 = 1B5 = 2B3 = 3B1 = 4B1; or even as 1BB3 or 1BB1B1. Also, children love to count in 3s, 4s, and in hands.

6. POST SETCENTRIC MATHEMATICS

Since children already communicate about Many before school, why not let mathematics education develop instead of reject the core mastery of Many that children bring to school?

Digits as Icons

Many exists in space as a total, and in time as counting. A total of 4 sticks may be rearranged as 1 4-icon containing 4 sticks, likewise with the other digits. So, we may say that a digit is a number-icon containing as many sticks or strokes as it represents if written less sloppy. The icon for zero is a magnifying glass finding nothing. Ten is a special number with its own name but with no icon. That is because we count by bundling in tens, which makes ten a double counting of bundles and singles, where 10 is short for 1B0, 1bundle and no unbundled. So, where letters are mere symbols unconnected to their sounds, digits are icons communicating directly about their degree of Many.

Bundle-Counting Makes Operations Icons Also

Counting a total of 9 in 2s, first we push away 2s, iconized by a division-broom, 9/2. The we stack the bundles 4 times, iconized by a multiplication-lift, 4x2 or 4*2. Then we pull away the stack to look for unbundled singles iconized by a subtraction-rope, 9 – 4*2. Finally, we use an addition-cross to iconize the two ways to unite boxes, on-top or next-to.

Outside, counting a total of 9 in 2s means boxing 9 as 4 2s with an unbundled single on-top. Inside we report this as T = 9 = 4B1 2s or T = 4.1 2s using a decimal point to separate the bundles from the unbundled. Or as T = 9 = 4 ½ 2s indicating that the single should also be counted in bundles. Or as T = 9 = 5B-1 2s indicating that 1 is needed for another bundle. In this way, decimals, fractions and negative numbers are connected to how the unbundled singles are treated and seen.

Bundling in tens, we may see 1 as a bundle of ten parts and write 9/4 as 2.5, or 0.9 tens/4 = 0.25 tens.

Recounting Creates a Recount-Formula

Recounting 8 in 2s by 8/2 times pushing away 2s may be written as a 'recount-formula' 8 = (8/2)*2, or T = (T/B)*B with unspecified numbers, saying 'From T, T/B times, B is pushed away.' This formula may also be called a proportionality formula used to change number units or physical units. Thus double-counting apples as 4$ and 5kg creates a 'per-number' 4$/5kg as a bridge between the units. So 12$ quickly change unit from $ to kg by recounting in the per-number:

T = 12$ = (12/4)*4$ = (12/4)*5kg = 15kg.

Likewise, the recount-formula may be used to solve equations occurring when asking 'How many 2s in 8' or 'u*2 = 8' immediately solved by u = 8/2 as seen by recounting 8 in 2s:

u*2 = 8 = (8/2)*2, giving u = 8/2.

Solving equations by moving a number to the opposite side with the opposite calculation sign works for all operations:

$u + 2 = 8$	$u*2 = 8$	$u\text{^}8 = 2$	$2\text{^}u = 8$
$u = 8 - 2$	$u = 8/2$	$u = 8\sqrt{2}$	$u = \log2(8)$

Totals May be Boxed, Ten'ed, or Squared

So, an outside total is boxed in the given unit. Thus 4 5s is reported inside as T = 4B0 5s. We may want to squeeze the box into tens, which of course decreases its height since it increases the base, T = 4 5s = 2 tens. If we want to squeeze it in 8s, inside we ask the question 'u*8 = 20' which is an equation easily solved by recounting 20 in 8s as u = 20/8.

Wanting to square the 4x5 box, the answer is a $\sqrt{}20$ square. To find $\sqrt{}20$ we observe that in a 4+t square, removing the 4 square leaves 20 - 4x4 = 4 shared by the two 4xt boxes, giving t = 0.5. A little less since we neglect the t square. Inside, a calculator confirms that $\sqrt{}20 = 4.472$.

In geometry, intersection points between lines and circles leads to quadratic equations as x^2 + 6x + 8 = 0, easily solved when rotating the upper of two x by x+3 tiles to create a square with sides x+3, which is zero apart from the 1 left in the upper 3-by-3 square after 8 is removed.

In total, (x + 3)^2 = 0 + 1 = 1 = $\sqrt{}1\text{^}2$,

so x+3 = ±1, giving x = - 2 and x = - 4.

Trigonometry: Recounting a Box Halved by its Diagonal

So, when counting and recounting, the core outside object is a rectangular form that may be called a stack, a tile, a block, or a box, with base, b or AC, and height, h or CB. Here the diagonal, d or AB, splits the box in two like right angled triangles, which may also be called 'half-boxes'. Alternatively, we may say that we have boxed the line AB so it becomes a diagonal with a base and the height, which may also be called its floor and wall, or horizontal and vertical shadow, or run and rise. We will not use the Greek name 'cathetus' since it does not distinguish between the two sides.

Besides its three sides, a half-box also has three angles counting the turning of the lines in degrees. The turning from the base to the diagonal is called the angle A. The quarter round turning from the base to the height is called the right-angle C, which is counted to 90 degrees since a full round is 360 degrees. With two like triangles in the full box we see that A + B = 90. Trigonometry finds formulas that predict the sides and angles in a half-box.

The sides may be counted in standard units. But they may also be recounted in the box's own units. Recounting the height in the base gives height = (height/base) * base = tangent A * base = tangent A bases, abbreviated to

h = (h / b) * b = tan A * b = tan A bs, thus giving the formula

tangent A = height / base, or tan A = h/b.

The word tangent is used since in a circle with center in A and the base as its radius, the height will be a tangent. This allows two ways to count angles, by degrees or by a tangent number.

Thus outside, in a 1x2 half-box, the angle A is counted on a protractor to 27 degrees; and in a 1x1 half-box, the angle A is counted to 45 degrees; and in a 2x1 half-box, the angle A is counted to 63 degrees. Inside these numbers may be predicted by the tangent formula

A = tan-1(1/2), A = tan-1(1/1), A = tan-1(2/1).

In the same we can create other trigonometry formulas as

sin A = height/diagonal, or sin A = h/d

cos A = base/diagonal, or cos A = b/d.

A Circle as Many Right Triangles

In an h x r half-box with area ½*h* r we look at the circle with center in A and r as its radius. Here h = (h/r)*r = tan A*r .

A half circle is 180 degrees that may split in 100 small parts as

180 = (180/100)*100 = 1.8 100s = 100 1.8s

With A as 1.8 degrees, the length of the circle and the tangent, h, are almost identical. So, with good approximation

Half the circumference C = 100 * h = 100 * tan 1.8 * r = 100 * tan (180/100) * r = 3.1426 * r

Or better, C = 1000 * tan (180/1000) * r = 3.1416 * r

Calling the circumference for 2 * π * r, we get a formula for the number π:

For n sufficiently large, π = tan (180/n) * n.

The area of the full circle then is

A = 2 * ½ * 3.1416 * r * r = 3.1416 * r^2 = π * r^2

Adding Sides in a Half-Box

Arranging 4 like half-boxes as a (base + height) square, it contains a diagonal square plus 4 half-boxes. However, it also contains a height square and a base square plus 2 full boxes. Consequently, the height square and the base square add up to the diagonal square as predicted inside by the Pythagorean formula h^2 + b^2 = d^2. So, in right angled triangles, the sides add by their squares.

Adding Two Half-Boxes

Two half-boxes ADB and CDB have the same height DB. They add as two right triangles into a non-right triangle with base AC and with BD as its height, h, and with AD = p and CD = q.

From the left half-box we get c^2 = p^2 + h^2. From the right half-box we get h^2 = a^2 – q^2, and q = a*cos C. In all we get

c^2 = a^2 + p^2 – q^2 = a^2 + p^2 + q^2 – 2*q^2

With b = p+q we get b^2 = p^2 + q^2 + 2*p*q. This gives

c^2 = a^2 + b^2 – 2*p*q – 2*q^2

= a^2 + b^2 – 2*(p+q)*q

= a^2 + b^2 – 2*b*a*cos C.

This formula is called the extended Pythagorean formula or the cosine formula.

Equating the two formulas for the height BD in the two half-boxes, we get

BD = AB * sin A = BC * sin C, or BD = c * sin A = a * sin C

This may be rearranged to the sine formula $\frac{a}{\sin A} = \frac{c}{\sin C}$

Intersection Line-Line

Extending a 3x4 box, ACBD, with a 3x2 box, CEFB, to a 3x6 box, AEFD, will allow finding intersection points between the diagonals in the three boxes. To find the height, PQ, of the intersection point between the diagonals AF and CD we notice that the angle A is part of a small triangle, AQP, and a large triangle, AEF, so that

tan A = QP/AQ = EF/AE = 3/6.

Likewise, since angle C is part of a small triangle, CQP, and of a large triangle, CAD, we get C = QP/CQ = AD/CA = 3/4.

Equating QP in the two triangles, and observing that with AQ = u, CQ = 4 – u, we get

QP = 3/6*AQ = 3/4* CQ, giving 3/6 * u = 3/4 * (4 – u), that is solved by u = AQ = 12/5 = 2.4, giving QP= 3/6*2.4 = 1.2.

Intersection Line-Circle

In a 3x5 box, ACBD, we look at a circle having its center in A and its radius r = AC. The intersection point P with the upper base BD and its point Q below on AC may be found by observing that the point is situated on the circle and the upper base line so that

AP^2 = AQ^2 + QP^2, and QP = 3. So, 5^2 = AQ^2 + 3^2, giving AQ^2 = 5^2 - 3^2 = 16 and AQ = $\sqrt{}$16 = 4.

So, to come from A to P we go out 4 units and up 3 units.

To find the intersection point P with the diagonal AB we see that the point is situated on the circle and the diagonal so that

tan A = QP/AQ = 3/5, and AQ^2 + QP^2 = 5^2.

With AQ = u, QP = 3/5 * u and u^2 = 5^2 – (0.6 * u)^2

This transforms into a quadratic equation

u^2 + 0.36 u = 5^2, or 1.36 u^2 = 25, or u^2 = 25/1.36 = 18.38

with u = AQ = $\sqrt{}$18.38 = 4.29, giving QP = 3/5*4.29 = 2.57.

So, to come from A to P we go out 4.29 units and up 2.57 units.

To find the intersection point P with the diagonal CD we observe that the point is situated on the circle and the diagonal. With AQ = u and QP = v, we get that

tan C = 3/5 = v/(5 + u) giving v = 3/5*(5 + u) = 3 + 0.6u; and u^2 + v^2 = 5^2, so

u^2 + (3 + 0.6 u)^2 = 25, or 1.36 u^2 + 3.6 u + 9 = 25, or

u^2 + 2.64 u – 11.8 = 0, solved by u = 2.36, giving v = 4.42

So, to come from A to P in this case, we go back 2.36 units and up 4.42 units.

Contact Points Tangent-Circle

In a 3x5 box, ACBD, we look at a circle having its center in A and its radius r = AC. The line from B to C is a tangent to the circle with the contact point C, and forming a right angle with the radius AC. From the point B, there is however also another tangent to the circle. We observe that the contact point P is situated both on the circle and on the tangent. Turning the horizontal line AC to AP means turning the vertical tangent to the PB direction. Consequently, the two angles CAP and QPB are like. With S as the intersection point between PQ and DB, with PQ = v, and with AQ = u, we have

tan P = SB/SP = (5 – u)/(v – 3) = tan A = v/u, giving

(5 – u) * u = (v – 3) * v, or

5u – u^2 = v^2 – 3v, or 5u = u^2 + v^2 – 3v = 5^2 – 3v, giving

u = 5 – 3/5v, or u = 5 – 0.6*v

At the same time, u^2 + v^2 = 5^2, so

(5 – 3/5v)^2 + v^2 = 5^2, giving -6v + 1.36v^2 = 0, giving v = 6/1.36 = 4.41, giving u = 5 – 0.6*4.41 = 2.35

So, to come from A to the second contact point P we go out 2.35 units, and go up 4.41 units.

Perpendicular Lines

In a 3x5 box, ACBD, we look at a line departing from A perpendicular to the diagonal CD intersecting it in S. In the right triangle ASC, tan A = CS/AS, and tan C = AS/CS. Consequently, tan A * tan C = 1. With tan C = 3/5, tan A = 5/3.

We now look at a circle having its center in A and its radius r = AC. Here AS intersects the circle in the point P. To find P and Q we observe that P is situated on the line AP and on the circle. With AQ = u and QP = v, we have

tan A = 5/3 = v/u, and u^2 + v^2 = 5^2, giving

(3/5v)^2 + v^2 = 25, or 1.36v^2 = 25, giving

v = 4.287, and u = 3/5*4.287 = 2.57

So, to come from A to the intersection point P we go out 2.57 units, and go up 4.29 units.

Technical Tasks

In the technical world, forms exist in abundance. Here are but a few examples.

● A six-edged screw head has two diameters. The big one is 27.7 mm. What is the little one?

● An axle-end increases the diameter from 15 mm to 25 mm over 30 mm. What is the angle?

● In a large tube with an internal diameter 10 cm must contain 3 equal tubes so that they touch each other precisely while each of the 3 tubes also touches the inside of the large tube. Calculate the outer diameter of the 3 tubes.

Astronomy

In astronomy you have sight-lines to objects far away in space. These lines may turn a little if they come from different places on the earth, e.g., from where the line is vertical and where it is horizontal. Knowing the diameter of the earth it is possible to calculate the distance from the earth to the sun, and then the diameter of the sun. Likewise, with the moon.

The amount of energy the earth receives from the sun is called the solar constant, 1.367 kW per square meter. However, if the square meter is turned v degrees away from the rays, the energy is reduced by cos v factor.

Triangles in a Coordinate System

In a coordinate system, a point has two coordinates x and y communicating how far out and how far up it is situated from a starting point, or how far back and how far down if using negative numbers also.

Two points thus creates a line with a length and a direction that can both be found by boxing the points. Thus the points A(2,1) and C(7,3) become the diagonal in a box with a horizontal length 7 – 2 = 5, and a vertical height 3 – 1 = 2. Also, the line constitutes a trip from A to C, called a vector AC, where the x coordinate has the change □x = 7 – 2 = 5, and the y-coordinate has the change □y = 3 – 1 = 2, thus giving the vector the coordinates AC = (Dy Dx) = (52). The Pythagorean and the tangent formula now allow finding the length of the line AC and its direction, also called its slope or gradient, as

AC = √(Δx^2 + Δy^2) = √((7-2)^2 + (3-1))^2 = √29 = 5.39; and

tan A = Δy/Δx = (3-1)/(7-2) = 2/5 = 0.4

Unless situated on the same line, 3 points as A(2,1) and C(7,3) and B(5,7) create a triangle with sides and angles that can be found in two ways, by boxing the triangle, or from its vectors.

Using boxing, the three points create a box with a horizontal length 7 – 2 = 5, and a vertical height 7 – 1 = 6. In this box, the triangle ABC is surrounded by three right triangles. Once these have their sides and angles determined, a simple subtraction gives the angles of the triangle. Likewise, with the area.

Using vectors is simpler. The triangle ABC is defined by the two trip vectors AC = (52) and AB = (36). A special formula called the dot-product formula now allows finding the angle between the two vectors:

AC*AB = (a1a2)*(b1b2) = a1*b1 + a2*b2 = IACI * IABI * cos A

Likewise, a formula called the cross-product formula allows finding the box area and the angle between two vectors:

AC x AB = (a1a2)x(b1b2) = a1*b2 – a2*b1 = IACI * IABI * sin A

Intersection in a Coordinate System

The three points A(2,1) and C(7,3) and B(5,7) create a triangle that may be boxed by a 6x5 rectangle. A line from a to the upper right corner M(7,7) will intersect the line BC in a point G with the coordinates u and v. To find it we equate tan A found in a lower and an upper triangle

tan A = 6/5 = (v – 1)/(u – 2) = (7 – v)/(7 – u)

This gives to equations with two unknowns that are solved by u = 5.75 and v = 5.5

6(u – 2) = 5(v – 1), and 6(7 – u) = 5(7 – v)

A quicker way is to find a formula for the points on the line AG by replacing u and v with x and y

6(x – 2) = 5(y – 1), giving y = 6/5(x – 2) + 1.

In the same way we may find the formula for the line through B and C: tan B = 4/2 = (7 – y)/(x – 5), giving y = -2(x – 5) + 7

Since the intersection point is situated on both lines, we equate their y's

y = 6/5(x– 2) + 1 = -2(x– 5) + 7, solved by x = 5.75 and y = 5.5

In a similar way, intersection points between a line and a circle are quickly predicted by their formulas.

Circular Motion

A stick with unit length rotates around its left endpoint. The right endpoint then moves in a circle if observed from the front. Seen from the side, its height becomes its sine-number. The endpoint moves quickly from height 0 and slows down around height 1 to return to height 0 again. Described in a coordinate system we then see that as the angle increases from 0 through 90 to 180 degrees, the sine-number increases quickly from 1 to 0 and slowly back to 1.

Allowing the rotation to continue after 180 through 270 to 360 degrees, we get a sine curve. Likewise, with a cosine curve where, seen from above, the endpoint decreases slowly from 1 to 0 and the quickly to -1. Seen from the front, the endpoint P's coordinates then will be P(cos v, sin v). Sine and cosine curves may be used to predict periodic phenomena as tide, solar height, alternating current, market fluctuations, etc.

7. EXPERIENCING THE POWER OF PREDICTION

The three trigonometry formulas and the Pythagorean formula allow us to experience the power of prediction. To find the diagonal in a 4x5 box we may use a 'formula table' to solve the equation occurring with only one unknown in the formula.

The table has two columns, one for the numbers, and one for the formula. On the top line we write the unknown number first and then the formula used to find it.

On the next lines we write the known numbers to the left, and to the right we write the equation occurring after the known numbers are inserted in the formula, leaving one unknown to be isolated by moving numbers to the opposite side with opposite calculation sign.

In the last line we test to see if the right-hand side and the left-hand side of the formulas produce the same numbers when inserting all the numbers.

c = ?	c^2 = a^2 + b^2
a = 4 b = 5	c^2 = 4^2 + 5^2 c = √41 c = 6.40
Test	6.40^2 = 4^2 + 5^2 ?? 41.0 = 41 ☺

On a graphing calculator, entering 'MATHSolver 0 = 4^2 + 5^2 – x^2' gives the same answer, x = 6.40.

Alternatively, c may be found from the two equations tan A = 4/5 giving A = 38.7; and sin 38.7 = 4/c giving c = 6.40.

Now books, tables, windows, doors, tiles, boxes, etc. may be used as examples.

Likewise, rotating a laser pointer 30 degrees form the 4-meter-wide floor allows predicting that the height of the spot on the wall will be 4 * tan 30, or 2.31 cm.

Distance to a Far Away Point

Wanting to predict the vertical distance PC across a river to an inaccessible point P, we use a known baseline, AB, to measure the angles A and B to P. With AB = m, BC = u, and PC = v, the two right angled triangles ACP and BCP give

tan A = v/(m+u), and tan C = v/u, giving

u = tan A/(tan B – tan A) * m, and

v = (tan B * tan A)/(tan B – tan A) * m

With AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees, we get

PC = v = (tan 55 * tan 34)/(tan 55 – tan 34) * 366 = 468.

Various Applications

In the outside world, a line will always be situated in space so it may often be of interest to box it to see how much is in vertical and horizontal direction. There are many such problems. They may be given as projects with open ended questions, or as specific exercises to be solved [16].

Project 1: What is the height of a high thing, e.g., a flagpole?

Project 2: How to move a thing around a corner?

Project 3: How to calculate shortcuts?

Project 4: How do you travel fastest from a point in one area to a point in another area when you move at different speeds in the two areas?

Project 5: How to calculate the different openings when a door opens ajar?

Project 6: How to build a path up a steep mountainside?

Project 7: How steep can a ladder be placed so as not to break glass or ice?

Project 8: How are astronomical distances calculated?

Project 9: How should a triangular bridge over a river be sized?

Project 10: On a swing, when to jump off to get a long jump?

Exercise 1. Determine the height of a high thing (a flagpole) in two different ways: a light way, where we can get close to the thing; and the hard way where we can't. Determine in the same way the width of a wide thing (a house wall or a ‘river’). If possible, check the calculations by measuring.

Exercise 2. A corridor with width 2 meters has a corner. A stick is turned around it without lifting. What is the maximum length of the stick? A box with a width of 1 meter is turned around the corner without lifting. What is the maximum length of the box? If possible, try with the other objectives also.

Exercise 3. A corridor with width 3 meters has a corner and continues with a width of 4 meters. A stick is turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 2 meters is turned around the corner without lifting. What is the maximum length of the box?

Exercise 4. A treasure can be found by first traveling 4 meters to the east, then 2.5 meters to the south, and finally 3.1 meter to the west. Find a shortcut to the treasure.

Exercise 5. A treasure can be found by first traveling 4 meters to the east in 32 degrees northerly direction, then 2.5 meters to the south in 68 degrees westerly direction, and finally 3.1 meter to the west in a 14-degree northerly direction. Find a shortcut to the treasure.

Exercise 6. A boat rows over a 50-meter-wide river at a speed of 20 meter/minute. The current in the river is 10 meter/minute. What angle should you choose for landing right across on the opposite side? How long does the trip take?

Exercise 7. Construct a straight line and two points on either side of the line. Go from one point to the other with steps equal to 1 shoe length. 1 second corresponds to 1 shoe length in one area and 2 shoe lengths in the other. Find the fastest route. Could the fastest route be figured out? If necessary, use a PC spreadsheet to calculate different routes. A formula applies to the incidence and refraction angle at the point on the boundary line which the fastest route passes. Which?

Exercise 8. Open a door ajar. There are now three openings, one perpendicular to the wall, one parallel to the wall and one parallel to the door. How big are these openings? How much will 10 degrees extra increase these openings by?

Exercise 9. Tip a plate 30 degrees and design a way up that can rise no more than 20 degrees (a hairpin bend). Repeat the exercise with other degree numbers. How much does the gravitational pull of a car increase when the rise of the road increases 10 degrees?

Exercise 10. Place a heavy book on a scale. Tilt it to different positions. What happens to the weight? Could this result be calculated? How much is the pressure against a vertical hand supporting it?

Exercise 11. Construct and load a triangular bridge and control the pressure against the surface by placing the bridge on two scales. Can these numbers be calculated?

Exercise 12. Distances in space cannot be measured, they must be calculated. How is the radius of the earth calculated? How far is it to the moon? What is the radius of the moon? How far is the sun? What is the radius of the sun?

Exercise 13 (difficult). A person sits in a swing that is suspended in ropes that are 3 meters long. Pull out the swing so that it is in height 1 meter above the bottom point. What angle does this correspond to?

If the swing is released, the speed v can be calculated from the formula: v^2 = 19.6 * h, where h is the distance from the maximum height in the outer position. The speed consists of a horizontal and a vertical part. Where do we have to jump off to get a long jump? (After the jump, the horizontal part of the speed will be unchanged, while the vertical part will grow downwards by 9.8 m/s every second.)

8. TRIGONOMETRY IN STEM

STEM is short for Science, Engineering, Technology and Mathematics that all see applications of trigonometry. The subjects are all rooted in a wish to predict the outside nature that we are part of.

To meet we must specify place and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is interacting matter in space and time.

A falling ball shows nature's three main ingrediences, matter and force and motion, like the three social factors, humans and will and obedience.

As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when moving in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases.

But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, core STEM problems could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water into electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to dissolving matter in water; to the orbit of a ball pulled down by gravity; to putting steam and electrons to work in a power plant creating an electrical circuit transporting energy from a source to many consumers.

Heavy things in motion are combined by the momentum = mass * velocity, a multiplication formula as most STEM formulas expressing re-counting by per-numbers:

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter, where force is the per-number change in momentum per second.

Thus, STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Science: Motion

Example 1. Pulled by gravity, a ball in a tube will start running the moment the tube is lifted 20 degrees. However, the gravity pull F must be boxed to see that only a part of it, F * sin 20, goes in the direction of the tube. Leaving the tube with a velocity vector v, this again must be boxed to see that v * sin20 is the vertical velocity and v * cos20 is the horizontal velocity.

Example 2. An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,73 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin 45 = cos 45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u * 0,707) * 0,73, solved as u = 9.69 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

Compared with the horizontal, the vertical distance is halved, but the speed changes from 9.69 to 9.69 * 0.707 = 6.85. However, the speed squared is halved from 9.69 * 9.69 = 94 to 6.85 * 6.85 = 47.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

Example 3. Light traveling from one medium to another with two different velocities v1 and v2 follows Snell's law of refraction. With angles of incidence w1 and refraction w2 both measured from the boundary normal, the law says that $\frac{\sin w1}{\sin w2} = \frac{v1}{v2}$

Engineering

Example 1, building a bridge. Over an 8-meter-wide canyon a suspension bridge made of steel is fastened to the points A and G on a cliff where A is 5 meters above G; and to the points B and F on a 3 meter vertical upright placed 1 meter from the edge, and fixed by a wire lifted 30 degrees. We want to determine the length of the 3 beams and the welding point C.

From the right-angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between line BG and line FA.

First in triangle EFB we use the sine-formula to find BE = 7.00. Then in the triangles FGA and GFB we use the Pythagorean formula to find AF = 10.30 and BG = 9.66.

Example 2, how many turns on a steep hill. On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u * cos(30). With Pythagoras we get

u^2 = CD^2 + BD^2 = CD^2 + u^2 * cos(30)^2, or

CD^2 = u^2(1–cos(30)^2) = u^2 * sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and AD = AC * cos(10). With Pythagoras we get

AC^2 = CD^2 + AD^2 = CD^2 + AC^2 * cos(10)^2, or

CD^2 = AC^2(1–cos(10)^2) = AC^2 * sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so Pythagoras gives

AC^2 = 1^2 + u^2, or AC = $\sqrt{}$(1 + u^2).

Consequently

u^2 * sin(30)^2 = AC^2 * sin(10)^2, or

u = AC * sin(10)/sin(30) = AC * r, or

u = $\sqrt{}$(1+u^2) * r, or

u^2 = (1+u^2) * r^2, or

u^2 * (1–r^2) = r^2, or

u^2 = r^2/(1–r^2) = 0.137, giving the distance

BC = u = $\sqrt{}$0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Technology

Pressure machines typically have a piston at the top of a cylinder with water that evaporated makes the piston go up, and vice versa go down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g., steam locomotives.

The force on the piston then has to be applied to a rotating disk in order to have vertical motion transformed into horizontal motion. So, we need to box the force twice. First to see how much goes in the direction of the pin connecting the piston to the disk. Next, boxing this to see how much goes in the tangent direction to turn the disk.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating lifts the

pressure by increasing both the temperature and the number of steam molecules. And vice versa in the cold cylinder where cooling lowers the pressure by decreasing both the temperature and the number of steam molecules that, when condensed to water, are pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to industries and homes.

9. CONCLUSION

We asked if also mathematics can be learned as communication. Well, only if there is something out there to communicate about. And indeed, there is, boxes. They transform a total into a Hindu-Arabic number. And they split in two by their diagonals creating trigonometry formulas when recounting the sides mutually. Or the other way around, boxing a line makes it a diagonal allowing trigonometry formulas predict its horizontal and vertical sides. Which leads to many core STEM examples.

This of course is to be expected given the original meaning of the word geometry and algebra. With geometry meaning earth-measuring in Greek, of course it may be learned as communicating about how to measure earth; and with algebra meaning re-uniting in Arabic, of course it may be learned as communicating about how to unite changing or constant unit- and per-numbers [17].

So, it seems that instead of following the ruling 'truth regime' [18] teaching one-dimensional line numbers instead of two-dimensional box- and bundle-numbers, and teaching plane geometry before trigonometry, we may be able to master outside boxes without learning plane geometry first. And the question is really, do we need to learn it at all? Of course, it has a quality that many theorems about general cases may be proven. Bet is it not far more interesting and important to use formulas to predict specific cases? And as long as no counter-example has been found, the formula works, and therefore it should be valid according North American pragmatism.

And since societies around the world scream for technicians and engineers able to master STEM problems, maybe the time has come to change the status of plane geometry from mandatory to elective for people finding it interesting.

Thus it seem possible that also the number-language should have its communicative turn as had the word-language in the 70s. Accepting the mastery of Many children develop before school will make a box the core outside subject to master, inside described by two-dimensional flexible bundle-numbers allowing fractions, decimals and negative numbers to communicate about what to do with the unbundled singles, and allowing number-language sentences, formulas, to inside predict what will happen outside.

And, experiencing the excitement of working with predicting formulas will dramatically increase the number of students wanting to go on to create a successful future within the STEM subjects.

Therefore, teaching trigonometry before geometry well help boys acquire a self-identity as masters of Many, thus improving the world economy to solve poverty and climatic issues. So maybe the time has come to realize that changing from pre- setcentric and setcentric to the child's own post-setcentric mathematics will make the number language a human right in accordance with fourth the United Nation's sustainable development goals, the right to quality education.

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TOWARDS TRUTH, BEAUTY AND GOODNESS IN MATHEMATICS EDUCATION

After more than half a century of research, many students still struggle with mathematics in spite of its many applications. And of course, to be applied, mathematics must first be learned. So, the goal of mathematics education must be to learn mathematics as defined by universities. However, people outside mathematics question the self-reference in this goal description asking why its many practical uses could be the end goal instead; or if the goal could be to contribute to the overall goal of education, formulated by Gardener as helping students understand and act on the basis of what is true, beautiful and good. Consequently, this paper asks which kind of mathematics can do precisely that. Often mathematics carries all three predicates, but sometimes it carries the opposite, false and ugly and bad. Therefore, difference research searching for differences making a difference looks for different kinds of mathematics in its history, in the viewpoints of the three basic human sciences, philosophy and psychology and sociology, and in the mastery of Many children develop before school when adapting to the physical fact Many. Apart from the present setcentric version following the New Math ideal, two other forms were found: a pre setcentric version rejecting it, and a post setcentric version respecting children's own mathematics. Two curricula in post setcentric mathematics are designed, one in counting and recounting and one in adding on-top and next-to, turning out to introduce core mathematics as proportionality, calculus and solving equations in grade one.

Keywords: calculus, equations, numeracy, proportionality, trigonometry

DECREASED PISA PERFORMANCE DESPITE INCREASED RESEARCH

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education, ICME, taking place each four year since 1969. Likewise, funding has increased as witnessed, e.g., by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result in mathematics decrease. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

To help students master the outside world, schools institutionalize education with subjects as inside means to outside goals. To each goal there are many means, to be replaced if not leading to the goal; unless a goal displacement (Bauman, 1990) makes a means the goal, thus preventing looking for alternative means that instead could lead to the original end goal.

Therefore, we ask: Is the end goal of mathematics education to master Many; or to master mathematics as a possible means to later master Many? And if institutionalized mathematics is too difficult a means to master, maybe other forms of mathematics should be looked for leading to the end goal.

Here a direction may be found in the general goal for education as described by Gardner, known for his theory on MI, multiple intelligences:

> In my book The Disciplined Mind, published in 1999, I put forth a simple educational agenda: To help students understand, and act, on the basis of what is true, what is beautiful and what is good. I believed – and still believe – in that agenda. (Gardner 2011, xiv)

With this end goal we can ask: which kind of mathematics can help students understand, and act, on the basis of what is true, what is beautiful, and what is good?

This question of course raises an immediate objection: "But, don't we teach mathematics precisely because it is both true, beautiful and good?"

Can Mathematics be False?

In addresses I often ask the audience reflect on the questions below. Then I show the expected answers with a cross to which the audience agrees. Then I show the correct answers with a dot to which the audience objects strongly asking for an explanation.

Figure 1. A Questionnaire to Teachers

This is true	Always	Never	Sometimes
2 + 3 = 5	x		●
2x3 = 6	x ●		
1/2 + 2/3 = 3/5		x	●
1/2 + 2/3 = 7/6	x		●

"In 2x3, 3 is the unit. And 2 3s may always be recounted as 6 1s. As to 2 + 3, it may be 5 with like units, else 2weeks + 3days = 17days, etc. When we count 3 apples, it means 3 times an apple. So, we cannot add without units as we see when writing out numbers fully:

345 = 3 x 100 + 4 x 10 + 5 x 1.

The same goes for fractions. The schoolbook answer 7/6 only holds if taken of the same total. Whereas the answer 3/5 holds in the following case: 1 red of 2 apples plus 2 red of 3 apples total 3red of 5 apples and certainly not 7 red of 6 apples.

So, we may coin the word mathe-matics for what is true always, and 'mathe-matsim' (Tarp, 2018) for what is true sometimes under certain assumptions, or else false.

The New Math also have some truth problems. Here the invention of the concept 'set' created a setcentric (Derrida, 1991) defining its concepts by self-reference, i.e., top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

Can Mathematics be Ugly?

Let us look at a saving established by monthly depositing a single amount c at a single interest rate r. After every month the ratio between two capitals C and c is the same as the ratio between the two interest rates R and r.

C/c = R/r

The proof is simple and beautiful: from an account with c/r$, each month the interest is moved to another account also receiving its own interest, thus containing (c/r) * R as a saving C, which gives C/c = R/r.

Textbooks substitute the total interest R with (1+r)^n and use geometric series to produce a proof. Ugly? Perhaps not, but certainly not as beautiful as this proof.

Can Mathematics be Bad?

The core of mathematics is inside calculations with specified or unspecified numbers as 2 * 3 or 2 * p or p * q predicting a number for an outside subject, typical a total, that of course should be included to have a full sentence in the number-language as in the word-language. Such number-language sentences could be called functions also

T = 2 * 3 or T = 2 * p.

And, of course, a calculation produces one result only. So, it seems like a joke when setcentric mathematics defines a function as an example of a set-product where first component identity implies second component identity.

Wanting to define an abstract concept as an example of a more abstract concept, you basically teach "bubibub is an example of babibab" thus preventing students from understanding and acting with the most important tool of the number-language, its calculating sentences.

This is bad. And it allows us to coin the phrase 'meta-matics' (Tarp, 2018) for a turning mathematics upside down by defining a concept as an example of an abstraction instead of as an abstraction from examples.

Yes, Mathematics can be Both False, Ugly and Bad

So yes, mathematics can be true, beautiful and good; but it can also be false, ugly, and bad if transforming mathematics linked to its outside roots into 'meta-matism' (Tarp, 2018) mixing self-validating mathe-matism including statements true inside but not necessarily outside a classroom with self-referring meta-matics defining concepts as examples of abstractions instead of as abstractions from examples.

Therefore, we now look for other forms of mathematics that allow students understand, and act, on the basis of what is true, beautiful, and good.

DIFFERENT KINDS OF MATHEMATICS

To get an impression of different kinds of mathematics we now look at world history and at the history of mathematics and at its most important application in proportionality.

A Short World History

When humans left Africa, some went west to the European mountains, some went east where the fertile valleys in India supplied everything except for silver from the mountains. Consequently, rich trade took place sending pepper and silk west and silver east. European culture flourished around the silver mines, first in Greece then in Spain during the Roman Empire. Then the Vandal conquest of the mines brought the dark middle age to Europe until silver was found in the Harz valley (in German, Tal, leading to thaler and dollar), transported through Germany to Italy. Here silver financed the Italian Renaissance, going bankrupt when Portugal discovered a sea route to India enabling them to evade Arab middlemen. Spain looked for a sea route going west and found the West Indies. Here there was neither pepper nor silk but silver in abundance, e.g., in the land of silver, Argentine. On their way home, slow Spanish ships were robbed by sailing experts, the Vikings descendants living in England, now forced to take the open sea route to India to avoid the Portuguese fortification of Africa's coast.

In India, the English found cotton that they brought to their colonies in North America, but needing labor they started a triangle-trade exchanging US cotton for English weapon for African slaves for US cotton. In the agricultural South, a worker was a slave fed to minimize cost, but in the industrial North a worker needed a salary to become a consumer at a free market. During the civil war, no cotton came to England that then conquered Africa to bring the plantations to the workers instead. Dividing the world in closed economies kept new industrial states out of the world market that it took two world wars to open for free competition.

A Short History of Mathematics

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With silver mines and slaves, the Greeks had ample time to develop art and philosophy; and in their general curriculum they soon found geometry and logic to be useful rhetoric tools in a democracy. This let to Euclidean geometry with its strict definitions and proofs that still today is taught as basic geometry, but that also has hampered the development of mathematics for centuries (Kline, 1972).

As to arithmetic, the Greeks used letters. Instead, they looked for geometrical forms to show the different degrees of Many.

With empty silver mines the Greek dominance stopped. Now the Romans took over and used silver from Spain to build an empire they could tax. So, they developed numbers for counting by tallying, and, stopping at thousand, they bundled each time they had five ones, tens or hundreds:

V, X, L, C, D, M for 5, 10, 50, 100, 500, 1000.

With the emperor as the only warlord there was no need for big armies to fight other warlords as in China needing formations of hundred by hundred called a 'wan'.

Although good at counting, Roman numbers cannot multiply. So, when the far east trade reopened with German silver financing the Italian renaissance, trade needed multiplication and division, so with

the advantage of bundling in tens only, Hindu-Arabic numbers quickly replaced the Roman numbers. So, arithmetic was replaced by Arab algebra, meaning 'to reunite'.

Furthermore, the wealth created through trade led to the creation of a bank system where interest calculation developed power and its two reverse operations, root and logarithm. Here the latter's ability to transform multiplication to addition was taught until the arrival of the pocket calculator. Arab traders also brough trigonometry to Europe as an effective tool for astronomy.

Italy went bankrupt when Portugal found a sea route to India allowing them to decreasing the price on pepper to 1/3 by evading the Arab middlemen. Spain then tried to find another way to India by sailing west where instead they found the West Indies and the Americas.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa they had to sail to India on open sea following the moon that according to the pope moved among the stars following the unpredictable will of the Lord. Newton objected: "No, the moon falls towards the earth as does the apple pulled down by a gravitational force predicted by its formula." The pope then argued that forces uphold motions. Newton objected: "No, they change motions. And since algebra calculate levels only, I had to develop calculus to calculate change."

Later in World War II, the US developed operational research to optimize the logistics between weapon factories and battle fields, which led on to the creation of computers.

Proportionality Illustrates the Variety of Mastery of Many

Proportionality is rooted in the two questions Q1 and Q2:

"2kg costs 5$, what does 7kg cost; and what does 12$ buy?"

Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, the divide.

So first we ask, Q1: "2kg cost 5$, 7kg cost ?$" to get to the answer (7 * 5/2)$ = 17.5$.

Then we ask, Q2: "5$ buys 2kg, 12$ buys ?kg" to get to the answer (12 * 2)/5$ = 4.8kg.

Then, two new methods appeared, 'find the unit', and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7 * (5/2)$ = 17.5$.

Q2: 1$ buys 2/5kg, so 12$ buys 12 * (2/5)kg = 4.8kg.

Q1: 2/5 = 7/x, so 2 * x = 7 * 5, x = (7 * 5)/2 = 17.5.

Q2: 2/5 = x/12, so 5 * x = 12 * 2, x = (12 * 2)/5 = 4.8.

The New Math chose modeling with linear functions to show the relevance of abstract algebra's group theory:

Let us define a linear function f(x) = c * x from the set of kg-numbers to the set of $-numbers, having as domain DM = {x∈R I x>0}.

Knowing that f(2) = 5, we set up the equation f(2) = c * 2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law:

c * 2 = 5, so (c * 2) * ½ = 5 * ½, so c * (2 * ½) = 5/2, so c * 1 = 5/2, so c = 5/2.

With f(x) = 5/2 * x, the inverse function is f-1(x) = 2/5 * x.

So with 7kg, f(7) = 5/2 * 7 = 17.5$

And with 12$, f-1(12) = 2/5 * 12 = 4.8kg.

In the future, we may simply 're-count' in the 'per-number' 2kg/5$ coming from 'double-counting' the total T. (Tarp, 2018)

Q1: T = 7kg = (7/2) * 2kg = (7/2) * 5$ = 17.5$

Q2: T = 12$ = (12/5) * 5$ = (12/5) * 2kg = 4.8kg

Or, we simply recount the units

Q1: $ = ($/kg) * kg = (5/2) * 7 = 17.5

Q2. kg = (kg/$) * $ = (2/5) * 12 = 4.8

GRAND THEORY LOOKS AT MATHEMATICS

After searching for different forms of mathematics by looking at history we now seek additional knowledge from the grand theories about human behavior, philosophy and sociology and psychology. This may be relevant since mathematics education means choosing supposedly true statements to mediate to humans forced to return to the same room daily to hopefully understand what is being communicated.

Here philosophy may inform us about the notion of truth as well as about the role of the language used to communicate; and psychology may inform us about concept formation in student brains; and sociology may inform us about the social consequences of institutionalizing truth and education.

Philosophy

Ancient Greece saw a controversy between two views on knowledge, called 'sophy' in Greek. The sophists warned: Know the difference between nature and choice to avoid hidden patronization presenting choice as nature. To their 'better-knowing' counterpart, the philosophers, choice was an illusion since to them, the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy.

But the philosophers themselves also disagreed on how to obtain knowledge, through experience or through reason, thus forming two schools, Anglo-Saxon empiricism and continental rationalism, today expressed in the debate between essentialism and existentialism defined by Sartre (2007) as holding that existence precedes essence.

As existentialist, Heidegger (1962) sees three of our seven basic is-statements describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip. So, by warning against predicates, Heidegger argues that true learning comes from openly meeting the subject, and show skepticism about predicates, even if institutionalized.

The two schools also disagree on whether the physical fact Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it.

Here, university mathematics holds that Many should be seen as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3 * BB + 4 * B + 5 * 1, resonating with what children bring to school, e.g. T = 2 5s.

So, according to philosophy, two different forms of mathematics exist. One sees a concept as abstracted from experience so that a 'function' labels a calculation including specified and unspecified numbers in order to distinguish between 2 * 3 and 2 * x (Euler, 1748). The other sees a concept as exemplified from abstractions and defines a 'function' as an example of a subset of a set product where first component identity implies second component identity.

Psychology

How the brain adapt to the outside world is theorized by psychology, often seen as the science of mind and behavior. Besides the reptile and mammal brains for routines and feelings, humans also have a third human brain for balancing and for storing and sharing information, made possible by transforming forelegs to arms with hands that can grasp (and share) food and things, which accompanied by sounds has developed a language.

Transmitting information may be called teaching, and receiving information may be called learning. Together, teaching and learning may be called education, that may be unstructured or structured by a social institution called education.

Asking if correct responses imply understanding created the educational psychology called constructivism focusing on what happens in the mind when constructing inside meaning to outside stimuli.

Piaget (1970) sees learning as inside schemas assimilating the outside world, or being accommodated to do so. Holding that no children master logical thinking before 11 years, and therefore needing to be taught using concrete objects and examples, Piaget warns against too much teaching by saying: "Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life" (azquotes.com).

Using the existentialist distinction between existence and essence, Piaget thus recommends adaption to outside existence, while Vygotsky recommends adaption to inside institutionalized essence, i.e., to enculturation allowing learners to expand their 'Zone of Proximal Development' (ZPD) under guidance of a more knowledgeable other. "What a child can do today with assistance, she will be able to do by herself tomorrow" (azquotes.com).

So, according to psychology, as in philosophy, two different forms of mathematics education exist. Piagetian radical constructivism sees learning constructed through peer-discussions about the outside subjects of mathematics, thus pointing to the need of quality guiding materials. And Vygotskyan social constructivism sees learning constructed through peer-discussions about of the inside essence of mathematics guided by a more knowledgeable teacher thus pointing to the need of quality teachers able to identify and stimulate the individual ZPDs (Tarp, 2020).

Sociology

Reviving the ancient Greek sophist skepticism about patronization masking choice as nature, the Enlightenment created two democracies, one in North America still having their first republic, and one in France now having their fifth republic. Later in Germany, Weber (1930) was the first to theorize the increasing social goal-oriented rationalization that disenchants the world and create an iron cage if carried to wide.

North American showed skepticism about rationalist philosophy by developing pragmatism (Menand, 1997) leading to symbolic interactionism (Blumer, 1998) and Grounded Theory (Glaser et al, 1967).

Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16).

Living together involves deciding upon which tasks to do individually and which to do collectively by creating rational institutions, "in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)." However, Bauman also warns against the danger of so-called goal displacement where "The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84)."

Such a goal displacement occurs if saying 'The goal of mathematics education is to master mathematics". By its self-reference such a goal is meaningless. So, if mathematics isn't the goal of mathematics education, it is but a means to reach the real end goal, its original roots, to master Many. Which of course should be reached by other means if a goal displacement has made institutionalized mathematics an obstacle instead of a means.

This quest may be inspired by the French post-structuralist thinker Foucault saying

It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

From a Foucauldian viewpoint, education may become a 'pris-pital' mixing power and surveillance techniques from a prison and a hospital. Prison, when forcing humans to return to the same room each hour, day, week, and month for several years as in the continental European Bildung education. And hospital, when forcing self-referring diagnoses upon the 'pati-mates' as "You lack knowledge about mathematics. Therefore, we will cure you with daily doses of mathematics." Therefore, to cure we must be sure the diagnose is not self-referring.

So, seeing mathematics education as an institution, a Weberian viewpoint would ask if the set-concept is a rationalization of Many gone too far leaving Many disenchanted and leaving the learners in an iron cage.

And a Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. Institutions are means, not goals. So, as an

institution, mathematics is a means, hence the word 'mathematics' must go from goal descriptions. If not, there is a risk that mathematics education will practice 'the banality of evil' (Arendt, 1963).

So, sociology would recommend replacing European line-organized education with the North American block-organized education changing cell each hour, and changing the diagnose twice a year; thus, complying with the OECD 2030 Learning Framework recommending individual learning.

And recommend that the truth regime saying 'the goal of teaching math is learning math' should be replaced by the real end goal, mastery of Many, e.g., by uncovering and developing the existing mastery of Many children create through adaptation to Many before school.

In short, it may be time to show skepticism about institutionalism and use difference research (Tarp, 2018) to look for differences that may make a difference.

PRE, PRESENT, AND POST SETCENTRIC MATHEMATICS

So, to further explore the difference between mathematics based upon essence and existence, first we look at a typical contemporary mathematics curriculum, next we return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

A Contemporary Math Curriculum

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g., that 2 + 3 = 5.

This view offers a 'natural' curriculum with one-dimensional linear multi-digit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

Reverse operations may create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions, decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as $\sqrt{2}$ and log 3 where $\sqrt{2} * \sqrt{3} = \sqrt{6}$, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g., that (a + b) * (a–b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles, and areas. Later, geometry and algebra are coordinated in coordinate geometry. To be followed by right-angled triangles relating the sides and angles with trigonometrical operations as sine, cosine and tangent where $\sin(60) = \sqrt{3}/2$.

In a formula, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified formula. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g., that (f * g)' /(f * g) = f' /f + g' /g. And with a reverse operation, integration, again creating additional education about results of operating on calculations, e.g., that $\int 6 * x^2 dx = 2 * x^3 + c$, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, the outside use of mathematics may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having the same three genres as has the qualitative: fact, fiction and fiddle (Tarp, 2001).

How Children Communicate About Many

How to master Many may be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, "That is not four, that is two twos'", thus describing what exists, and with units: bundles of 2s, and 2 of them.

So inside, children use two-dimensional bundle-numbers as representation for outside boxes, which resonates nicely with the box-structure of Hindu-Arabic numbers where 345 = 3 * B^2 + 4 * B + 5 * 1, but not with the one-dimensional cardinal numbers that schools teach.

Children also use box-numbers when talking about snap cubes as '2 3s' and '4 5s'. When asked "How many 5s when united?" they typically say '5 5s and 1 extra'; and when asked "How many 3s?" they may say '9 3s less 1'; and, placing them next-to each other, they typically say '3 8s and 2 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s & 5, 2 2s & 3, 3 2s & 1, 4 2s less 1, 1 double2s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as T = 7 = 1B5 = 2B3 = 3B1 = 4B-1; or even as 1BB3 or 1BB1B1. Also, children love to count in 3s, 4s, and in hands.

In short, children love numbering without having learned about numbers. Apparently, children learn mathematics through communication (Widdowson, 1978).

A POST SETCENTRIC COUNTING CURRICULUM

So, since children already talk about Many before school, why not let mathematics education develop instead of reject the core mastery of Many that children bring to school?

Digits as Icons

Many exists in space as a total, and in time as counting repetitions where adding a stick each time gives a spatial representation.

A total of 4 sticks may be rearranged as 1 4-icon containing 4 sticks; and likewise, with the other digits. So, we may say that a digit is a number-icon containing as many sticks or strokes as it represents if written less sloppy. The icon for zero is a magnifying glass finding nothing. Ten is a special number with its own name but with no icon. That is because we count by bundling in tens, which makes ten a double counting of bundles and singles, where 10 is short for 1B0, 1bundle and no unbundled. So, where letters are mere symbols unconnected to their sounds, digits are icons communicating directly about their degree of Many.

Bundle-Counting Makes Operations Icons Also

Counting a total of 9 in 2s, first we push away 2s, iconized by a division-broom, 9/2. Then we stack the bundles 4 times, iconized by a multiplication-lift, 4 x 2 or 4 * 2. Then we pull away the stack to

look for unbundled singles iconized by a subtraction-rope, 9 – 4 * 2. Finally, we use an addition-cross to iconize the two ways to unite boxes, on-top or next-to, 9 = 4 * 2 + 1.

Iconizing the counting operations allows a calculator predict the result.

Entering 9/2 gives the result 4. some predicting that from 9 we can push 2 away 4 times. Entering 9 – 4 * 2 gives the result 1 predicting that 9 = 4 * 2 + 1 as can be observed.

Outside, counting a total of 9 in 2s means boxing 9 as 4 2s with an unbundled single on-top. Inside we report this as T = 9 = 4B1 2s or T = 4.1 2s using a decimal point to separate the bundles from the unbundled. Or as T = 9 = 4 ½ 2s indicating that the single should also be counted in bundles. Or as T = 9 = 5B-1 2s indicating that 1 is needed for another bundle, thus creating an outside underload, where T = 9 = 3B3 2s creates an outside overload.

This again allows the outside total 9 to inside become a 'flexible bundle-number'

T = 9 = 3B3 2s = 4B1 2s = 5B-1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Decimals, fractions and negative numbers thus describe how to see unbundled singles.

Recounting Creates a Recount-Formula

Recounting a total of 8 1s in 2s by 8/2 times pushing away 2s to be stacked as (8/2) * 2 allows writing a 'recount-formula' 8 = (8/2) * 2, or T = (T/B) * B with unspecified numbers, saying 'From T, T/B times, B is pushed away.'

Recounting a total of 5 6s in 7s, again operations allow a calculator predict the result. Entering (5 * 6)/7 gives the result 4. some predicting that from 5 * 6 we can push 7 away 4 times. Entering 5 * 6 – 4 * 7 gives the result 2 predicting that 5 * 6 = 4 * 7 + 2 as can be observed.

Outside, counting a total of 8 1s in 2s means squeezing the 8 x 1 box into a 4 x 2 box. Thus, increasing the width means decreasing the height.

And when squeezing a 5 x 6 box into a 4.2 x 7 box, increasing the width means decreasing the height.

Recounting or squeezing the total 5 6s in tens we miss a ten-button on the calculator. However, multiplication gives the result directly:

T = 5 6s = 5 * 6 = 30 = 3.0 tens

Again, we see that increasing the width means decreasing the height.

Recounting or squeezing the total 40 in 6s, a calculator predict that 40/6 = 8.some, and 40-8 * 6 = 2, so 40 = 8 * 6 + 2 = 8 2/6 6s.

However, using the letter u for an unknown number allows recounting 40 in 6s to be formulated as an equation u * 6 = 40, quickly solved by recounting, u = 40/6.

Recounting Solves Equations

The recount-formula may be used to solve equations. Thus, rephrasing recounting 8 in 2s as 'How many 2s in 8?' gives the equation 'u * 2 = 8', immediately solved by recounting 8 in 2s, u = 8/2:

u * 2 = 8 = (8/2) * 2, giving u = 8/2.

By definition of the opposite operations, we see that solving equations by moving numbers to the opposite side with the opposite calculation sign works for all operations.

The subtraction 8 – 2 is defined as the number u that added to 2 gives 8.

The division 8/2 is defined as the number u that multiplied to 2 gives 8.

The root 8√2 is defined as the factor u that applied 8 times gives 2.

The logarithm log2(8) is defined as the number u of 2-factors that gives 8.

Figure 2. Basic Equations Solved by Moving to Opposite Side with Opposite Calculation Sign

u + 2 = 8	*u* * 2 = 8	*u*^8 = 2	2^*u* = 8
u = 8 – 2	*u* = 8/2	*u* = 8√2	*u* = log2(8)

Recounting Changes Units, Called Proportionality

The recount-formula T = (T/B) * B changes the unit to Bs. The formula may also be called a proportionality formula used to change both number units and physical units.

Thus double-counting apples as 4$ and 5kg creates a 'per-number' 4$/5kg as a link between the units.

So 12$ and 10kg quickly change unit by recounting in the per-number:

T = 12$ = (12/4) * 4$ = (12/4) * 5kg = 15kg.

T = 10kg = (10/5) * 5$ = (10/5) * 4$ = 8$.

With like units, per-number become fractions, 4$/5$ = 4/5 = 4part/5total.

Again, 12part and 10total quickly change unit by recounting in the per-number:

T = 12part = (12/4) * 4part = (12/4) * 5total = 15total.

T = 10total = (10/5) * 5total = (10/5) * 4part = 8part.

Likewise, with percentages, e.g., 40% = 40/100.

Again, 12part and 10total quickly change unit by recounting in the per-number:

T = 12part = (12/40) * 40part = (12/40) * 100total = 30total.

T = 10total = (10/100) * 100total = (10/100) * 40part = 4part.

Totals May be Squeezed or Squared

So, outside to squeeze boxes changing unit from icons to tens or the opposite means inside multiplying or solving equation.

Wanting to square the 4 x 5 box into a b x b box, the equation b * b = 4 * 5 = 20 is solved by b = √20.

To find √20 we observe that in a 4 + t square, removing the 4 square leaves 4 * 5 – 4 * 4 = 4 shared by the two 4xt boxes, i.e., 4 = 2 * 4 * t, giving t = 0.5. A little less since we neglect the t square. Inside, a calculator confirms that √20 = 4.472.

In geometry, intersection points between lines and circles leads to quadratic equations as x^2 + 6x + 8 = 0, easily solved when rotating the upper of two x by x + 3 tiles to create a square with sides x + 3, which is zero apart from the 1 left in the upper 3 x 3 square after 8 is removed.

In total, (x + 3)^2 = 0 + 1 = 1 = $\sqrt{}$ 1^2

So x + 3 = ±1, giving x = - 2 and x = - 4.

Trigonometry: Recounting a Box Halved by its Diagonal

So, when counting and recounting, the core outside object is a rectangular box that may be called a stack, a tile, or a block. In a h x b box, we call the base b or AC, and the height h or CB. Here the diagonal d or AB links the box in two like right angled triangles, which may also be called 'half-boxes'.

Alternatively, we may say that we have boxed the line AB so it becomes a diagonal with a base and a height that may also be called its floor and wall, or horizontal and vertical shadow, or run and rise. We will not use the Greek name 'cathetus' since it does not distinguish between the two sides.

Besides its three sides, a half-box also has three angles counting the turning of the lines in degrees. The turning from the base to the diagonal is called the angle A. The quarter round turning from the base to the height is called the right-angle C, which is counted to 90 degrees since a full round is 360 degrees.

With two like triangles in the full box we see that A + B = 90.

Trigonometry finds formulas that predict the sides and angles in a half-box.

The sides may be counted in standard units. But they may also be recounted in the box's own units.

Recounting the height in the base gives

height = (height/base) * base = tangent A * base = tangent A bases, abbreviated to

h = (h / b) * b = tan A * b = tan A bs, thus giving the formula

tangent A = height / base, or

tan A = h/b.

The word tangent is used since in a circle with center in A and the base as its radius, the height will be a tangent. This allows two ways of counting angles, by degrees or by tangent numbers.

Thus outside, in a 1 x 2 half-box, the angle A is counted on a protractor to 27 degrees; and in a 1 x 1 half-box, the angle A is counted to 45 degrees; and in a 2 x 1 half-box, the angle A is counted to 63 degrees. Inside, these numbers may be predicted by the tangent formula

Tan A = ½ gives A = opposite-tan ½ = tan-1(½) = 27

In the same we can create other trigonometry formulas as

sin A = height/diagonal, or sin A = h/d

cos A = base/diagonal, or cos A = b/d.

A Circle as Many Right Triangles

In an h x r half-box with area ½ * h * r we look at the circle with center in A and r as its radius. Here h = (h/r) * r = tan A * r.

A half circle is 180 degrees that may split in 100 small parts as

180 = (180/100) * 100 = 1.8 100s = 100 1.8s

With A as 1.8 degrees, the length of the circle and the tangent, h, are almost identical. So, with good approximation

Half the circumference C = 100 * h = 100 * tan 1.8 * r = 100 * tan (180/100) * r = 3.1426 * r

Or better, C = 1000 * tan (180/1000) * r = 3.1416 * r

Calling the circumference for 2 * π * r, we get a formula for the number π:

π = tan (180/n) * n for n sufficiently large

The area of the full circle then is

A = 2 * ½ * 3.1416 * r * r = 3.1416 * r^2 = π * r^2

Adding Sides in a Half-Box

Arranging 4 like half-boxes as a (base + height) square, it contains a diagonal square plus 4 half-boxes.

However, it also contains a height square and a base square plus 2 full boxes.

Consequently, the height square and the base square add up to the diagonal square as predicted inside by the Pythagorean formula h^2 + b^2 = d^2.

So, in right angled triangles, the sides add by their squares.

Thus, in a 3 x 4 half-box, the sides add by their squares as 3 * 3 + 4 * 4 = 25 giving a 2.5by10 box that squared into a d x d box gives the equation d^2 = 25, giving the solution d = √25 = 5.

A POST SETCENTRIC ADDING CURRICULUM

After totals have been counted and recounted to be represented as outside boxes and inside flexible bundle-numbers, the time har come to ask how to unite and split totals.

Writing out a number fully as T = 345 = 3BB + 4B + 5 = 3 * B^2 + 4 * B + 5 uncovers the general bundle-formula T = a * x^2 + b * x + c called a polynomial.

It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration.

The tradition teaches addition and multiplication together with their opposite operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

We might call this beautiful simplicity ‘the algebra square’.

Figure 3. The Algebra Square Reunite into Total to be Split into the Four Number-Types

Operations unite/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\,dx$ $dT/dx = f$	$T = a\hat{}b$ $b\sqrt{T} = a \quad loga(T) = b$

Next-To Addition

An exemplary question to answer could be

“With T1 = 2 3s and T2 = 4 5s, what is T1 + T2 when added next-to as 8s?”

Outside we see that next-to addition of boxes means adding by areas where multiplication precedes addition. This shows that next-to addition is an example of integral calculus.

Inside, the recount-formula predicts the result:

Entering (2 * 3 + 4 * 5)/8 gives the answer ‘3.some’.

Entering (2 * 3 + 4 * 5) – 3 * 8 gives the answer ‘2’.

The calculator thus predicts that 2 * 3 + 4 * 5 = 3 * 8 + 2 which is also observed.

Reversed Next-To Addition

An exemplary question to answer could be

“If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?”

Outside we see that by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Inside, the recount-formula predicts the result:

Entering (4 * 7 – 2 * 3)/(7 – 3) gives the answer ‘5.some’.

Entering (4 * 7 – 2 * 3) – 5 * 4 gives the answer ‘2’.

The calculator thus predicts that 2 * 3 + 5 * 4 + 2 = 4 * 7 which is also observed.

On-Top Addition

An exemplary question to answer could be

“With T1 = 2 3s and T2 = 4 5s, what is T1 + T2 when added on-top as 3s; and as 5s?”

Outside we see that on-top addition of boxes means changing the units to be the same by using the recount-formula. Thus, on-top addition applies proportionality.

Inside, the recount-formula predicts the result:

Entering (2 * 3 + 4 * 5)/3 gives the answer ‘8.some’.

Entering (2 * 3 + 4 * 5) – 8 * 3 gives the answer ‘2’.

The calculator thus predicts that 2 * 3 + 4 * 5 = 8 * 3 + 2 which is also observed.

Likewise, the calculator predicts that 2 * 3 + 4 * 5 = 5 * 5 + 1 which is also observed

Reversed On-Top Addition

An exemplary question to answer could be

“If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?” Outside we see that by removing the initial block and recounting the rest in 5s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Inside, the recount-formula predicts the result:

Entering (4 * 5 – 2 * 3)/5 gives the answer ‘2.some’. Entering (4 * 7 – 2 * 3) – 2 * 5 gives the answer ‘4’.

The calculator thus predicts that 2 * 3 + 2 * 5 + 4 = 4 * 5 which is also observed.

Adding Tens

An exemplary question to answer could be

“With T1 = 23 and T2 = 48, what is T1 + T2 when added as tens?”

We may begin by rewriting the totals as flexible bundle numbers:

“With T1 = 2B3 and T2 = 4B8, what is T1 + T2 when added as tens?”

Outside, we get 2 + 4 boxes and 3 + 8 unbundled written as T = 6B11 or 7B1.

Inside, we get

Figure 4. Bundle Addition

***T*1 + *T*2**	**23 + 48**	**236 + 487**
*T*1 *T*2	2 *B* 3 4 *B* 8	2 *BB* 3 *B* 6 4 *BB* 8 *B* 7
T	6 *B* 11 7 *B* 1	6 *BB* 11 *B* 13 6 *BB* 12 *B* 3 7 *BB* 2 *B* 3
T	**71**	**723**

Subtracting Tens

An exemplary question to answer could be

“If T1 = 23 and T2 add to T = 71, what is T2?”

We may begin by rewriting the totals as flexible bundle numbers:

“With T1 = 2B3 and T2 add to 7B1, what is T2?”

Outside, we get 7-2 boxes and 1-3 unbundled written as T = 5B-2 or 4B8

Inside, we get

Figure 5. Bundle Subtraction

***T*1 – *T*2**	**71 – 23**	**956 – 487**
*T*1	7 *B* 1	9 *BB* 5 *B* 6
*T*2	2 *B* 3	4 *BB* 8 *B* 7
T	5 *B* -2	5 *BB* -3 *B* -1
	4 *B* 8	5 *BB* -4 *B* 9
		4 *BB* 6 *B* 9
T	**48**	**469**

Since 19 = 2B-1 we get: T2 = 50 – 19 = 5B0 – (2B-1) = 3B-(-1) = 31, so -(-1) = + 1.

Multiplying Tens

An exemplary question to answer could be

"What is 6 7s and 6 43s recounted in tens?"

Outside, we can squeeze the boxes so they get the length 10, which may increase or decrease the height. Inside, we may write 1digit numbers as a flexible 2digit numbers, 6 = B-4, and 7 = B-3.

Figure 6. Bundle Multiplication

***T* =**	**6 * 7**			**6 * 43**		
	B	-4		*B*	-4	
	BB	-4*B*	*B*	4*BB*	-16*B*	4*B*
	-3*B*	12	-3	3*B*	-12	3
1*BB*	-7*B*	12		4*BB*	-13*B*	-12
1*BB*	-6*B*	2		4*BB*	-15*B*	8
	4*B*	2		2*BB*	5*B*	8
***T* =**	**4**	**2**		**2**	**5**	**8**

Dividing Tens

An exemplary question to answer could be

"What is 348 and 349 recounted in 6s?"

With multiplication we work up-down, so with division we work down-up.

Figure 7. Bundle Division

T = 348 =		58 * 6			T = 349 = 58 * 6 + 1 = 58 1/6 * 6			
		6				6		
			30*B*	5*B*			30*B*	5*B*
			48	8			48	8
		30*B*	48			30*B*	48	+ 1
		34	8			34	9	
T	**3**	**4**	**8**	/6	**3**	**4**	**9**	/6

Adding Per-Numbers

An exemplary question to answer could be

"2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?"

Here we see that the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 first multiply to $-numbers thus creating areas.

T = Σ ($/kg * kg) = (Σ kg) * $/kg

So, per-numbers add by areas under the per-number graph, also called integral calculus in the case of 'locally constancy'.

Here, the per-numbers are piecewise constant.

A formal definition of globally constancy would be: y is globally constant c if for any positive number n, the distance between y and c is less than n.

A formal definition of piecewise constancy would be: y is piecewise constant c if an interval k exists so that for any positive number n, the distance between y and c is less than n in k.

A formal definition of locally constancy would be: y is locally constant c if for any positive number n, an interval k exists so that the distance between y and c is less than n in k.

Locally constant may also be called continuous, and n and k may also be called epsilon and delta.

Subtracting Per-Numbers

An exemplary question to answer could be

"2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?"

Here we see that unit-numbers 6 and 2 subtract directly, whereas the per-numbers 3 and 5 first multiply to $-numbers thus creating areas.

T = 2 * 3 + 4 * u = 6 * 5

u = (6 * 5 – 2 * 3)/4 = ΔT/Δkg

In the case of locally constancy, subtracting per-numbers is called differential calculus.

Finding Common Units

An exemplary question to answer could be

"Only add with like units, so how to add T = 4ab^2 + 6abc?".

Here the units come from factorizing

Figure 8. Finding Common Units

Total	4*ab*^2	+ 6*abc*
Units	2 * 2 * *a* * *b* * *b*	3 * 2 * *a* * *b* * *c*
Unit in common	2 * *a* * *b*	2 * *a* * *b*
Rest	2 * *b*	+ 3 * *c*
Total	(2*b* + 3*c*)	* 2*ab*

So, with like units,

T = 4ab^2 + 6abc = (2b + 3c) * 2ab = 2b + 3c 2abs.

Intersection in a Coordinate System

The three points A(2,1) and C(7,3) and B(5,7) create a triangle that may be boxed by a 6x5 rectangle. A line from a to the upper right corner M(7,7) will intersect the line BC in a point G with the coordinates u and v. To find it we equate tan A found in a lower and an upper triangle

tan A = 6/5 = (v – 1)/(u – 2) = (7 – v)/(7 – u)

This gives to equations with two unknowns that are solved by u = 5.75 and v = 5.5

6(u – 2) = 5(v – 1), and 6(7 – u) = 5(7 – v)

A quicker way is to find a formula for the points on the line AG by replacing u and v with x and y

6(x – 2) = 5(y – 1), giving y = 6/5(x – 2) + 1.

In the same way we may find the formula for the line through B and C:

tan B = 4/2 = (7 – y)/(x – 5), giving y = -2(x – 5) + 7

With intersection point placed on both lines, we equate their y's

y = 6/5(x– 2) + 1 = -2(x– 5) + 7, solved by x = 5.75 and y = 5.5

In a similar way, intersection points between a line and a circle are quickly predicted by their formulas (Tarp, 2021).

CONCLUSION

We asked: how to make mathematics education comply with the three classical virtues, truth and beauty and goodness. Using sociological imagination (Mills, 1959) on pre-school observations we now suggest that the following differences may be worthwhile testing.

It is Truth

To reserve names for what exists. And if the end goal of mathematics education is to master Many, Many should be present in each statement as the total T, thus using full sentences with a subject, a verb and a predicate in both the word-language and the number-language. It is correct that 2 * 3 = 6, but this only states the inside fact that 2 3s may be recounted as 6 1s. What is missing is the outside total that may be predicted by a multiplication as, e.g., 2 3s, or 2 kg at 3$/kg, etc. So, to be a true statement about the world it must take the form of a full sentences, T = 2 * 3.

To include the word 'bundle' when using a counting sequence to find the degree of Many in a given total: 0B1, 0B2, …, 0B9, 1B0, 1B1, etc.

To use other bundle-sizes than ten to at once meet bundle-of-bundles when counting ten fingers in 3s as 0B1, 0B2, 0B3 or 1B0, 1B1, …,2B2, 2B3 or 3B0 or 1BB0B0, 1BB0B1. Thus, allowing the inside names ten and hundred and thousand to be seen also outside as bundles and bundle-of-bundles and bundle-of-bundles-of-bundles.

To see the unbundled accounted for in three different ways using a decimal, a fraction, or a negative number as in T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

To always include units when adding digits and fractions, thus seeing both as, not numbers, but operators needing a number to become a number. Omitting this will transform grounded mathematics

true both inside and outside into ‘mathematism’ always true inside but seldom outside where, e.g., 2weeks + 3days = 17 days.

To present integral calculus as adding piecewise or locally constant per-numbers by their areas, and as preceding differential calculus.

It is Beauty

To transform a collection of sticks into one icon with eight sticks in one 8-icon that then can be used as one unit when recounting, instead of using 4 symbols as did the Romans, VIII.

To see a total become box after box when too many transforms into 1 bundle, and into 1 bundle-of-bundles etc.

To see a total as a flexible rectangular box with a height that increases if squeezed, and decreases if expanded to hold the same total. And to see a total transformed into a square thus allowing finding close values of a square root.

To see the operation ‘push-away’ named division iconized as a broom brushing away bundles; and to see the operation ‘stacking’ named multiplication iconized as a lift; and to see the operation ‘pull-away’ named subtraction iconized as a rope; and to see the operation ‘uniting stacks’ named on-top or next-to addition iconized as a cross showing the 2 directions; and to see the operation of bundling bundles named power iconized as a hat.

To see how formulas predict. How uniting 2 and 3 apples is predicted by addition, how 2 apples 3 times is predicted by multiplication, how 12 apples recounted in 3s is predicted by division.

To see subtraction defined by an equation: the numbers u that added to 2 gives 6 is 6-2, or if u + 2 = 6 then u = 6-2, thus showing inverse calculation as an equation solved by moving a calculation to the opposite side as the opposite calculation.

To see division defined by an equation: the numbers u that multiplied to 2 gives 6 is 6/2, or if u * 2 = 6 then u = 6/2, again showing inverse calculation as an equation solved by moving a calculation to the opposite side as the opposite calculation.

To see the factor-finding root defined by an equation: the factor u of which 3 gives 20 is $3\sqrt{}20$, or if u^3 = 20 then u = $3\sqrt{}20$, again showing inverse calculation as an equation solved by moving a calculation to the opposite side as the opposite calculation.

To see the factor-counting logarithm defined by an equation: the number u of 3factors in 20 is log3(20), or if 3^u = 20 then u = log3(20), again showing inverse calculation as an equation solved by moving a calculation to the opposite side as the opposite calculation.

To see the process of re-counting a total of 8 in 2s written as T = 8 = (8/2) * 2 that, with letters for unspecified numbers, becomes a recount formula, T = (T/B) * B, also called a proportionality formula present all over mathematics and science, and enabling a calculator predict the result of changing units by recounting before performing it.

To see that double-counting totals creates per-numbers as 3$/4kg allowing the units to be linked by recounting in the per-number: 20kg = ?$; 20kg = (20/4) * 4kg = (20/4) * 3$ = 15$. Or by recounting the units: $ = ($/kg) * kg = (3/4) * 20 = 15.

To see fractions as per-numbers coming from double-counting parts and totals in the same unit, again allowing them to be linked by recounting in the per-number.

To see trigonometry occurring as mutually recounting the three sides in a box halved by its diagonal.

To see the algebra square showing the 2 by 2 numbers occurring in the outside world, constant and changing unit- and per-numbers; together with the four ways to unite them into a total, addition and multiplications uniting changing and constant unit-numbers, and integration and power uniting changing and constant per-numbers; as well as the five ways to split totals, subtraction and division splitting into changing and constant unit-numbers, and differentiation and root and logarithm splitting into changing and constant per-numbers. Thus, again seeing reversed calculation formulated as equations with the answer predicted by moving to opposite side with opposite sign.

To use a tile divided with horizontal and vertical rubber bands to multiply multi-digit numbers as well as algebraic expressions.

To see the quadratic equation solved by two identical tiles turned a quatre round.

To see the Pythagorean theorem coming from rearranging 4 identical tiles.

To see the different change formulas as special examples of the number formula T = a * x^2 + b * x + c; the proportionality formula as T = b * x, the linear formula as T = b * x + c, the exponential formula as T = a * p^x, the power formula as T = a * x^p, and the quadratic formula as it stands.

To use two tiles to show the most beautiful formula in mathematics: when saving, the ratio between the total and single $-and %-input are identical.

To treat integral before differential calculus, using that when adding differences, middle terms disappear leaving only one difference between the start- and end-numbers.

To see integral calculus rooted, first in next-to addition of two stacks asking, e.g., “2 3s + 4 5s = ? 8s” thus letting multiplication precede addition; accompanied by differential calculus as reversing next-to addition of two stacks asking, e.g., “2 3s + ? 5s = 3 8s” thus letting subtraction precede division; and found by performing the action: ? = (T2– T1)/5 = ΔT/5

To see integral calculus rooted, next in adding piecewise constant per-numbers asking, e.g., “3kg at 2$/kg + 5kg at 4$/kg = (3 + 5)kg at ?$/kg” thus again letting multiplication precede addition, T = □ $/kg * kg; accompanied by differential calculus as reversing adding piecewise constant per-numbers asking, e.g., “3kg at 2$/kg + 5kg at ?$/kg = 8kg at 4$/kg” thus letting subtraction precede division: ? = ($ – $o)/(kg – kgo) = Δ$/Δkg.

To see integral calculus rooted, third time in adding locally constant per-numbers asking, e.g., “3kg at 2$/kg increasing to 4$/kg = 3kg at ?$/kg” thus again letting multiplication precede addition, T = ∫ $/kg * d(kg); accompanied by differential calculus as reversing adding locally constant per-numbers asking, e.g., “3kg at 2$/kg + 5kg at ?$/kg = 8kg at 4$/kg” thus letting subtraction precede division: ? = d$/dkg.

To see the multiplication formula of differential calculus appearing as tile-shadows.

To introduce the Euler-number as continuous compound interest e = (1+1/n)^n, also since many proof may be derived from this.

To see how much mathematics comes out of rearranging a few playing cards. (Tarp, 2019)

It is Goodness

To accept and develop the flexible-bundle numbers children bring to school.

To allow totals to be linked in time and space as bundle-numbers and boxes.

To postpone addition to after division, multiplication and subtraction has changed a total into an outside box described by an inside flexible bundle-number.

To allow calculus in primary school as direct and reversed next-to addition of two bundle-numbers; and in middle school as adding per-numbers asking 2kg at 3$/kg + 4kg at 5$/kg total 6kg at ?$/kg.

To accept the word 'per-number' when double-counting in two units.

To treat fractions as, not numbers, but operators needing numbers to become numbers, i.e., as per-numbers with like units.

To parallel theoretical words with linking words: function/formula, root/factor-finder, logarithm/factor-counter, divided by/counted in, continuous/locally constant, differentiable/locally linear, equation/reversed calculation, fraction/per-number, etc.

RECOMMENDATION

For all students to succeed in mathematics education, it should allow them understand and act with true, beautiful and good mathematics. So, the time has come to leave out false, ugly and bad mathematics. Consequently, we should stop

To present digits and fractions as numbers since they are in fact operators needing a number to become a number.

To introduce addition before subtraction before multiplication before division since outside, the opposite order is the natural when transforming a total into boxes described inside by flexible bundle-numbers.

To introduce 8/2 as 8 split in 2 since 8/2 is defined as the solution to the equation u * 2 = 8 asking how many 2s there are in 8. Therefore, 8/2 means 8 recounted in 2s which later may be applied to share 8 between 2.

To present 456 as one number obeying a place-value system when it is indeed three numberings of unbundled, bundles and bundles-of-bundles

To silence that addition has two forms, on-top and next-to

To silence that on-top box-addition leads directly to proportionality.

To silence that next-to box-addition leads directly to calculus.

To silence that 8 recounted in 2s leads to the proportionally formula T = (T/B) * B.

To silence per-numbers coming from double-counting in two units.

To silence fractions as per-numbers.

To silence that equations are by definition solved by moving to opposite side with opposite calculation sign.

To introduce differential calculus before integral calculus when you would never introduce subtracting per-numbers before adding them.

To introduce Plane geometry before trigonometry when trigonometry comes directly from mutual recounting of the three side in a box halved by its diagonal.

To silence that many concepts may be named both from a theoretical and a practical view, thus allowing linear and exponential functions be called change by adding and by multiplying also; and allowing a function be called a number-language sentence also; and allowing calculus be called change-calculations also; and allowing solving equations be called reverse calculation also; and allowing root and logarithm be called factor-finder and factor-counter also.

To silence that algebra in Arabic means reuniting numbers.

To silence the algebra square showing the four operations that unite changing and constant unit- and per-numbers, as well as the five opposite operations splitting them.

To silence that as quantitative literature, mathematical models have two genres also, fact and fiction.

To silence that change calculations have three parts: constant change, changing but predictable change, and changing but unpredictable change.

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REPLACING ACADEMIC NUMBER SENSE WITH CHILDREN'S OWN MANY SENSE

Background: Why do we colonize children's number-language with one-dimensional place-value numbers? After all, their biological capacity to adapt to their environment make children develop a number-language of their own that may be developed instead of replaced by present pre-setcentric or set-centric curricula? Objectives: Is there a different third 'post-setcentric' mathematics education that allows children develop their own number-language? Design: Difference research is chosen as a method, inspired by four grand theories looking at mathematics education: biology, philosophy, psychology and sociology. Setting and Participants: In a kindergarten, children are asked to perform and discuss different counting problems. Data collection and analysis: The uncovered counting-curriculum is compared with a traditional curriculum and a special discussion compares number-sense with many-sense. Results: In the children's own counting curriculum digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Double-counting in two units creates per-numbers. Addition occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas. Conclusions: Instead of colonizing it, mathematics education should develop and learn from the number-language children bring to school allowing counting, recounting and double-counting to take place before adding on-top and next-to. This will finally allow the communicative turn to take place within the number-language as it did within the word-language 50 years ago.

Keywords: early childhood; number sense; counting; adding; proportionality.

POOR PISA PERFORMANCE DESPITE INCREASED RESEARCH

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed, e.g., by the creation of a National Centre for Mathematics Education in Sweden.

However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level

2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

MATHEMATICS AND ITS EDUCATION

The Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

This 'pre-setcentric math' was replaced by the present setcentric 'New Math' in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Consequently, continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But without excelling in the PISA tests.

In their 'Learning framework 2030', OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the 'Cinderella question': with pre-setcentric and setcentric mathematics unable to deliver, is there a different third post-setcentric alternative, that may 'make the prince dance'? Difference research (Tarp, 2018) may give an answer.

DIFFERENCE RESEARCH

Difference research uses sociological imagination described by Mills (1959) as the core of sociology. Bauman (1990) agrees by saying

Sociological thinking is, one may say, a power in its own right, an anti-fixating power. It renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could

be different from what it is now. (p.16) Rational action (..) is one in which the end to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical. (p.79) Last but not least, the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance (p.84).

It is a general opinion that the goal of mathematics education is to learn mathematics. However, this goal is self-referring. So maybe traditional mathematics has a goal displacement hiding a different more fruitful way to the outside goal, to master Many as it occurs in space and time?

This scepticism towards a world 'hitherto oppressive in its apparent fixity' is the basis for postmodern and post structural thinking (Lyotard, 1984). But instead of rejecting the grand theories let us hear what they have to say about mathematics education to see if differences will appear.

BIOLOGY LOOKS AT EDUCATION

As a life science, biology sees life as built from green, grey and black cells.

Green cells form plants able to perform photosynthesis that store the energy form solar photons in carbon hydrate molecules by replacing oxygen with water in carbon dioxide molecules. To survive, plants must access light and water where they are situated since they are unable to move.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

Black cells exist individually in oxygen depleted areas on the bottom of lakes or in the stomach of animals, surviving by removing oxygen from carbon hydrate molecules, thus being transformed to carbon or carbon hydrogen or oil allowing energy to be used by machines.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms. Reptiles have one brain allowing it to transform outside information into a choice between alternative actions. Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients in life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured, e.g., by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations.

To answer this, we now turn to three other sciences: philosophy, psychology and sociology.

PHILOSOPHY LOOKS AT EDUCATION

Philosophy looks at the relation between outside existence, ontology, and inside representation, epistemology, or, in other words, the 'it-we' relation. Within philosophy, precedence is given to outside phenomena by empiricism, to inside rationality by rationalism, and to questioning ruling knowledge regimes by scepticism.

A controversy within philosophy began in ancient Greece where the sophists pointed out that to practice democracy, a population must be enlightened, especially about the difference between nature and choice to avoid being patronized by political choice masked as unpolitical nature. In opposition to this, Plato saw choice as an illusion since all physical is but examples of metaphysical forms only visible to philosophers educated at the Plato academy. (Russell, 1945)

Later, the Christian church changed the academies into monasteries, where some changed back into universities after the reformation; and after Newton's discovery of physical laws controlling nature without being physical or metaphysical patronized. This inspired the Enlightenment Century installing two republics, one in North America taking over empiricism when developing 'it is right if it works' pragmatism, symbolic interactionism, grounded theory, and action research; and one in France taking over scepticism after seeing its republic turned over several times because of resistance from its German speaking neighbours.

Today, opposition against rationalism is seen within existentialism, claiming with Sartre (2007) that existence precedes essence. And explicated by Heidegger (1962) arguing that in defining verdict-sentences "subject is predicate", the subjects should be respected for naming what exist outside, whereas the predicates should be questioned and appealed since they represent an inside choice between alternatives in risk of being masked as nature.

In contrast to this, continental rationalism gives precedence to inside essence developed by rational universities through peer-reviewed research.

PSYCHOLOGY LOOKS AT EDUCATION

Psychology looks at cognitive aspects of learning, or, in other words, the 'it-I' relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially

constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development. Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

SOCIOLOGY LOOKS AT EDUCATION

Sociology looks at the social aspects of human interaction, or, in other words, the 'they-I' relation. Here, the philosophical controversy between outside existence and inside essence carries on as a controversy between different forms of social theory emphasizing individual agency or social structure.

Individual agency is emphasized in the first Enlightenment republic in North America showing strong resistance against institutional answers since they may lead to a goal displacement (Bauman, 1990) becoming an inside goal itself instead of staying as an inside means to an outside goal, thus suppressing the 'sociological imagination' (Mills, 1959) that might keep the answer fluid instead of fossilizing into what Weber (1930) calls an 'disenchanting Iron Cage'.

Structuralism is preeminent in continental Europe seeing established science as being a true inside representation of the outside environment if accepted by the society's knowledge institutions, the universities.

In the second Enlightenment republic in France, institutional scepticism inspired by Heidegger led to French post-structuralism where Derrida, Lyotard, Foucault and Bourdieu warn against hidden patronization in our most basic institutions: words, correctness, diagnosing discourses, curing institutions, and education especially that might make mathematics so difficult it may serve as symbolic violence to establish a new knowledge nobility.

Foucault (2006: 41) recommends criticizing modern institutions:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them.

As to education, Foucault (1995) sees schools as 'pris-pitals' mixing social power techniques from a prison and a hospital. Here students are 'pati-mates' forced to return to the same class, hour after hour, day after day, month after month for several years. Furthermore, self-reference is used to create diagnose 'ignorant' without specifying of what: 'you don't know math, so we must teach you math'. Consequently, humans are placed in a Kafkaesque (2015) situation where they, unable to have the diagnose defined, finally accept it and therefore also accept being retained and treated to be cured.

In Germany, Habermas (1981) theorizes the possibility of creating a third Enlightenment republic in post-war Germany. Inspired by Weber's warning that rationalization carried too far might lead to an iron cage dis-enchanting the world, Habermas warns against system-worlds tending to colonize the life-world and recommends a power free communication rationality to prevent this from happening where "peers exchange views".

Likewise, Arendt (1963) points out that by definition, institutions lack competition, forcing employees to follow order, which might lead to the banality of evil, where ordinary citizens must act evilly to keep the job.

Agency-based education sees knowledge construction as best taking place in symbolic interaction between peers exchanging views about the sentence subject in order to negotiate a common view. This resonates with Piagetian constructivism, and with philosophical existentialism giving precedence to existence; and with the construction of social knowledge by using Grounded Theory (Glaser and Strauss, 1967). Secondary education should be block-organized to support the student's identity work through self-chosen half-year blocks with teachers teaching only one subject.

In contrast to this, structuralism sees democratically controlled educational institutions as the best way to mediate the university knowledge heritage. This resonates with Vygotskyan constructivism. Secondary education should be line-organized to supply the state with skilled academic and non-academic workers, thus forcing students to choose career line early, and to start all over if changing career.

FOCUSING ON EARLY CHILDHOOD EDUCATION

A good beginning is half done, an old saying goes. So why not look more closely into what preschool children already know and what they are taught. In accordance with this, the National Council of Teachers of Mathematics (2000) revised the Principles and Standards for School Mathematics to include attention to prekindergarten, since they appreciated the importance of providing students with a solid and deep mathematics knowledge at this stage. Later, The National Association for the Education of Young Children (NAEYC) and NCTM (2002) issued a joint position statement on teaching mathematics to young children. Furthermore, the research literature is not overwhelming in this area.

MEETING MANY, CHILDREN COUNT WITH BUNDLES AS UNITS

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units. And also using a full sentence as in the word-language with a subject 'that' and a verb 'is' and a predicate '2 2s', which abbreviated makes a formula a number-language sentence 'T = 2 2s'.

Likewise, children use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 more'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 more'.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in, e.g., 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different colour for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new colour for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don't mind writing using 'bundle-writing' with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering '8/2', a calculator predicts they can have 4 each; they thus meet the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering '4x2', a calculator predicts they have a total of 8; they thus meet the multiplication sign as an icon for a lift lifting up 2s.

And again, they smile they see that entering '8 – 4x2', a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; they thus meet the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, lifting up, and pulling away, that can be performed by a broom, a lift, and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription 'From the total 8, 8/2 times, 2s can be pushed away' may be shortened to the calculation formula '8 = (8/2)x2', or '8 = (8/2)*2', later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula 'T = (T/B)*B'.

Exposed to counting, children adapt a natural way to the three basic operations division, multiplication and subtraction; and typically enjoy using a calculator, or even the recount-formula, to predict the counting result. We may call this the child's natural 'counting curriculum'.

A TYPICAL CONTEMPORARY MATH CURRICULUM

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g., that 2+3 = 5.

This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication

is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

Reverse operations create new numbers calling for additional education about operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about operating on such numbers, e.g., that (a+b)*(a-b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

To be followed by halving a rectangle by its diagonal to create a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent.

In a calculation, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about operating on calculations, e.g., that (f*g)' /(f*g) = f' /f + g' /g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g., that ∫ 6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

A math curriculum always silences that it is not teaching mathe-matics, but 'meta-matism', a mixture of 'meta-matics' and 'mathe-matism'. Here 'Meta-matics' defines a concept a function as an example of subset of a set-product where first-component identity implies second-component, instead of as a name for a calculation containing both constant and variable numbers; i.e., defining a concept from above as an example of an abstraction instead of from below as an abstraction from examples. And 'mathe-matism' is mathematics that is always true inside, but seldom outside a classroom where '2+3 = 5' meets countless counter-examples as 2 weeks + 3 days = 17 days.

Likewise, fractions are treated as numbers despite the fact that they are operators needing a number to become a number as shown by the fraction-paradox: 1 red apple of 2 plus 2 red apples of 3 total 3 red apples of 5, and not 7 red apples of 6 as the school teaches.

THE DIFFERENCE TO THE CHILD'S COUNTING CURRICULUM

Thus, typically the core of a contemporary mathematics curriculum is how to operate on specified and unspecified numbers and calculations.

Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles, bundles, and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never

children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Seldom children may say 'bundle' instead of ten, or to say 'ten-ten' or 'bundle-bundle' instead of hundred.

Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Using flexible bundle-numbers to rename 17 as 2B-3 tens and 24 as 1B14 tens is not allowed.

Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction.

The children never experience the thrill of seeing a calculator predict a recounting result.

Or the fun of recounting to a different unit predicted by the recount-formula and tested on a western ten by ten abacus.

Or the fun building a multiplication table on a pegboard using rubber bands to show the numbers as flexible bundle-numbers thus seeing that - - is + since the upper right corner is pulled away twice and therefore must be added again.:

6*8 = (B-4)*(B-2) = BB – 4B – 2B - - 4x2 = (10-4-2)B + 8 = 4B8 = 48

Or the powerful technique of double-counting in two different units to create per-numbers as 2$ per 5kg, allowing answering proportionality questions as '8$ buys ?kg' by recounting in the per-number: T = 8$ = (8/2)*2$ = (8/2)*5 kg = 20 kg.

Or the fun to add two totals once they are counted as blocks to see that added on-top recounting first must make the unites the same; and that added next-to mean adding areas, i.e., integration where multiplication precedes addition, and becoming differentiation when reversed where subtraction precedes division since the start block must first be removed before counting bundles (Tarp, 2018, 2020a).

In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and basic quantitative competence in counting and later adding.

DISCUSSING NUMBER SENSE AND NUMBER NONSENSE

The basic question in grade in early childhood education is: shall education be about numbering or about numbers? Shall education guide and support the development of the children's already existing adaption to quantity, or shall education teach numbers? Shall the 'I' keep on adapting to the 'it' directly, or indirectly by having the adaption replaced by what is mediated by the 'they'?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics in their publication 'Principles and Standards for School Mathematics' (2000) says: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense (p. 7)."

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children's whole number arithmetic development?" (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept 'number sense' is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term 'number sense' for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or 'more-ness'. Using the term 'cardinality' just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term 'many sense' is more precise than the term 'number sense'. Especially since, with its reference to numbers, 'number sense' becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word 'understanding' makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense as many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the 'missing link', the bundle and the unit, absent in everyday use: T = 67 = 6B7 tens.

Therefore, a short definition could be: Having number-sense as many-sense means including the word 'bundle' as a unit for the numbers when counting totals. That will allow

To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.

To bridge an outside total with an inside numbering by bundling expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g., T = 2 3s.

To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with 'flexible bundle-numbers' with overload, underload or fraction, i.e., as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s. And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 3 1/3, 1BB0B1, or 1BB1B-2. And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens. And 678 = 67B8 = 6BB7B8. (Tarp, 2020a)

To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift lifting up bundles into a stack, where subtraction iconizes a rope pulling away the stack to find unbundles singles, and where addition iconizes placing stacks next-to or on-top.

To see the counting process predicted by the recount-formula T = T/(B)*B, saying 'From the total T, T/B times, B-bundles can be pushed way'; and to use a calculator to enter '9/4' giving '2', and '9-2*4' giving '1' to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.

To see totals as double-described both as outside stacks and as inside bundles.

To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.

To see a multiplication task as recounting from icons to tens, facilitated by using flexible block&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

To see a multiplication equation 4*u = 20 as recounting from tens to icons, solved by the recount-formula: 4*u = 20 = (20/4) * 4, thus giving the solution u = 20/4 coming from moving a number to the opposite side with the opposite calculation sign.

Table 1. A counting table that includes the bundles in the number names

1*BB*0	1*BB*1	1*BB*2	1*BB*3	1*BB*4	1*BB*5	1*BB*6	1*BB*7	1*BB*8	1*BB*9	1*BB*10
10*B*0	10*B*1	10*B*2	10*B*3	10*B*4	10*B*5	10*B*6	10*B*7	10*B*8	10*B*9	10*B*10
9*B*0	9*B*1	9*B*2	9*B*3	9*B*4	9*B*5	9*B*6	9*B*7	9*B*8	9*B*9	9*B*10
8*B*0	8*B*1	8*B*2	8*B*3	8*B*4	8*B*5	8*B*6	8*B*7	8*B*8	8*B*9	8*B*10
7*B*0	7*B*1	7*B*2	7*B*3	7*B*4	7*B*5	7*B*6	7*B*7	7*B*8	7*B*9	7*B*10
6*B*0	6*B*1	6*B*2	6*B*3	6*B*4	6*B*5	6*B*6	6*B*7	6*B*8	6*B*9	6*B*10
5*B*0	5*B*1	5*B*2	5*B*3	5*B*4	5*B*5	5*B*6	5*B*7	5*B*8	5*B*9	5*B*10
4*B*0	4*B*1	4*B*2	4*B*3	4*B*4	4*B*5	4*B*6	4*B*7	4*B*8	4*B*9	4*B*10
3*B*0	3*B*1	3*B*2	3*B*3	3*B*4	3*B*5	3*B*6	3*B*7	3*B*8	3*B*9	3*B*10
2*B*0	2*B*1	2*B*2	2*B*3	2*B*4	2*B*5	2*B*6	2*B*7	2*B*8	2*B*9	2*B*10
1*B*0	1*B*1	1*B*2	1*B*3	1*B*4	1*B*5	1*B*6	1*B*7	1*B*8	1*B*9	1*B*10
0*B*0	0*B*1	0*B*2	0*B*3	0*B*4	0*B*5	0*B*6	0*B*7	0*B*8	0*B*9	0*B*10

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in the set of rational numbers, etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is four numbers: 6, 7, 8 and bundles, which typically is ten. Or scores in France and Denmark both counting four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 67 = 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, 'this sentence is false', since the set of sets not belonging to itself will belong if and only if it will not.

Russell's point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept 'number-sense' we should remember that sociologically, a school is a pris-pital. So, the moment you introduce a new construct you risk also introducing a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses is an example of what Foucault calls pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault describes the salvation promise of the generalized church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e., the church, the school, the correction center, the hospital, and do exactly what we tell you. If not, you only have yourself to thank for your decline'.

Introducing diagnoses may also be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu, 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Bauman calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means, thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight 'mathematical competencies' (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002). Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

CONSEQUENCES FOR FUTURE TEXTBOOKS: FROM INSTITUTIONALIZED INSTRUCTION TO SELF-INSTRUCTION

Seeing the goal of mathematics education as developing further the mastery of Many children build up before school will, of course, call for different textbooks that

- guide numbering tasks instead of teaching numbers,
- bridge inside division, multiplication and subtraction to the outside actions pushing away bundles to count, stacking bundles to boxes, and pulling away boxes to identify unbundled singles, inside described as decimals, negative numbers or fractions
- treat counting, re-counting, solving equations, roots, double-counting, per-numbers and trigonometry before adding
- treat next-to and on-top addition as introduction to calculus and proportionality
- treat the letters in algebraic expressions as units.

Guiding questions could be: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g. if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "What to do if a bundle is not full?", "How to bundle-count seconds, minutes, hours, and days?", "How to double-count lengths in centimeters and inches?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s in 8 may be written as u*2 = 8, how can

this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when recounted stacks are placed on a squared paper?", "What happens when turning or stacking stacks?"

Activities springing from these questions may be found in the booklet "Learn core math through your kid's tile-math, recounting bundle-numbers, early trigonometry, calculus, STEM" (Tarp, 2020b)

Or in the booklet 'A Refugee Camp Curriculum' (Tarp, 2021) containing 30 foci like, e.g.,

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s.

● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say "seven ones recounts as 2B1 3s" and write T = 2B1 3s.

● Outside, place the single next-to the stack. Inside say " Placed next-to the stack the single becomes a decimal-fraction '.1' so now seven recounts as 2.1 3s" and write T = 2.1 3s.

● Outside, place the single on-top of the stack. Inside say "Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s" and write T = 2 1/3 3s. Now, inside say "Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s" and write T = 3.-2 3s. Finally, inside say "With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s."

Activity 02. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 5s.

Thus, the time has come to leave the tradition of institutionalized instruction to develop textbooks that as self-instructing allows the learner to ask the outside subject described instead of an instructor.

TESTING WITHIN SPECIAL EDUCATION

Observing and listening to students in special education students is very helpful when looking at when and why students drop out.

The first drop out under addition of mono-digit numbers. Unable to memorize the results they have to count on their fingers. They flourish when hearing about flexible bundle numbers:

7+5 = B-3 + 5 = B+2 = 1B2 = 12.

The next drop out under multi-digit numbers. The mix up 35 and 53 and show little understanding of what hundred and thousand is apart from names. They flourish when hearing about bundle writing:

35 = 3B5, and 53 = 5B3, and

456 = 4BB5B6

where BB means a Bundle-of-Bundles, met when asked to recount the ten fingers in 3s.

The next drop out under addition of multi-digit numbers. They find carrying difficult. They flourish when hearing about flexible bundle writing:

35 + 27 = 3B5 + 2B7 = 5B12 = 6B2 = 62.

The next drop out under subtraction of multi-digit numbers. They find carrying difficult. They flourish when hearing about flexible bundle writing:

65 – 27 = 6B5 + 2B7 = 4B-2 = 3B8 = 38.

The next drop out under multiplication of mono-digit numbers. Unable to memorize the multiplication tables they rely on written tables or a calculator. They flourish when hearing that 6*7 is 6 7s, i.e., a box with base 7 and height 6 that, if squeezed to base ten will loose some height thus becoming only 4.2 tens. And when seeing that flexible bundle-numbers allow overflow:

6 * 43 = 6 * 4B3 = 24B18 = 25B8 = 258.

And when seeing the multiplication on a pegboard shown with flexible bundle numbers as

6*7 = (B-4)*(B-3) = BB – 4B – 3 B + 3*4 = 3B + 1B2 = 4B2 = 42

where 3*4 is added since it was subtracted twice.

The next drop out under division. Unable to memorize the multiplication tables they rely on written tables or a calculator. They flourish when hearing that 18/6 is means 18 counted in 6s, thus seeing a 1.8*10 box that when squeezed to a box with base 6 increase it height to 3 to give the result: 18 counted in 6s gives 3 6s or 3*6.

And when seeing that flexible bundle-numbers allow overflow before dividing:

84 /3 = 8B4 /3 = 6B24 /3 = 2B8 = 28

And they are excited when seeing that 8 counted in 2s can be predicted by a recount formula as 8 = (8/2)*2, thus allowing them to solve the equation u*2 = 8 by recounting:

u*2 = 8 = (8/2)*2, so u = 8/2 = 4

They enjoy manually performing recounting tasks as ‘3 4s = ? 5s’ after having predicted the result on a calculator while explaining what is going on:

I enter the 3 4s as 3*4 and count them in 5s by dividing them with five. This gives 2.some, so I know that I can push away 5s 2 times. To see if there are any unbundles left I pull away the 2 fives from the 3 4s. So I enter 3*4 pull away 2*5. The result is 2. So, I can now predict that 3 4s can be recounted as 2 5s and 2, or 2B2 5s.

And their happiness continues when they see that to add 2 3s to 4 5s they need to choose a common unit which means that one must be recounted using the recount formula which they gladly accept the name proportionality for.

And when they see that 2 3s and 4 5s may also be added next-to as 8s, again predicting the result with the recount-formula. And they gladly accept the word calculus for next-to addition.

And when they found out that next-to addition may be reversed to asking '2 3s + ? 5s gives 4 8s', they do not hesitate to first pull away the initial box with 2 3s before recounting the rest in 5s thus using subtraction before addition on the calculator. Asking what this is called they accept the is a difference between integral calculus adding boxes next-to and differential calculus reversing it precisely by beginning with a difference.

Some now feel so confident they want to return to their normal class where they get special status when teaching their peers about flexible bundle-numbers, solving equations by recounting, and adding boxes on-top and next-to with proportionality and calculus, long before the teacher intends to teach it.

PROPORTIONALITY SHOWS A VARIETY OF MASTERING MANY

The need to change units has made the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g., Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

- Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.
- Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.
- Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x= (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.
- Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.
- Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.
- Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.
- Modeling a linear function f(x) = c*x, with f(2) = 5, f(7) = ?, and f(x) =12. Set-centrism prefers this to show the relevance of abstract algebra's group theory: Let us define a linear function f(x) = c*x from the set of kg-numbers to the set of $-numbers, having as domain DM = {x∈R I x>0}. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: c*2 = 5, (c*2)*½

= 5*½, c*(2*½) = 5/2, c*1 = 5/2, c = 5/2. With f(x) = 5/2*x, the inverse function is f-1(x) = 2/5*x. So with 7kg, f(7) = 5/2*7 = 17.5$; and with 12$, f-1(12) = 2/5*12 = 4.8kg.

WITH BOX&BUNDLE NUMBERS, ROOTS COME FROM SQUEEZING BOXES

To number an outside total, it is boxed in a given unit, which may then be changed by squeezing the box, e.g., into tens or into a square. We may want to squeeze the box 4 5s into tens, which of course decreases its height since it increases the base, T = 4 5s = 2 tens. If we want to squeeze it in 8s, inside we ask the question 'u*8 = 20' which is an equation easily solved by u = 20/8 since recounting 20 in 8s gives 20 = (20/8)*8.

Wanting to square the 4*5 box, the answer is a$\sqrt{}$20 square. To find$\sqrt{}$20 we observe that in a 4+t square, removing the 4 square leaves 20 ‑ 4*4 = 4 shared by the two 4*t boxes, giving t = 0.5. A little less since we neglect the t square. Inside, a calculator confirms that $\sqrt{}$20 = 4.472.

In geometry, intersection points between lines and circles leads to quadratic equations as x^2 + 6x + 8 = 0, easily solved when rotating the upper of two x by x+3 tiles to create a square with sides x+3, which is zero apart from the 1 left in the upper 3-by-3 square after 8 is removed.

In total, (x + 3)^2 = 0 + 1 = 1 = $\sqrt{}$1^2,

so x+3 = ±1, giving x = - 2 and x = - 4.

TRIGONOMETRY: RECOUNTING IN A BOX HALVED BY ITS DIAGONAL

So, when counting and recounting, the core outside object is a rectangular form that may be called a stack, a tile, a block, or a box, with base, b or AC, and height, h or CB. Here the diagonal, d or AB, splits the box in two like right angled triangles, which may also be called 'half-boxes'.

Alternatively, we may say that we have boxed the line AB so it becomes a diagonal with a base and the height, which may also be called its floor and wall, or horizontal and vertical shadow, or run and rise. We will not use the Greek name 'cathetus' since it does not distinguish between the two sides.

Besides its three sides, a half-box also has three angles counting the turning of the lines in degrees. The turning from the base to the diagonal is called the angle A. The quarter round turning from the base to the height is called the right-angle C, which is counted to 90 degrees since a full round is 360 degrees. With two like triangles in the full box we see that A + B = 90. Trigonometry finds formulas that predict the sides and angles in a half-box.

The sides may be counted in standard units. But they may also be recounted in the box's own units. Recounting the height in the base gives

height = (height/base) * base = tangent A * base = tangent A bases, abbreviated to

h = (h / b) * b = tan A * b = tan A bs, thus giving the formula

tangent A = height / base, or tan A = h/b.

The word tangent is used since in a circle with center in A and the base as its radius, the height will be a tangent. This allows two ways to count angles, by degrees or by a tangent number.

Thus outside, in a 1x2 half-box, the angle A is counted on a protractor to 27 degrees; and in a 1x1 half-box, the angle A is counted to 45 degrees; and in a 2x1 half-box, the angle A is counted to 63 degrees. Inside these numbers may be predicted by the tangent formula

A = tan-1(1/2), A = tan-1(1/1), A = tan-1(2/1).

In the same we can create other trigonometry formulas as

sin A = height/diagonal, or sin A = h/d

cos A = base/diagonal, or cos A = b/d.

A CIRCLE AS MANY RIGHT TRIANGLES

In an h x r half-box with area ½*h*r we look at the circle with center in A and r as its radius. Here h = (h/r)*r = tan A*r .

A half circle is 180 degrees that may split in 100 small parts as

180 = (180/100)*100 = 1.8 100s = 100 1.8s

With A as 1.8 degrees, the length of the circle and the tangent, h, are almost identical. So, with good approximation

Half the circumference C = 100 * h = 100 * tan 1.8 * r = 100 * tan (180/100) * r = 3.1426 * r

Or better, C = 1000 * tan (180/1000) * r = 3.1416 * r

Calling the circumference for 2 * □ * r, we get a formula for the number □□□

For n sufficiently large, □ = tan (180/n) * n.

The area of the full circle then is

Area = 2 * ½ * 3.1416 * r * r = 3.1416 * r^2 =□□ * r^2

ADDING SIDES IN A HALF-BOX

Arranging 4 like half-boxes as a (base + height) square, it contains a diagonal square plus 4 half-boxes. However, it also contains a height square and a base square plus 2 full boxes. Consequently, the height square and the base square add up to the diagonal square as predicted inside by the Pythagorean formula h^2 + b^2 = d^2. So, in right angled triangles, the sides add by their squares.

OUTDOOR PROJECTS

Mastering trigonometry opens the door to many different outdoor activities, as does learning mathematics through Tile math (Tarp, 2020b)

Project 1: What is the height of a high thing, e.g., a flagpole?

Project 2: How to move a thing around a corner?

Project 3: How to calculate shortcuts?

Project 4: How do you travel fastest from a point in one area to a point in another area when you move at different speeds in the two areas?

Project 5: How to calculate the different openings when a door opens ajar?

Project 6: How to build a path up a steep mountainside?

Project 7: How steep can a ladder be placed so as not to break glass or ice?

Project 8: How are astronomical distances calculated?

Project 9: How to size a triangular bridge over a river?

CONCLUSION

This paper asked if there is a third hidden alternative post-setcentric mathematics education.

The answer is yes, and it is easy to see by observing the mastery of Many children develop before being educated. So, we may want to respect and develop this further by accepting the children's two-dimensional bundle-numbers with units instead of colonizing their adaption to quantity with one-dimensional line-numbers forced upon them by setcentric or pre-setcentric mathematics education.

The time has come to realize that the goal of mathematics education is not to master mathematics first to later master many, but to master many directly, or indirectly by means of one of the different versions mathematics has had over several thousand years.

The time has come to enjoy a paradigm shift (Kuhn, 1962) in mathematics education using design research (Bakker, 2018) to explore what happens when mathematics education allows the children to develop the number-language they bring to school using inside flexible bundle-numbers always linked to outside blocks and tiles, thus connecting the physical with the mental, and connecting geometry with algebra, and modeling the outside inside or de-modeling the inside outside (Tarp, 2020a). And to allow core mathematics as linearity and calculus to be performed in year one in primary school instead of postponing them to year one at the university level.

The numerous action learning (Kolb, 1984) and action research projects springing from the child's own counting curriculum may be difficult to carry out in a traditional classroom having so much mental and economical capital bound to the traditional line-numbers. But outside this orthodoxy plenty of opportunities present themselves in preschool, home education, special education, immigrant camps and refugee camps to give some examples.

The time has come where mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

After all, the prime goal of education it to allow the learner develop inside tools to master the outside world. And here the physical fact many plays a so dominant role that children already have a solid adaption to it that education then can choose to develop through proper guidance, or to reject for not being in agreement with the contemporary truth regime.

RECOMMENDATION: MATHEMATICS AS A NUMBER-LANGUAGE

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: 'This is a chair'. As does the number-language assigning numbers instead: 'the 3 chairs each have 4 legs', abbreviated to 'the total is 3 fours', or 'T = 3 4s' or 'T = 3*4'. Unfortunately, the tradition hides the

similarity between word- and number-sentences by leaving out the subject and the verb by just saying '3*4 = 12'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is an auxiliary verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is a commutative operation'.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead, its self-referring set-based form has turned mathematics into a grammar labelling its outside roots as 'applications', used as means to dim the impeding consequences of teaching a grammar before its language.

A core question in language education is the following: should education develop further the children's own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978, and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its New-math revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish a sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing are seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift in early childhood education in by developing the children's already existing many-sense created through their adaption to Many.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

Which here appears to be the curers in the form of teachers and teacher educators eager to force upon children the present 'math before applications' regime. Instead learning from children's rooted math provides an excellent learning opportunity for teachers to become decolonized and to join the children in celebrating and enjoying the enchantment of visiting a quantitative universe.

DATA AVAILABILITY STATEMENT

Data Sharing does not apply to this article as no new data has been created or analyzed in this study.

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