Chapter CC

Demodeling Calculus: from Deriving Functions to Adding Per-numbers

Abstract. The fourth United Nations Development Goal wants to ensure that all youth and most adults achieve literacy and numeracy. In mathematics education, numeracy is at the beginning and calculus is at the end, so apparently, calculus has nothing to do with numeracy. This is indeed the case within the present 'essence-based' paradigm built on one-dimensional linear numbers without units. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3 per bundle thus containing both a time-based counting number, 2, and a space-based per-number, 3s. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas, and for reverse addition asking "2 3s and how many 5s total 4 8s?". And here, a bundle-bundle square changes to the next when adding 2 bundles and 1 unbundled, which suggests that later 2 bundles will be the result of deriving a bundle-bundle square. So, a communicative turn is now possible also within number-language education if letting existence precede essence as philosophical Existentialism recommends.

CC.1 Background: The 4th UN Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy".

The background of this goal is the fact that on our planet, life takes on the form of single black cells, and of green and grey cells combined as plants and animals. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a low number per year. Humans only need a few children in their

lifetime since transforming the forelegs to hands and fingers allows humans to grasp and share the food, and to share information through communication and education by developing a language when associating sounds to what is grasped or pointed to.

Where food cannot be reused as in a zero-sum game, information can as in a plus-sum game through teaching and learning taking place in the family and in the workplace and in educational institutions with primary, secondary, and tertiary education for children, teenagers, and adults. Continental Europe uses words for education that do not exist elsewhere such as 'Bildung', 'Unterricht', 'Erziehung', meaning forming, handing down, and pulling up thus seeing learners as in need of institutionalized knowledge.

So, Europe needs its institution-directed line-organized office-preparing Bildung education that was created by the German autocracy shortly after 1800 under the Napoleon wars to mobilize the population against the French democracy built on enlightening education instead as in the other Enlightenment republic in North America with its individual-directed block-organized talent-developing education from secondary school. Despite these differences, all over the world education is seen as a necessary condition to achieve literacy and numeracy for all, according to the UN goals. But, as with education, within numeracy different views also seem to exist.

CC.2 Different Views on Numeracy

As to the meaning of the word 'numerate', the English Oxford Dictionary defines it as being 'competent in the basic principles of mathematics, esp. arithmetic'. In contrast, the American Merriam-Webster dictionary defines it as 'having the ability to understand and work with numbers'. The difference in the definitions is interesting.

As to the difference between 'competent' and 'work', we can ask if assessing competence and work will be equally valid? The word 'competent' is a predicate, not an action verb, I cannot competent something, I can only be judged as competent by someone who is already competent. In contrast, 'work' is an action verb, since I can work on something with my body and mind.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, we can ask if assessing mathematics and numbers will be equally valid? Mathematics is not an action verb, I cannot 'mathematics' or even 'math' something. In contrast 'number' is both a verb and a noun since I can number something to produce a number for later calculations. So, who is to decide if I am numerate, already examined mathematicians or the accuracy of my predicting calculations?

Finally, the existence of singular and plural in language indicates that 'Many' exists in the outside world where humans see and name it differently. In contrast, mathematics doesn't do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions. Therefore, Many exists in nature and space while any institutionalized essence is a constructed 'knowledge-regime' (Foucault 1995) that varies in time.

It thus seems that with the English definition of numeracy, the assessment must be carried out by persons seen as experts on the predicates 'competent' and 'mathematics'

and other social constructions. In contrast, the American definition of numeracy allows laymen to judge the actions carried out when working with numbering and numbers. And, in their common history, England once colonized America, so we may ask if the different views are the views of a former colonizer and a former colonized? But, in the end, will there be any difference between beginning with numbers or mathematics, will not a study of numbers automatically lead to mathematics?

CC.3 Will Studying Numbers Automatically lead to Mathematics?

In the tradition, number education begins with the number symbols as found on a number-line, 0, 1, 2, etc. Then we learn how to add them, then how to subtract them, then how to multiply them, and finally how to divide them, where subtraction is seen as reversed addition, and multiplication as repeated addition, and division as reversed multiplication. Then reversed calculations called equations show the need for more numbers. First, the equation $3+x=2$ leads on to negative numbers, which makes the whole numbers expand to integer numbers to again be added, subtracted, multiplied and divided.

Then the equation $3*x=2$ leads on to fractions, which makes the integer numbers expand to rational numbers to again be added, subtracted, multiplied and divided.

Then the equations $x\text{^}3=2$ and $3\text{^}x=2$ lead on to roots and logarithms, which makes the rational numbers expand to real numbers to again be added, subtracted, multiplied and divided.

Then the equation $x\text{^}2=-1$ leads on to imaginary numbers, which makes the real numbers expand to the complex numbers to again be added, subtracted, multiplied and divided.

Then letter-calculations also need to be added, subtracted, multiplied and divided.

Then tables of 'standby' calculations with an unspecified number leads on to formulas or functions to again be added, subtracted, multiplied and divided. Here $y=x\text{^}2$ is a specified function while in $y=f(x)$, $f(x)$ stands for an unspecified function with x as its unspecified number, which makes $f(2)$ meaningless since, as specified, 2 is not unspecified.

Then the ratio between a change in x and a change in a function $y=f(x)$ leads on to the gradient, $\Delta y/\Delta x$, that then via the limit concept leads on to the derivative, dy/dx, that then leads on to the anti-derivative or primitive function that turns out to be the area under the function graph.

So, it turns out that studying numbers will indeed lead to traditional mathematics with calculus in the end, and with differential calculus preceding integral calculus, and where the ability to calculate arears with anti-derivatives becomes the 'fundamental theorem of calculus'.

So, building on the number line and the assumption that $1+1=2$, mathematics seems to be a well-defined and well-proven science that can be further developed at the universities and taught in schools without worrying about being accused of being a temporary truth regime colonizing a more natural regime.

Of course, inspired by the classical liar-paradox 'This sentence is false' that is false if true and vice versa, Russell formulated a set-paradox pointing to 'The set of sets not belonging to itself' that belongs to itself if it doesn't and vice versa.

This may worry university mathematics, but not school mathematics, so here mathematics may still be taught in its 'universal' from year after year after year.

Russell solved his paradox by introducing a type-theory avoiding self-refence. And we may now do the same by asking a 3year-old "How many years next time?"

CC.4 Being Educated by a 3year-old Child

"No, that is not four, that is two twos". Said a 3year-old child when asked "How many years next time?"; and when seeing four fingers held together two by two. This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time, i.e., 2 bundles with 2 per bundle. The number 'two' thus exists both in and in time but in very different forms.

In space, 2 exists as 2s, a space-number, a per-number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Either horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 1B2 3s, or 2B-1 3s with B for bundle, and with '-1' for 'less 1'.

In time, 2 exists together with the unit that is counted, as 2 units. So, 2 may be called a time-number or a unit-number. Here, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit.

So, as space-numbers, 2+3 is 5, but as time-numbers 2+3 is undecided until their units are known. In contrast, 2*3 is 2 3s that may always be recounted as 6 1s.

With fingers, a collapsed V-sign shows that 1 1s + 1 1s = 1 2s, which together with a V-sign's 2 1s total 1 2s + 2 1s = 1 4s, and not 3 3s as may be expected if 1+1 = 2 always. Adding with units we get 1*2 + 2*1 = 4. Here we multiply before adding, thus adding areas, which is what integral calculus does.

We see that reaching the unit-number means changing from one counting of singles to two countings, a counting of bundles and a counting of unbundled singles. Later we have three countings of a total T in bundle-bundles, bundles and unbundled singles.

T = 3tens & 4 = 3B4 tens = 34

T = 6 hundreds & 7 & tens 8 = 6BB 7B 8 tens = 678

Instead of saying '3ten4' we should really say '3 bundles with ten per bundle, and 4 unbundled' to show that the bundle-size is a per-number that also add by their areas:

2 3s + 4 5s = (2*3+4*5)/8 8s.

Likewise, instead of saying '6hundreds7ten8' we should really say '6 bundle-bundles and 7 bundles with ten per bundle, and 8 unbundled'.

The bundles may be rearranged as icons with as many sticks or strokes as the represent (Tarp 2018). As the unit, ten needs not one but two icons for the bundles and unbundled, ten = 1B0 = 10.

By using bundle-numbers with units, children open our eyes to a different mathematics that, freed from its present essence-bounds, may return to its original identity as a natural science about the natural fact Many existing in space and time. This decolonization may create a natural number-language to communicate about outside existence also instead of about inside essence only.

If existence comes before essence, then counting comes before adding, where normally numbers are given to add. So, let now the adult educator be educated by the uneducated child to see how much math comes from counting. These discoveries may later be transformed into micro-curricula in existence-based 'many-math' where counting outside existence comes before adding inside essence (Tarp 2018).

Looking at a total of five fingers we see that the inside essence 'five' outside may exist in different forms with each their label.

In space, five may exist as five ones, 0B5, or as a bundle of one fives, 1B0. Counted in twos, five may exist as one bundle and three, 1B3, as 2B1, or as 3B-1 needing 1 to become an extra bundle or having 1 pulled away. Or even as 1BB 0B 1 since two twos held together exist as a bundle of bundles, one bundle-bundle, 1BB, 1B^2, or 1B-square becoming so with centi-cubes.

In time, one by one the five fingers are counted in 5s with the bundles included as a unit: 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0 5s. Counting only '1, 2, …' would mean that the first single should be 1 5s instead of 0B1 5s. Digits without units are not numbers but follower-names as the names of the weekdays or the months.

We count by bundling, so a number has at least two digits counting the unbundled and the bundles; and perhaps the bundle-bundles. We cannot count in 1s since, as a single, 1 is not a bundle, and since a bundle of bundles of 1s will stay a 1s, 1 1s = 1 1s, instead of becoming a new unit as, e.g., 2 2s becoming a new unit, 4s, as seen when counting five fingers in 2s: 0B1, 0B2 or 1B0, 0B3 or 1B1, 0B4 or 1B2 or 2B0 or 1BB 0B 0, and finally 0B5 or 1B3 or 2B1 or 1BB 0B 1.

Therefore, when counting and adding Many, we will no longer use a ruler's one-dimensional line-numbers without units as 5 and 42. Instead we now will use two-dimensional bundle-numbers with units as 0B5, and 4B2 that exist as rectangular or squared totals on a ten-by-ten bundle-bundle pegboard, a 'BBBoard'. Which allows learning mathematics indirectly when formulating inside tales about outside totals as, e.g., 6 7s existing on the BBBoard limited by two rubber bands, and that may be recounted in tens as a total of four tens and two, shortened to 'T = 4B2' tens. This number-language sentence or formula contains an outside subject linked to an inside predicate, just as does a word-language sentence as 'This is a glass'.

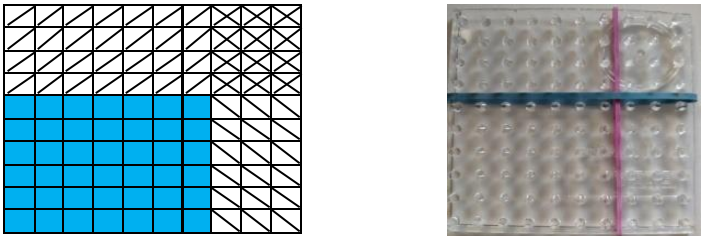

Fig. CC.1 On a ten-by-ten bundle-bundle pegboard, a 'BBBoard', 6*7 is seen as (B-4)*(B-3), which is left when from 10B pulling away 3B and 4B and adding the 4 3s pulled away twice, so 6*7 = 3B12

Flexible bundle-numbers with units allow the same total, forty-two, to be recounted with over-load or under-load so that T = 42 = 4B2 = 3B12 = 5B-8, which makes unneeded the place value system. As well as carrying and borrowing since now

17 + 28 = 1B7 + 2B8 = 3B15 = 4B5 = 45, and

57 – 28 = 5B7 – 2B8 = 3B-1 = 2B9 = 29.

When bundle-counting outside totals we find that not only digits but also the operations are icons, but in the reverse order.

Power now is the first operation we meet as a bundle-bundle hat when counting in 3s will change 9 into 3 3s, a bundle of bundles, a bundle-bundle, a BB, or a B^2, that on a BBBoard is a square where 2 3s is a rectangle that may be transformed into almost a square with the rectangle's square root as the side by moving half the excess from the top to the side. 6 4s thus is almost 5 5s.

Division and multiplication then follow as a broom and a lift to push away and to push back to stack bundles. Here, recounting in 2s will change 8 into (8/2)*2, or T = (T/B)*B telling that the total, T, contains T/B Bundles. This proportionality 'recount-formula' is used all over to shift units.

Also, it solves equations as 'u*2 = 8' asking "How many 2s in 8?" by recounting 8 in 2s as 8 = (8/2)*2, so that the solution is u = 8/2 found by moving 'to opposite side with opposite sign'. This follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, so if u*2 = 8 then u = 8/2. This means that now disappears the balancing method solving equations by doing the same to both sides.

Subtraction now follows as a rope to pull away the stack to locate unbundled singles, thus splitting the total in two, T = (T–B) + B, the 'split-formula'. Finaly in the end, addition is a cross showing the two ways to unite stacks, next-to and on-top.

The split-formula solves addition equations as 'u+2 = 8' asking "What is the number that with 2 added becomes 8?" which of course is found when splitting 8 by pulling away the 2 that was added, 8 = (8-2)+2, so that the solution is u = 8-2, again found by moving 'to opposite side with opposite sign'. Also, this follows the formal definition: 8-2 is the number u that with 2 added gives 8, so if u+2 = 8 then u = 8-2. So again the balancing method is unneeded.

Recounting 8 in 3s, a calculator may inside predict the outside result. Entering '8/3' gives '2.more', and entering '8-2x3' gives '2' unbundled as the 'more'.

This prediction is validated when outside pushing away 3s from 8 twice. Included on-top of the bundles, the unbundled becomes decimals if writing T = 2B2 3s; or fractions if counted in bundles also, T = 2 2/3 3s, or replaced by a negative number, T = 3B-1 3s, telling what is missing in space for an extra bundle, or what in time was pulled away from it. So, counting the unbundled leads to decimals, fractions, and negative numbers.

Recounting from one icon-unit to another may be predicted by a calculator. To inside predict the answer to the outside question "2 3s = ? 4s", entering '2*3/4' gives '1.more'. So, the unbundled are found when pulling away 1 4s predicted by entering '2*3 – 1*4' giving '2'. The calculator thus predicts that 2 3s = 1B2 4s, which may be validated outside.

Recounting from tens to icons when asking 'How many 6s in 24?' leads to the equations 'u*6 = 24' solved by u = 24/6 since 24 recounts in 6s as '24 = (24/6)*6', so the solution again follows the 'opposite side & sign' rule.

Recounting from icons to tens when asking 'How many tens in 6 7s?' leads to early algebra when placed on a BBBoard as (B-4)*(B-3) as on the figure above. So, 6 7s are left when we pull away the side 3B and the top 4B and add the 4 3s pulled away twice, so

(B-4)*(B-3) = B*B – 3B – 4B -- 4*3 = (10-3-4)B + 4*3 = 3B0 + 1B2 = 4B2 = 42.

This clearly shows that minus times minus must be plus. Also, the four rectangles on the BBBoard illustrate the FOIL-method, First, Outside, Inside, Last.

Recounting rectangles in squares, we may ask "How to square 6 4s by finding its square root as its side?" Moving half the excess from the top to the side gives the first guess as 5 5s. Then, to fill out the 1x1 top right corner we pull away a slice, u, from the top and side.

Here, 2*4*u = 1 gives u = 1/8, so 4 7/8 = 4.88 is our second guess, which is close to the calculator's answer, 4.90. Since two squares add as the square created by their Bottom-Top BT line we now have a way to add rectangular areas.

A split square may also be used to solve quadratics: In a (u+3) square, a vertical and horizontal dividing line will split a BBBoard into two squares, u^2 and 3^2, as well as two 3*u rectangles. So (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6 *u + 9.

Now, with the quadratic u^2 + 6*u + 8 = 0 they all disappear except 9-8 = 1. So (u+3)^2 = 1, which gives -2 and -4 as the two solutions. On the other hand, if u^2 + 6*u + 10 = 0, then 9-10 = -1, and then the equation (u+3)^2 = -1 has no solution.

Also, a split square with y = x^2 may show that, with dy and dx as small changes of y and x, dy will be 2*x*dx if we neglect the tiny-tiny upper right corner.

So, with y = x^2, dy/dx = 2*x.

Recounting in 2s, we may ask 41 = 2^?, or how many 2-factors in 41. Here we need a factor-counter called a logarithm. We choose the ln-button to see that ln 41 = 3.more.

We now try from 4 onwards: 2^4 = 16, and 2^5 = 32, and 2^6 = 64.

Choosing 2^5 = 32, we now have that 41 = 32 + 9, and 9 = 8+1 = 2^3+1.

So, recounted in 2s, 41 = 1B^5 + 0B^4 + 1B^3 + 0B^2 + 0B + 1 = 101001.

Recounting physical units, e.g., from \$ to kg, \$ = (\$/kg)*kg, gives a 'per-number', \$/kg, to connect them: with 3 \$/5 kg, 12 \$ = (12/3)*3 \$ = (12/3)*5 kg = 20 kg.

With like units, per-numbers become fractions or percentages,

3 \$/5 \$ = 3/5, and 3 \$/100 \$ = 3/100 = 3%.

Recounting mutually the sides in a stack split by its diagonal, the per-numbers gives trigonometry that connects the sides and the angles, e.g.,

rise = (rise/run)*run = tangent-Angle*run. Likewise,

rise = (rise/diagonal)*diagonal = sine-Angle*diagonal, and

run = (run/diagonal)*diagonal = cosine-Angle*diagonal.

Here the tangent-Angle describes the steepness of the sloping diagonal. In an x-y coordinate system the run and the rise become changes in x and in y, Δx and Δy. So here the tangent-Angle describes the steepness as the per-number Δy/Δx called the local slope of the curve.

Once counted and recounted, totals may finally be added on-top or next-to.

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like. Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process asking, e.g., "2 3s and how many 5s total 4 8s" is called differential calculus since you must find the difference between the two known totals before recounting it in 5s, (T2-T1)/5, or ΔT/5.

Adding 2 kg at 3 $/kg to 4 kg at 5 $/kg, the unit numbers 2 kg and 4 kg add directly. But the per-numbers 3 $/kg and 5 $/kg must first be multiplied to unit-numbers before adding, thus adding as areas, i.e., as integral calculus. Likewise with fractions where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 apples, and of course not 7 red of 6 apples as taught by in school. Per-numbers and fractions and digits thus are not numbers, but operators needing a number to become a number.

Here the per-numbers are piecewise constant, but they may also be locally constant as in the case of a falling object with an increasing meter/second number. Calculus thus occurs three times, as next-to addition of stacks in primary school, as adding piecewise constant per-numbers in middle school's mixture problems, and as adding locally constant per-numbers in high school where the tiny area-strips if written as changes, $p*dx = dA$, will profit from the fact that adding many changes makes all middle changes disappear leaving only the total change of A from the start to the end.

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives 106%^10 or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

Looking inside my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bend the two outer fingers. So, 1/3 of the Ls are bent, and ½ of the Rs. Does that mean that 1/3 of the bent are Ls? No, ½ is. So, in cross-tables we must also see fractions as per-number operators needing numbers to become numbers.

The first months the children thus meet the core of mathematics: functions, equations, proportionality, trigonometry, and calculus.

As well as the four operations that unite unlike and like unit-numbers and per-numbers: addition, multiplication, integration, and power, seen in an 'Algebra-square' that is named after the Arabic word 'algebra' meaning 'to reunite'.

And that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations unite/ *split Totals into*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n $T - n = a$	T = a * n $T/n = a$
Per-numbers m/s, $/100$ = %	T = ∫ f dx $dT/dx = f$	T = a^b $b\sqrt{T} = a$ $\quad log_a(T) = b$

Fig. CC.2 An 'Algebra Square' shows how to unite and split our four number-types, unlike and like unit- and per-numbers, and how to solve equations by moving 'to opposite side with opposite sign'

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres, fact and fiction and fake models, that are also called since-then and if-then and what-then models, or room and rate and risk models (Tarp 2001).

Fact models typically talk about the past and present and only need to check the units. Fiction models typically talk about the future and need to be supplemented with models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that '2+3 = 5' always despite 2 weeks + 3 days = 17days, thus changing mathematics to 'mathematism' true inside its 'no-unit greenhouse' but seldom outside where it is falsified by examples as 1 week + 1 day = 8 days, in contrast to multiplication where 2*3 = 6 simply states that 2 3s may be recounted as 6 1s.

CC.5 What is calculus

The tradition sees calculus as working with functions to find their limits and derivates and primitives (Bressaud et al. 2016). As difficult to learn, it is reserved for committed learners only. But once learned, it is very useful for modeling real-world change problems within the natural and social sciences and economy. So, calculus comes at the end of K-12 education as something that is chosen by a few only.

But is calculus so by nature or by choice? Here we quote the ancient Greek sophists warning that unable to tell nature from choice we may be patronized by choices masked as nature. And thus, we may be forced to serve an unnoticed 'truth-regime' to use a phrase of the French thinker Foucault (1995).

Together with Derrida, Foucault is part of the modern-day sophism called French post-structuralism that is inspired by the German existentialist thinker Heidegger (1962). Who warns against the judging is-statements 'A is B' where the subject A is naming an outside existence that must be respected while the predicate B is a chosen inside essence that could be different and that may come from preconceived prejudices. Consequently, Derrida (1991) recommends deconstructing predicates by returning to their outside root and finding a more proper name, preferably an action word, a verb, instead of another judging predicate.

So, we may try to deconstruct calculus in order to find a different answer to the question 'what is calculus'. Here, we may also want to use the word 'demodel' (Tarp 2020) getting its meaning from reversing the modeling ability of calculus. When modeling we go from outside concrete real-world things and actions to an inside abstract mathematical representation. When demodeling we do the opposite, we go from an inside abstract mathematical concept to outside concrete examples, some of whom may be the original roots from which the concept was originally abstracted.

As to the historical roots of calculus the tradition points to Newton's famous no's.

"Box [Newton] starts" Wanting to find a sea route to India to avoid the Portuguese fortification of the African coast, the English discussed how to navigate on open sea.

The latitude was no problem, but the longitude was, which created the question "How does the moon move on the sky?" "The moon moves among the stars, and they all follow the will of the Lord above that will be done on earth as in heaven," said the church. "No, it falls towards the earth," said Nowton. "But then it will hit the earth as do falling apples," said the church. "No, since a strong horizontal push may make the apple curve in the same way as does the earth ," Newton said. "But still it follows the Lord's unpredictable will," said the church. "No, it follows its own will that is predictable by a formula describing the universal gravity force existing between bodies," said Newton. "But Aristotle says that forces uphold states," said the church. "No, forces change states," said Newton. "So, we can use Arabic algebra for the change-calculations," said the church. "No, algebra only works with static situations, so we need a new change-calculation, which I invented and called calculus," said Newton. "Box [Newton] ends"

In Germany, Leibniz discovered calculus at the same time by looking at tangent curves to pendulum motions. The long controversy on how to denote a derivate, *y*' or d*y*/d*x*, is today settled by using both notations. But still there is agreement that differential calculus should precede integral calculus, even if the uniting operations addition, multiplication, and power precede their inverse operations subtraction, division and root and logarithm.

But, once the Hindu-Arabic numbers replaced the Roman numbers in Renaissance Italy it became possible to use multiplication and division to write out bills as an example of a mixture problem where you add areas by multiplying before adding in the case of per-numbers: 2 kg at 3 $/kg + 4 kg at 5 $/kg = 6 kg at (2*3+4*5)/6 $/kg

Likewise, bridging units could have introduced per-numbers. Instead, a 'regula-de-tri' rule-of-three method was used to solve a problem like: "With 2 $ per 3k g, what can we get for 6 $, and what is the price for 12 kg?". To answer, the three numbers were arranged with shifting units to first multiply and then divide:

Question, 6 $ = ? kg. Order: 6 $, 3 kg, 2 $. Result: 6*3 = 18, and 18/2 = 9.

So, 6 $ buys 9 kg.

Question, 12 kg = ? $. Order: 12 kg, 2 $, 3 kg. Result: 12*2 = 24, and 24/3 = 8.

So, 12 kg costs 8 $.

Using per-numbers, the answers come from recounting in the per-numbers which makes division precede multiplication:

6 $ = (6/2)*2 $ = (6/2)*3 kg = 9 kg, and

12 kg = (12/3)*3 kg = (12/3)*2 $ = 8 $

The usefulness of per-numbers is seen in the basic recount formula that recounts a total, T, with B-bundles as the unit, T = (T/B)*B, which portrays T as an area formed by a per-number and a unit.

So, by containing both an area and a per-number, recounting could be seen as the root of calculus that could have flourished with the arrival of multiplication and division in the Renaissance, but that first becomes possible now where bundle-numbers with units create the recount formula. And where 8/2 then means 8 counted in 2s in time instead of 8 split in 2 in space.

CC.6 The Role of Calculus in Existence-based Many-math

Existence-based Many-math counts and adds in time and space totals coming from the physical fact Many that exists as duration in time and extension in space. It needs calculus for next-to addition of its bundle-numbers with units since these are two-dimensional areas. But, with existence before essence, counting comes before adding. Does that mean that calculus first occurs when adding occurs? Of course that depends on how you view calculus.

Seeing it as working with areas and per-numbers, calculus occurs from the very start since a bundle-number with units as, e.g., 2 3s exists as 2 bundles with 3 per bundle. This makes the 2 3s a two-dimensional area, a rectangle with 3 as a per-number, with the described total T = 2*3 as its area-number that may or may not be recounted in tens as 2*3 = 6, and with the 2up-by-3out diagonal as a sloping line with the per-number 2/3 as the tangent-number that allows recounting the height in the base as well as finding half the perimeter of the circle as the limit-number n*tan(180/n) for n large. And that later may be added next-to horizontally, e.g., to 4 5s as 8s thus adding the per-numbers as areas, which is what integral calculus does.

And, finding that the bundle-bundles are squares leads to meeting and constructing and calculating a close value of the square root to recount a rectangle as, e.g., 6 4s, into a square by moving areas twice, first from the top to the side, then from the sides to fill the empty upper right empty square. Later, placing four stacks after turning them creates a square that by inside containing two squares and two stacks show that squares may add next-to as the square created by their mutual bottom-top line.

And, with multiplication of numbers, two rubber bands on a BBBoard allow the multiplication 6*7 to be seen as a (B-4)*(B-3) stack with a total coming from pulling away the side 3 B's and the top 4 B's and adding the 4 3s in the corner that was pulled away twice. And allows multiplication 36*47 to be seen as 3B6*4B7 by showing two stacks for the B's and one stack for the BB's and for the unbundled singles.

CC.6.1 Calculus in primary school

So, seeing it as working with areas and per-numbers, calculus occurs from the very beginning. But before we begin adding the bundle-numbers with units, we want to produce them by counting process where an outside total comes inside as a number-language sentence, e.g., T = 2 3s, connecting the total with a bundle-number with a unit. By using the recount-formula T = (T/B)*B telling that with the bundle B as a unit, T/B is the number of Bs in the total T, and showing the height, T/B, and the base, B, of the rectangle that represents the outside total on a BBBoard.

And, before adding the totals we may want to recount a total in another unit.

Recounting from one icon-unit to another will ask, e.g., '6 7s = ? 8s'. Finding the answer can be done manually on a BBBoard by moving a 7-bundle from the top to the side giving, 5B2 8s. Also, 5B2 could have been predicted on a calculator by entering '6*7/8' giving the result '5.more', followed by finding the more by entering '6*7-5*8' giving '2'.

Recounting from an icon-unit to tens will ask, e.g., '6 7s = ? tens'. This question comes when looking at the multiplication tables. To predict the answer on a calculator, we only enter 6*7 to get the result 42 that we then read as 4B2 tens. The answer may also be found manually by seeing 6*7 as (B-4)*(B-3) so that 6*7 occurs on a BBBoard as 6 7s found by subtracting the areas 4B and 3B from its 10B, and then adding the area 4 3s that was subtracted twice:

6*7 = (B-4)*(B-3) = BB – 3B – 4B + 4*3 = (10 – 3 – 4)B 12 = 3B12 = 4B2 = 42.

Recounting from tens to an icon-unit will ask, e.g., '60 = ? 7s'. Finding the answer can be done manually on a BBBoard by moving the unwanted 6 3s from the side to the top giving, 6B18 7s, or 8B4 7s. Also, 8B4 could have been predicted on a calculator by entering '60/7' giving the result '8.more', followed by finding the more by entering '60-7*8' giving '4'.

Rephrased as the equation 'u*7 = 60' the question asks "How many 7s in 60?" which is answered immediately by recounting 60 in 7s as 60 = (60/7)*7. The answer to a multiplication equation thus is simply found by moving the known number to opposite side with opposite sign to isolate the unknown number u, so the equation 'u*7 = 60' is solved by 'u = 60/7'.

Recounting from rectangles to squares may ask "How to square 6 4s by finding its square root as its side?" Moving half the excess from the top to the side gives the first guess as 5 5s. Then, to fill out the 1x1 top right corner we pull away a slice, u, from the top and side. Here, 2*4*u = 1, or u = 1/8, so 4 7/8 = 4.88 is our second guess, which is close to the calculator's answer, 4.90. Since two squares add as the square created by their Bottom-Top BT line we now have a way to add rectangular areas.

Once counted and recounted, totals may be added next to or vertically on-top.

Next-to addition of 2 3s and 4 5s as 8s means adding two areas, which is an example of integral calculus. Finding the answer can be done manually on a BBBoard by filling up the low stack from the high stack giving, 3B2 8s. Also, 3B2 could have been predicted on a calculator by entering '(2*3+4*5)/8' giving the result '3.more', followed by finding the more by entering ''(2*3+4*5) – 3x8' giving '2'.

Next-to addition of two squares as a square gives the square formed by the bottom top line between the two squares. This follows from the Pythagoras rule seen when adding four hxb stacks after each other after turning them a quarter round. Inside the figure we find a h-square and a b-square and two stacks. But we also find the diagonal square and two stacks split in four half stacks. So, h^2 + b^2 = d^2.

A BBBoard thus can show stationary forms in space. And a BBBoard may also illustrate motion in time where horizontal and vertical change-steps create different orbits showing straight lines or curved lines, with different names.

To follow the orbits, we assign to each point an out-number, x, and an up-number, y, so that each point on the BBBoard has an (out, up) or (x, y) label. Going clockworks round, the corners thus have the (x,y) labels (1,1), (1,10), (10,10) and (10,1). An orbit then is created by horizontal and vertical (out, up) change-steps that mean (back, down) if negative.

A straight line is created if each +1 change-step out is followed by (+1, +1, +1, …) change-steps up. If we begin in the point (1,1) then *n* steps out will result in *n* steps up so that the final up-number, *y*, will be $y = x$, called the formula or function for y.

A steeper line created by the (+2, +2, +2, …) change-steps up will have the formula $y = 2*x$. And a less steep line created by the (+1, +1, +1, …) change-steps up per each +2 change-step out will have the formula $y = 1/2*x$.

So, a line's steepness is described by the per-number in front of x, which is called the slope-number. The line points upwards with a positive slope and downwards with a negative slope, and horizontally with a zero slope.

The two lines $y = 2*x$ and $y = 9 - x$ intersect at the point $(x,y) = (3,6)$. This is seen on the BBBoard, or predicted by finding identical y's where $2*x = 9 - x$. Moving to opposite side with opposite sign we find that

$2*x + x = 9$, so, $3*x = 6 = (6/2)*2$, so, $x = 6/2 = 3$, so, $y = 2*x = 2*3 = 6$.

A hyperbola is an upwards bending curve created by the top right corner of the stacks when recounting twelve ones in 2s, 3s, 4s, 6s, and twelves.

A parabola occurs as a downwards bending curve created if, beginning in the point (1,1), each +1 change-step out is followed by the up change-steps (+4, +3, +2, +1, +-1, +-2, +-3, +-4).

A parabola occurs as an upwards bending curve created if, beginning in the point (1,10), each +1 change-step out is followed by the up change-steps (+-4, +-3, +-2, +-1, +1, +2, +3, +4).

A double-parabola occurs as a downwards and upwards bending curve created if, beginning in the point (1,1), each +1 change-step out is followed by the up change-steps (+3, +2, +1, +-1, +-2, +-1, +1, +2, +3).

A wave is created if, beginning in the point (1,5), each +1 change-step out is followed by the up change-steps (+2, +1, +-1, +-2, +-2, +-1, +1, +2, +2, +1).

A hill is created if, beginning in the point (1,1), each +1 change-step out is followed by the up change-steps (+1, +2, +3, +2, +1, +-1, +-2, +-3, +-2).

A circle is almost created if, beginning in the point (5,9), we use the (out, up) change-steps (+1, +0), (+2, +1), (+1, +-2), (+0, +-1), etc.

An ellipse is almost created if, beginning in the point (5,9), we use the (out, up) change-steps (+1, +0), (+2, +-1), (+0, +-1), (+-2, +-1), (+-1, +0), etc.

The area under the curves may be found as a sum of stacks and half-stacks.

A BBBoard may also show a simple prey-predator simulation on an island where cats eat mice. We expect a cycle in time since many cats and many mice leads to many cats and few mice, which leads to few cats and few mice, which leads to few cats and many mice, which leads to many cats and many mice once again.

One model will be to assume that a mice-population at 7 and 2 will make the cat-population change with 7-5 and 2-5 respectively. Likewise, a cat-population at 7 and 2 will make the mice-population change with 5-7 and 5-2 respectively. We see that initial populations at level 5 will give a stable model. Here we assume that the initial populations for the cats and the mice are 8 and 1 respectively.

The following period the two populations will then be 8 + (1-5) = 4, and 1 + (5-4) = 2 respectively. Continuing, we see that the cat population will change as 8, 4, 1, 2,

6, 9, 8; and that the mice population will change as 1, 2, 6, 9, 8, 4, 1. This allows the points (8,1), (4,2), etc., to be marked on a BBBoard, showing a cycle continuing again and again. Different initial numbers will give different cycles.

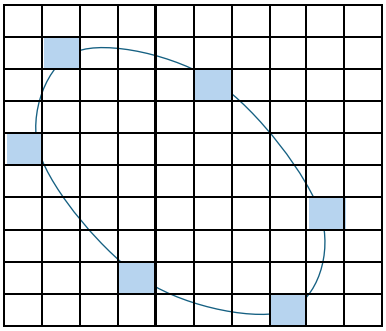

Fig. CC.3 Cats eat mice, so the mice-number decreases, so the cat-number decreases, so the mice-number increases, so the cat number increases, so the mice-number decreases, etc.

Numeracy after primary school. At the end of primary school the students' original bundle-numbers with units have developed into counting-numbers in time as unit-numbers, into per-numbers in space as Bundle-numbers, into decimal-numbers and fractions for bundle-parts and negative numbers for what is missing in space or pulled away in time, into square roots for squaring rectangles, into logarithms for recounting from bundles of tens to bundles of 2s, into per-numbers also as bridging physical units with fractions when the units is like, and into tangent-numbers describing the angle and steepness of the diagonal in a stack. They have seen digits and operations as icons. And seen place-values and carrying as unneeded if including units to show numbers' polynomial nature as a union of unbundled singles, bundles, bundle-bundles, etc.

As to operations, the students have met power as bundling bundles and bundle-bundles, division as pushing away bundles to count an outside total in bundle-units, multiplication as pushing bundles back to a stack, subtraction as pulling away a stack to find unbundled singles to place on top of the stack as a decimal, a fraction or a negative number. And they have seen equations solved by recounting using the core recount-formula T = (T/B)*B, also expressing proportionality used to change units.

So, the students have met all the operations in the Algebra-square as tools to communicate about things and actions on a BBBoard.

As to geometrical forms they have met lines, hills, circles, parabolas, double-parabolas, ellipses, hyperbola, and waves.

As to calculus, integral calculus is used for next-to addition of stacks and for finding the area inside a circle and under curves. Where differential calculus is used to see how bundle-bundle squares change to the following square, to describe the steepness of lines on a BBBoard and of diagonals inside a stack, to find the length pi of a half-circle through a limit process, and to create orbits on a BBBoard with change-steps formulas.

As to tools, a BBBoard and a pocket calculator are used for recounting manually and for using calculations for predictions. And, the BBBoard is also used for multiplying single digit and multi-digit numbers.

CC.6.2 Middle school

In middle school the BBBoard is replaced by a coordinate system, and the pocket calculator is replaced by a simple graphical CAS calculator.

To coordinate geometry and algebra in coordinate geometry, a coordinate system now has two number-line containing also zero and negative numbers. And in algebra total-formulas now are replaced by formulas predicting y from a calculation containing numbers and an x-variable called a function where an unspecified function is written as $f(x)$.

And here change-steps is repelled by difference equations allowing change calculation to split in two parts, predictable change and unpredictable change leading to and statistics and probability.

Predictable change includes both situations where the change-number and change-percent is constant, or constantly changing, i.e., both linear and exponential change and quadratic change and logistic change, as well as the combination of a constant change number and change percent in savings. Exponential change allows meeting the next example of a limit process when a yearly interest at 100% is split into n parts and added n times, which leads to the Euler-number, e, defined by a limit process as

$e = (1+1/n)\text{^}n$ for n large enough.

Change equations are treated at a spreadsheet using the formula

end-place = start-place + velocity, and

end-velocity = start-velocity + acceleration.

This allows different orbits to be studied by changing the initial values for the place and the velocity. Here a straight line becomes bended into a parabola when the acceleration changes from zero to a constant, and into a double parabola when the acceleration changes constantly.

In the inner space close to the earth, the acceleration may be seen as constant, which allows studying the orbit of a golf ball, the orbit when jumping from a swing, and the oscillating orbit created by one or more springs.

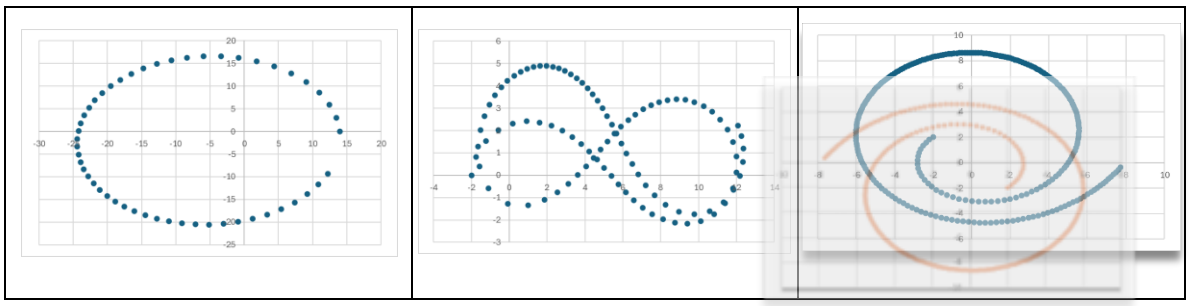

Fig. CC.4 On a ten-by-ten bundle-bundle pegboard, a 'BBBoard', difference equations on a spreadsheet is used to show orbits of a particle around one body, two bodies, as well as of two bodies

In outer space far from earth the acceleration is in inverse proportion to the distance to the particle and the pulling body. Trigonometry then allows splitting the total pulling acceleration in a horizontal and vertical part that changes the horizontal and vertical velocity that again changes the horizontal and vertical position.

In the same way it will be possible to study possible a particle's orbit around two fixed bodies, and to study possible orbits of the two bodies if allowed to move freely.

Also, the predator-prey interaction may now be studied with change equations.

In primary school integral calculus adds per-numbers occurring as bundle-numbers. In middle school integral calculus adds per-numbers from in mixture problems as

2 kg at 3 $/kg + 4 kg at 5 $/kg = 6 kg at ? $/kg. Or as

2 $ at 2/3 $/$ + 4 $ at 4/5 $/$ = 6 $ at ? $/$ with fractions.

Or as 1/2 + 2/3 = ?, if taken of the same total.

Also integral calculus may be useful to add numbers with different units, as 2 $ + 3 £ = ? €. Likewise, when adding percentages in cross-tables where 2% + 3% rarely is 5% as shown by Bayes Theorem.

Furthermore, Carnot cycles two hyperbolas may be areas telling about the efficiency of steam engines.

In middle school, a CAS graphing calculator allows modeling to take place where statistical tables are transformed into formulas that may be expressing a unit number where the per-number then is found by using the calculator to differentiate. Or a per-number as meter pr second where the meter-number then is found as the area under the per-number graph by the calculator, and where the differentiation will give the acceleration.

In geometry itself, constructions are carried out by using a ruler and a drawing compass. And in algebra itself, the different word problems will be given enough time. And, inspired by the British curriculum, also mechanical physics could be modeled. And economics looks, e.g., at markets with an equilibrium where the demand meets the supply, and where per-numbers occur marginal numbers to totals.

As to letter-calculations, '$2*a + 3*a = (2+3)*a$' no longer is an example of a distributive law, but an example of identifying the like units.

CC.6.3 High school

Geometry expands geometry from two to three dimensions by introducing linear algebra and vectors.

Algebra introduces complex numbers

Integral calculus introduces areas under a locally constant curve where the formal definitions of piecewise and locally constancy (continuous) -as well as local linearity- occur by interchanging epsilon and delta.

A variable y is globally constant c	$\forall \varepsilon > 0$:	$\lvert y-c \rvert$	$< \varepsilon$ all over
A variable y is piecewise constant c	$\exists \delta > 0 \ \forall \varepsilon > 0$:	$\lvert y-c \rvert$	$< \varepsilon$ in the interval δ
A variable y is locally constant c	$\forall \varepsilon > 0 \ \exists \delta > 0$:	$\lvert y-c \rvert$	$< \varepsilon$ in the interval δ

Fig. CC.5 The formal definition of the three kinds of constancy. It is strange that the university defines local constancy as piecewise constancy, and not when defining local and piecewise linearity

Differential calculus is now given a precise definition of the limit and shows how different change formulas follow from studying the shades from a book and from the definition of the Euler number

e = $(1 + 1/n)^n$ for *n* large enough

Differential calculus is used for optimizing functions of one variable and two variables. Game theory shows the existence of 3D saddle points.

CC.7 Consulting the three Grand Theories

To understand the differences between essence-based mathematism and existence-based Many-math we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

In this way the three grand theories may help to enlighten and discuss the core of a teacher job formulated as 'teach learners something'. Here the nature of 'something' may be discussed in philosophy. And the power relations in the textbook-teacher-learner interaction may be discussed in sociology. Finally different learning possibilities in working with the subject or hearing about its predicate may be discussed in psychology. And, discussing may be a better word than enlightening since alle three areas contain conflicting theories.

Within philosophy, a discussion between existentialism (Sartre 2007) and essentialism about the precedence of existence or essence has taken place since philosophy began in ancient Greece (Russell 1945). Here the 'knowing' sophists argued that to practice democracy we must tell nature from choice to avoid being patronized by choice masked as nature. In other words, we must be able to tell outside existence from its many chosen inside essences and especially the ones that have been institutionalized as the tradition. Against this, the 'better knowing' philo-sophists argued that choice is nonexistent since everything physical is but imperfect examples of metaphysical essence as illustrated in Plato's Cave allegory and that is only accessible to philosophers educated at the Plato Academy.

This disagreement between sophists and philosophers about nature versus choice, and existence versus essence runs through history. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. In the counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe's line-organized office-preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the French Enlightenment republic's post-structuralism inspired by Heidegger (1962) who argued that "P is Q" is a statement judging an outside existence, P, with an inside constructed essence-predicate that may be a preconceived prejudice, e.g., gossip (Gerede), which therefore should be met with skepticism and possibly be deconstructed. With 'essence' coming from Latin 'esse' meaning 'being', Heidegger gave four answers to the question "What is is?" pointing down, up, over, or nowhere: is for example, is an example of, is like, or is period. A

function thus may be an example of a subset in a set-product where first-component identity implies second-component identity, or a name for expressions combining specified and unspecified numbers, or a number-language sentence including a subject, a verb, and a predicate as in a word-language sentence, or simply what it is, a stand-by calculation, e.g., '3*x' in contrast to '3*5'.

Later, the 20th century two clashes in Europe between highly institutionalized countries were both terminated by the low institutionalized American Enlightenment republics. This made thinkers as Baumann (1990) and Arendt (1963) warn against a built-in tendency in institutions that, despite created to reach a goal, they become tempted to perform a 'goal displacement' by using 'not reaching the goal' as a means to become the goal itself in need of more funding and more research. And, reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job may force you to carry out evil orders, called the banality of evil. As an example of an institutionalized knowledge-regime (Foucault 1995) this also is an issue to mathematics education.

Psychology (Skemp 1971) has a controversy within constructivism where Vygotsky (1986) sees education as adapting to the institutionalized essence-regime by building ladders from it to the learners' proximal learning zones. Piaget (1969) replaces this top-down view with a bottom-up view inspired by American Grounded Theory (Glaser and Strauss 1967) allowing categories to emerge from concrete experiences. This conflict has sociological consequences. Presented top-down from above as examples of inside abstractions, concepts become hard to learn, which forces many learners to stop learning what is meaningless to them and to accept patronization by those who accept such meanings. In contrast, bottom-up concepts grounded from below in the outside world are easier to learn for children through the concrete examples that exemplify and root the concepts; and for teenagers since knowing the subject in a sentence makes learning gossip-like.

Within Sociology, mathematics education and its research may be discussed by asking the basic question: in a social space, do we need patronization, or can we find mutual solutions by using the threefold information-debate-choice method of a democracy (Foucault1995, Kuhn 1962)? As pointed out, the debate on patronization began in ancient Greece between the philosophers and the sophists; and the debate is still with us today between socialist top-down critical theorists and skeptical bottom-up existentialist theorists. The power exercised by the social institution education can be patronizing; or it can be emancipating if providing self-supporting resilience ('Mündigkeit' in German). Europe maximizes the power-component by using line-organized office-preparing education to force learners to stay on the line as long as possible, and to accept that their difficulties are caused by their inferiority to the children of the public officeholders helping their children inherit their offices created to support the non-self-supported. Since they cannot use the alphabet as in ancient mandarin China, they use concepts instead. Whereas North America from secondary school minimizes the power-component by using daily lessons in self-chosen half-year blocks to uncover and develop the individual talent of the learner.

Mathematics may also serve both purposes. Presented from above as a top-down self-referring essence regime, mathematics becomes so hard to learn that it forces many learners to stop learning it. This is a minor problem with half-year blocks since leaving math does not mean leaving school, but it is a big problem at line-organized schools where leaving the line means leaving school for good. Presented bottom-up from below grounded in the existing fact Many, mathematics becomes easier to learn; and the learner can keep on choosing more blocks until the interest may disappear. The need to keep the job forces teachers to follow the orders of their specific institution. But when trained, teachers should as potential change agents be informed about the choices in an educational institution and within mathematics, so the individual teacher knows the difference between choice and nature, i.e., what can be changed and what cannot, to prevent being forced to become an agent practicing the tradition's banality of evil by preventing the children from developing their own number-language.

CC.8 Discussion and Conclusion

The UN Sustainable Development Goals wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, it turned out that two different views exist on numeracy, one pointing to knowing basic mathematics, and the other to using numbers.

Seeing numbers as one-dimensional line-numbers without units obeying the assumption that 1+1=2, these two views do not conflict since they both have addition using place value systems and carrying in the beginning, and calculus in the end to find derivatives and primitives. Therefore, calculus cannot be a part of numeracy.

Unless numbers are the two-dimensional bundle-numbers with units that children use before school, and that may become areas on a ten-by-ten BundleBundleBoard. Here adding the bundle-stacks 2 3s and 4 5s as 8s means adding areas thus rooting a primary school integral calculus that becomes differential calculus when asking the reverse question: "2 3s and how many 5s total 4 8s".

Here digits have different roles in space and time: 2 3s describes bundles of 3s existing in space, and 2 of them when counted in time. So, with 2 3s as 2 bundles with 3 per bundle, 2 may be called a counting-number or a unit-number since it only has meaning as attached to a unit describing what was counted, 3s, or meters, or dollars, etc. And 3 may be called a bundle-number or a per-number also without meaning unless connected with a counting-number telling how many there are. Furthermore, neither obey the assumption that 1+1 = 2 as seen when four fingers arranged as an open and a closed V-sign show that 2 1s and 1 2s total 1 4s and not 3 3s.

These two views on numbers may be called 'essence-based' and 'existence-based' mathematics when using the two core concepts of philosophical Existentialism holding that existence should precede essence which entails that counting and recounting outside totals should precede adding them inside.

Essence-math has calculus in the end. In contrast, existence-math has calculus in the beginning as the core of numeracy. It is needed in space for horizontal direct or

reversed next-to addition of stacks. And for recounting rectangular stacks as quadratic bundle-bundles with the square root as the side. And when adding squares to the square formed by their bottom-top line. And to ease multiplication tables by seeing 6*7 as (B-4)*(B-3) on a BBBoard where the 6 7s are found by subtracting and adding areas. Also, it is helpful for finding the area of a circle.

Furthermore, in time differential calculus shows how one bundle-bundle square changes to the next by adding two bundles and one unbundled thus introducing how to derive quadratics later. And, it occurs as change-formulas creating orbits as straight or bended lines on a BBBoard or in a coordinate system. And used to show a simple prey-predator simulation.

Where essence-based mathematics is textbook-bound with a lot of written exercises, existence-math is physical with totals occurring both as extension in space and duration in time, and with counting preceding adding to bring outside totals inside as bundle-numbers with units on a BBBoard.

Consequently, it will be possible to realize the UN numeracy goal, but only if the United Nations also declare that a number-language is a human right to prevent a colonization of the concept 'numerate' by the essence-based tradition claiming that numerate means 'being competent in the basic principles of mathematics' with its one-dimensional line-numbers built on the assumption that 1+2 = 3 always, despite the fact that 1 week + 2 days = 9 days, and that 1 ones + 2 ones = 1 threes, whereas 1*2 = 2 always, which shows that "Outside, addition folds but multiplication holds".

Instead, a number-language will become a human right only if its education accepts and develops the children's own two-dimensional bundle-numbers with units with its numerate concept expressed by the alternative existence-based view as 'having the ability to understand and work with numbers.' But only if the numbers are the bundle-numbers that children bring to school. And that has calculus in the sense of 'adding per-numbers' in their very nature since a bundle-number with units as 2 3s is 2 bundles with 3 per-bundle that add with 4 5s as areas when adding the two per-numbers to 8s. And since bundle-bundle numbers are squares needing two bundles to change to the following square, except for a small corner.

Before making the choice between the essence- and existence-based views on numeracy it may be a good idea to listen to the three grand theories, philosophy and sociology and psychology. Here philosophical Existentialism discusses the difference between essence and existence. And here sociology warns that despite institutions are created as goal-directed means, a goal displacement will occur if the institutionalized essence-based mathematic will use its monopoly to use 'not reaching the goal' as a means to become the goal itself. And here psychology points out that meeting concrete existence and listening to institutionalized essence are two different ways of being educated.

In the 1970s a communicative turn brought general literacy to word-language education (Widdowson 1978). Calculus now has the chance to let a communicative turn bring general numeracy to number-language education also by helping children to add the per-numbers they use to communicate about what exists in space and what happens in time on a BundleBundleBoard.

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