Lessons in BBM Bundle-Bundle Math

Fairy-told by Bo, a self-educated pre-teen child still living in an enchanted world with Bundle-numbers as 2 3s and 4B2 5s existing on a BundleBundleBoard.

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Lesson 01. I count my fingers in 2s

5 = 0B5 = 1B3 = 2B1 = 3B-1 2s. Also 5 = 1BB 0B1 2s, and ten = 2BB0B2 = 1BBB 0BB 1B 0 2s. Likewise 37 = 3B7 = 2B17 = 4B-3. Otherwise, ten = 3B1 = 1BB 0B1 3s.

Lesson 02. I build squares with BundleBundles

I see that 3 3s = 1BB 2B 1 2s = 1BB -2B 1 4s. So now I can learn the square-numbers 1, 4, 9, 16, 25 from the bottom and 81, 64, 49, and 25 from the top, of course sharing the same last digits.

Lesson 3. I square 6 4s to find its square-root

I square 6 4s to find its square-root by moving half the excess form the top to the side to give 5 5s. So my first guess is that the square root of 6 4s is 5, which is too much since both 5s must give away a slice to fill the empty corner.

Lesson 04. I will re-count 8 1s in 2s to change the units

I will re-count 8 1s in 2s to change the units, so from 8 I push-away 2s and write 8/2 with division as an icon for a broom. Then 4 times I push-back and stack the 2s and write 4x2 with multiplication as an icon for a lift. So, 8 1s counted in 2s is 8 = 4B0 2s. Then I write 8 = (8/2)*2, or T = (T/B)*B with T and B for Total and Bundle, which I call a re-count formula used to change units. My parents call it a proportionality and linearity formula, but I don't mind.

Lesson 05. I re-count 8 1s in 3s to change units

First, a calculator predicts that 8/3 = 2.more. To find the more I look for the unbundled by pulling-away the stack and write 8 – 2x3 with subtraction as in icon for a rope. Placed on-top of the stack, the unbundled 2 becomes a decimal number, a negative less number or a fraction if also counted in 3s as 2 = (2/3)*3. So, 8 1s recounts in 3s as 8 = 2B2 = 3B-1 = 2 2/3 3s. In the same way I can re-count in tens: 64 = 6B4 = 7B-6 = 6 4/10 tens. My parents write 6.4 instead of 6B4, but I don't mind.

Lesson 06. The solution to the equation 'u*2 = 8'

My parents want me to find the solution to the equation 'u*2 = 8'. A strange way to ask me how many 2s there are in 8, but I don't mind. Of course, I just recount 8 in 2s as 8 = (8/2)*2 and get 4 2s, so the solution is u = 8/2. When I don't know how many times to pull-back 2 to get 8, I just pull-away the 2s from 8 instead. So, in the equation I could simply move 4 to the opposite side with the opposite sign.

Lesson 07. A better guess at the square root of 6 4s

To get a better guess at the square root of 6 4s, I now take two u*4 slices away from the top and side to fill the empty corner. This gives the question 'u*8 = 1' with the solution t = 1/8. So, the second guess will be 5 – 1/8 that is 4.88 on my calculator, which also gives the correct numbers 4.90 that is close.

In the same way I now square 5 3s and find its square root, first as 4 before I remove the two u*3 slices to solve the equation 'u*6 = 1' to get u = 4 – 1/6 = 3.83 close to the real number 3.87.

Lesson 08. Splitting a BBBoard into two squares and two stacks

It is fun to use two rubber-bands to split a BBBoard into two squares and two stacks, e.g., 3 3s and 7 7s and 3 7s twice. And it is fun to write is with letters instead of numbers to see that an (a+b)-square includes an a-square, and a b-square, and two a bs, which my parents write (a+b)^2 = a^2 + b^2 + 2*a*b, and call a square-rule. I asked: "Is there also are square equations?" They showed me 'u^2 + 6 u + 8 = 0' and told me I would learn to solve such quadratic equations in highshool. Strange, because I just place the two rubber bands, so they form a large square, u^2, and a small square, 3^2 = 9, and two stacks, 3*u. Then the whole BBBoard, (u+3)^2, is u^2 + 6*u + 9. But u^2 + 6 *u + 8 disappears since it is zero. So, what is left is (u+3)^2 = 1. Here, 1 is a V1-square, and u+3 is 1 if u is -2, so the solution must be u = -2. My parents say there is also another solution, but I don't mind. Instead, I saw that the equation u^2 + 6 u + 7 = 0 will give (u+3)^2 = 2 where 2 is a V2-square, so here, u+3 = V2 gives the solution u = V2-3, and the other solution that my parents talk about. But the equation u^2 + 6 u + 10 = 0 will give (u+3)^2 = -1 which is impossible since -1 has no square root, so here are no solutions.

Lesson 09. Adding squares as squares

I really like squares, so now I will try to add squares as squares. On my BBBoard I placed two rubber bands after 7, and see that 7^2 and 3^2 add to (7+3)^3, but only if I add the two 3 7s stacks also. It would be nice if they could add directly to a new square without any stacks. Then I placed two extra rubber bands after 3. Now I have a square in the middle surrounded by 4 stacks that can be halved by interior lines that my parents call diagonals. But the diagonals also form a square, which is surrounded by 4 half-stacks that can become 2 full stacks. So, inside a stack of 3 7s, the 3^2 and 7^2 can be added as the diagonal's square, which I can measure to 7B6 cm. Or find as the square-root of the sum of 3^2 and 7^2. First I recount 9 in 7s as 9 = (9/7)*7 = 1B28 7s. Then I add half of this to the top and the side of the 7 7s to find my first guess to be 7 + ½B14 = 7B64, which is close enough. Then the fun begins. "What happens if a 3-square eats a 4-square and a 5-square and a 6-square? I place the 4-square next-to the 3-square and see that the 'bottom-top' line is a diagonal in a 4 3s stack with a length that I measure to 5. So, a 3-square and a 4-square add to a 5-square. That eats a 5-square and becomes a 7-square, that eats a 6-square and becomes a 9B2-square.

Lesson 10. 6 times 7 is 6 7s

"Yes, there are many days in the 6 weeks your uncle is away, you will learn how many later" said my parents to which I replied "Why, 6 times 7 is 6 7s. Here I have 7 fingers as '1 half-bundle and 2' when I close my hand. So, 6 half-bundles are 3 bundles, and 6 2s is 1 bundle and 2 when I close both hands and include my arms. Therefore, 6 7s is 4B2 or 42."

CHANGING UNITS FROM 7S TO TENS is what my parents want me to do. So, I use my BBBoard with two rubber bands at the half-bundles, and with two extra bands to show the 6 7s. FIRST, I pull away 7s to the right to 5B-8 tens. After 2 7s I have the answer, 6 7s = 4B2 tens. THEN, I count with my fingers on the board. Inside the 6 7s there are 6 half-bundles to the left of the half-bundle band, and 2 half-bundles to the right. With 2 in the corner it gives a total of 4B2, or 42. THEN, I calculate: '6*7 = 6*½B2 = 3B12 = 4B2 = 42'. THEN, I use less-numbers where '6 = B-4' and '7 = B-3'. First, I pull away 3 ten-bundles from the right and 4 ten-bundles from the top, which leaves 3 ten-bundles. Then I add the 4 3s that I pulled away twice. So, minus times minus gives plus. Then I write this down as '6*7 = (B – 4)*(B – 3) = 1BB – 3B – 4B + 4*3 = 3B12 = 4B2 tens'. My parents call this the FOIL method to multiply binomials, First-Outside-Inside-Last, but I don't mind.

CHANGING UNITS FROM TENS TO ICONS I did in lesson 06. Asking "How many 2s in 10?" leads to the equation 'u*2 = 10' that is solved by recounting 10 in 2s as 10 = (10/2)*2 = 5 2s, or simply by moving '*2' to the opposite side as '/2', u = 10/2 = 5. But, I want a real answer with Bundle-Bundles where 1B = 2, 1BB = 1B^2 = 4, 1BBB = 1B^3 = 8, etc. With 10 as 8 & 2, I see that 5 2s = 1BBB 0BB 1B 0 2s. My calculator predicts this with its log2 button that counts 2-factors: log2(4) = 2 since 4 = 2^2, and log2(8) = 3 since 8 = 2^3. First, I find 'log2(10) = 3.more'. Then I see that 2^3 = 8, which pulled away from 10 leaves 10 – 8 = 2. Next, I find 'log2(2) = 1'. Then I see that 2^1 = 2, which pulled away from 2 leaves 2 – 2 = 0. So, the calculator agrees that 10 = 1BBB 0BB 1B 0 2s. In the same way I can use the log2-button to change 25 to 1BBBB 1BBB 0BB 0B 1 2s. And I can use the log3-button to change 25 to 2BB 2B 1 3s.

CHANGING UNITS FROM ICONS TO ICONS can be asking "Ten 1s is how many 2s or 3s or 4s or 5s or 9s?" On the BBBoard I see that increasing the width will decrease the height of the stack. Or asking "4 5s is how many 6s?". To predict the result, I use the recount formula on my calculator: Entering '4x5/6' gives '3.more', and entering '4x5 – 3x6' gives '2'. So, the calculator predicts that 4 5s recounts as 3B2 6s. This result I then test on my BBBoard.

Lesson 11. Recounting between physical units

Now I will recount between physical units: "With 2$ per 3 kg, which I call a per-number 2$/3kg, what does 6$ buy, and what does 12kg cost?" Since I know something about 2$, I recount the dollar-number in 2s to see how many times I have 3 kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Likewise with the kg-number where I know something about 3kg: 12kg = (12/3)*3kg = (12/3)*2$ = 8$. I could also use the equation '$/kg = $/kg' that in the first case gives '2/3 = 6/kg'. Here I move to the opposite side with opposite sign and get '2*kg = 6*3', or kg = 6*3/2 = 9; and that in the second case gives ' 2/3 = $/12' , or '2*12 = 3*$', or $ = 2*12/3 = 8.

When I share a total and get 2$ per 3$, the units are the same, and then the per-number becomes a fraction. So, I may ask: "With 2$ per 3$, what does 15$ give, and what did 8$ come from?" Again I recount in the per-number 2$/3$ or 2/3: 15$ = (15/3)*3$ gives (15/3)*2$ = 10$; and 8$ = (8/2)*2$ came from (8/2)*3$ = 12$. Also, I can use words as 'part' and 'Total' to set up the equation 'part/Total = p/T'. The first equation then gives '2/3 = p/15'. Again, I move to opposite side and get '2*15 = 3*p', and p = 2*15/3 =10. The per-number can also be per-hundred or percent. Here, I can ask "With 20 per 100, what does 300 give, and what did 80 come from?" Again I recount in the per-number: 300 = (300/100)*100 gives (300/100)*20 = 30; and 80 = (80/20)*20 came from (80/20)*100 = 400.

Lesson 12. Recounting the sides in a bundle-number stack

Now I will recount the sides in a bundle-number stack, 2 3s, where 3 is the width or the 'out-number' and 2 is the height or the 'up-number'. So, I may ask: "With 2 up per 3 out, what is the height if the out-number is 15, and what is the width if the up-number is 8?" Again I recount in the per-number: 15out = (15/3)*3out = (15/3)*2up = 10up; and 8up = (8/2)*2up = (8/2)*3out = 12out. With equations I can use the words 'up' and 'out' to set up the equation 'up/out = up/out'. The first equation then gives '2/3 = up/15'. Again, I move to the opposite side and get '2*15 = 3*up', or up = 2*15/3 =10.

The up/out per-number2/3 is also called a tangent-number, which connects the per-number to the diagonal-angle: 'tangent(Angle) = up/out'. My calculator shows that the angle 30 degrees has a per-number 'up/out = 58/100', and that the per-number ½ has the angle 27 degrees.

In a circle with radius 1, the length is called pi. In the beginning the circle follows its tangent, the line that shows its direction if it was not to hold a constant distance to its center. So here the circle and its tangent have almost the same length. And my calculator shows that tangent(180/1000) = 3.1416/1000. This indicates that pi must be close to 3.1416.

LESSON 13. Totals with other forms than rectangles

Now I will look at totals with other forms than rectangles. I saw in lesson 07 how a stack can be squared, and I saw in lesson 10 how 8 1s change form when becoming 2s or 3s or 4s. My parents say that is because they have the same area, a strange name for how much they fill, but I don't mind. First, I see that a diagonal splits a stack in two right triangles with half the area to each. Then I see that any triangle can be split into two right triangles surrounded by a stack. So again, the triangle has half the area of the surrounding stack.

In lesson 12 I saw that half a circle can be split up in n small almost right triangles with the area half of pi/n times the radius. If they are stacked two by two in opposite directions, we get a stack of n*pi/n radii. So, a full circle has the area, 2*pi*radius. Now the fun begins with pearls forming a triangle on a BBBoard.

I place pearl A on the (2,3)-dot in the upper right corner of a 3 2s stack. Pearl B on the (3,8)-dot, and pearl C on the (7,5)-dot. Now I want to find the length and the angles and the area in the triangle ABC.

The AB line is diagonal in a 5 1s stack with a 5-per-1, or 5/1 diagonal angle, or 79 degrees with the tangent button, and 90 – 79 = 11 degrees to the other angle. The length of the diagonal I find as V26 = 5.1 by adding the squares of the stack's sides. And the area I find as half of 5*1, i.e., 2½.

The BC line is diagonal in a 3 4s stack with a 3/4 diagonal angle, or 37 degrees with the tangent button, and 90 – 37 = 53 degrees to the other angle. The length of the diagonal I find as V25 = 5 by adding the squares of the stack's sides. And the area I find as half of 3*4, i.e., 6.

The AC line is diagonal in a 2 5s stack with a 2/5 diagonal angle, or 22 degrees with the tangent button, and 90 – 22 = 68 degrees to the other angle. The length of the diagonal I find as V29 = 5.4 by adding the squares of the stack's sides. And the area I find as half of 2*5, i.e., 5.

To find the area of the ABC triangle, I begin with the full area of the surrounding 5 5s stack, 5*5 = 25. Then I pull away the three outer right triangles and get 25 – 5 – 6 – 2½ = 11½.

To find the angles in the triangle ABC I begin with 90 degrees for A and 180 degrees for B and C. Then I pull away the two neighbor angles in the outer right triangles and get: A = 90 – 22 – 11 = 57 degrees, and B = 180 – 79 – 37 = 64 degrees, and C = 180 – 68 – 53 = 59 degrees. Then, I tested the results by adding the three angles: 57 + 64 + 59 = 180.

Finally, I repeat this experiment on a squared paper to allow testing the answers by measurement.

LESSON 14. A BBBoard as a time-board

Now I will look at my BBBoard, not as a space-board with fixed forms, but as a time-board where I change place with different trips. I will travel on a line to see when it meets other lines or circles or parabolas. On a space-board, the first dot is number 1, but on a time-board it is number 0 since I have not yet changed place.

Now, a 2 3s stack will be called a 3x2 box, where 3 is the run- or base-number, and 2 is the rise- or height-number. In a 3x2 box, the two inside diagonals slope up 2 per 3, 2/3, or down -2 per 3, -2/3. Its angle can be found on a calculator to 33.7 degrees up or down.

A 2/4 trip from the (0,0)-dot to ends at the (4,2)-dot where 4 is the out- or x-number and 2 is the up- or y-number. Now the trip continues with the same direction to an unknown (x,y)-dot. Since the angle hasn't changed we have that y/x = 2/4, or y = 2/4*x = ½*x. This formula, which is called the line's equation, can find one number if the other is known.

If the x-number is 9, then the y-number is y = 1/2*x = 1/2*9 = 4.5. And if the y-number is 4, then the x-number is found in the equation 4 = 1/2*x giving x = 4*2 = 8.

Another line connects the (0,6)-dot to the (6,0)-dot. Inside this 6x6 box the diagonal slopes -6/6 or -1/1. So after x steps y have decreased to y = 6-x.

The two lines then meet where y = 1/2*x = 6-x. Here, we change the unit by recounting x in 2s as x = (x/2)*2 = 2*u with u = x/2. Now, 1/2*2*u = 6 – 2*u, or u = 6 – 2*u, or u + 2*u = 6, or 3*u = 6, or u = 6/3 = 2. Which gives x/2 = 2, or x = 2*2 = 4. Here, y = 6 – 4 = 2, so the two lines meet in point (4,2).

The circle with radius 10 and center in the (0,0)-dot consists of (x,y)-points, where x^2 + y^2 = 10^2. On its way the y = 1/2*x line meets the circle in the (x,y)-dot that is placed both on the line and on the circle. So, y = ½x makes x^2 + y^2 = x^2 + (½x)^2 = 100, or x^2 + 1/4x^2 = 100.

Again, we change the unit by recounting x in 2s as x = (x/2)*2 = 2*u with u = x/2.

Then (u*2)^2 + ¼*(u*2)^2 = 100, or 4*u^2 + u^2 = 100, or 5*u^2 = 100, or u^2 = 100/5 = 20, or u = V20, or u = 4.5.

Which gives x/2 = 4.5, or x = 2*4.5 = 9. Here y = 2/4*9 = 4.5. So, they meet in point (9,4.5).

A trip where y = (x-3)^2 = x^2 – 6*x + 9 is a bent line called a parabola. It meets the y = 1/2*x line in point (x,y) that is placed both on the line and on the parabola.

So, y = ½x makes ½*x = x^2 – 6*x + 9, or x^2 – 6.5*x + 9 = 0.

On the board we see they meet in point (2,1). To find the other meeting point we write x^2 – 6.5*x + 9 as (x – 2) * (x – u) = x^2 – u*x – 2*x + 2*u = x^2 – (u+2)*x + 2*u.

Here we see that u = 4.5, so x^2 – 6.5*x + 9 = (x – 2)*(x – 4.5) = 0 gives x = 2 and x = 4.5, where y = ½*2 = 1 and y = ½*4.5 = 2.25.

So, the line and the parabola meet in points (2,1) and (4.5, 2.25).

LESSON 15. A time BBBoard used for storytelling, called modeling

Now I will use my time BBBoard for storytelling, called modeling. First, I see that the points on my parabola are connected by trips where the up-number rise with 2 each time: (+1, -5), (+1, -3), (+1, -1), (+1, +1), (+1, +3), and (+1, +5).

• I use this to model throwing a ball. A constant up-number will give a straight line. But here gravity makes the up-number fall so the line bends as a parabola.

From (0,0) the ball takes a '1 out, 5 up' (+1, +5) trip followed by a (+1, +3) trip, etc.. So, it passes through points (1,5), (2,8), (3, 9), (4,8), (5, 5), and (6,0).

Since y = 0 for x = 0 and for x = 6, I guess that the formula is y = a*x*(6-x). In the point (1,5) this gives 5 = a*1*(6-1), or 5 = a*5, solved by a = 1. So, my guess is y = x*(6-x), or y = -x^2 + 6*x. This holds in the other points, e.g. (3,9): y = 3*(6-3) = 3*3 = 9

The height 8 is reached after x steps found by the equation 8 = -x^2 + 6x, or x^2 – 6*x + 8 = 0, or x^2 – 2*3*x + 8 = 0, or (x-3)^2 – 9 + 8 = 0, or (x – 3)^2 = 1 = 1^2, solved by x = 3+1 = 4, and x = 3-1 = 2.

The height 10 is never reached since there are no solutions to the equation: 10 = -x^2+6*x, or x^2 – 6*x + 10 = 0, or (x – 3)^2 – 9 + 10 = 0, or (x – 3)^2 = -1.

The top-point is found in the middle between 0 and 6, i.e., in x = 3 that gives

y = 3*(6 – 3) = 3*3 = 9.

• I now model an infection in a group with 6 children that first speeds up and then down when all are infected. From (0,0) the trips will be (+1, +1), (+1, +2), (+1, +2), and (+1, +1). But, rolling dice tells me that this is not realistic, since the chance for infecting the next child decreases from 6 per 6 to 1 per 6 for the last child if the virus dies after 6 rounds. So typically, only 4 per 6 will be infected as I can see when rolling dice gives, e.g., (4, 3, 2, 3, 5, 5) and (4, 2, 6, 2, 2, 4). So, always let a virus die out.

In a group with very many children, infection will change by doubling: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. We already know these numbers from counting in 2s as 1, B, BB, BBB. Also, we know that about ten doublings gives around 1000 since log2(1000) = 9.97.

• I now model an empty island where cats eat mice. I assume that 5 cats and 5 mice are in balance. And that, with 8 cats and 1 mouse in the point (8,1), the cats change to 8 + (1-5) = 4, making the mice change to 1 + (5-4) = 2. Now, with 4 cats and 2 mice in the point (4,2), the cats change to 4 + (2-5) = 1, making the mice change to 2 + (5-1) = 6, etc. On a space BBBoard I get a cycle going round and round where the cats will change from 8 to 4, 1, 2, 6, 9, 8; and where the mice will change from 1 to 2, 6, 9, 8, 4, 1. New start numbers will give new cycles.

LESSON 16. After counting it is time to subtract and add

After counting and recounting, it is now time to subtract and add. I count in 2s by pushing-away 2s, i.e., by pulling-away or subtracting 2 many times. So, "How to subtract?" I ask my now 4-year-old baby brother who taught me that four fingers held together two by two is not four, but 2 2s. He hates adding because he runs out of fingers, which he never does when subtracting.

● Let us begin with our five fingers here, he said. First, I pull-away one finger so I have 5 – 1 finger. Then I push it back again, so I have 5 = (5 – 1) + 1. That also works with 2, see. Here is 5, I pull-away a 2-bundle, and now I push it back again, so I have 5 = (5 – 2) + 2. And it works with all bundles, 5 = (5 – B)+B. Also, it works with all totals, T = (T – B) + B, which I call a 'split formula'. You get it? Great, now we will find unknown numbers.

● I have 5, but I had 3, so which Bundle did I add? We write '3 + B = 5'. Now look at my fingers. Her I have 5, and now I have 3 when I pull-away 2, which I also get by pulling-away 3 from 5. So, B = 5 – 3. Instead, I can split 5 as 5 = (5 – 3) + 3, which with 5 = B + 3 also shows that B = 5 – 3. So, we just move 3 to the opposite side with the opposite sign. You get it? Great, now we will also work with the big second-hand numbers.

● I want to find 7 – 4. I see it is 2 + 1 = 3. But 7 is also 5 +2, or half bundle-ten +2 if I close my fist, 7 = ½B 2. Likewise, 4 is a half bundle less 1, 4 = ½B -1. So, if I pull-away a full ½B, I must push back the one that should not have been pulled away, so 7 – 4 = ½B 2 – ½B -1 = 2 - -1 = 2 + 1 = 3. So, you see that minus minus is plus. You get it? Great, now we will use the BBBoard to subtract numbers.

● I want to find 8 – 6. We place 8 on the first row and 6 on-top on the next row. Then we see, that 8 – 6 = 2, and that 6 – 8 = -2. Also, we see that 8 – 6 = 1B 2 6s = 1B -2 8s.

● Now we subtract 2digit numbers: 74 – 46 = 7B 4 – 4B 6 = 3B -2 = 2B 8 using a bundle to get rid of the underload.

● Finally, we look at addition: 8+6. As second-hand numbers we can write them with half bundles: 8 + 6 = ½B 3 + ½B 1 = 1B 4 = 14, or 8 + 6 = 2B 2 6s = 2B -2 8s.

● Now we add 2digit numbers: 28 + 46 = 2B 8 + 4B 6 = 6B 14 = 6B 1B 4 = 7B 4. Here we call 14 an overload. All numbers can be written with an overload or an underload, 74 = 7B 4 = 6B 14 = 8B -6.

● We can also multiply 2digit numbers, 3 * 46 = 3 * 4B 6 = 12B 18 = 13B 8 = 138. Sorry for using my calculator, but I have not yet learned multiplication by heart. And I don't need it because I can see on the BBBoard that 6 * 7 = 6 * ½B 2 = 3B 12 = 4B 2 = 42. Or, I can use underload and write 6 * 7 = 6 * 1B -3 = 6B -18 = 4B 2 = 42. Or, 6*7 = (B – 4) * (B – 3) = = 10B – 3B – 4B + 4*3 = 3B12 = 4B2. Again, minus minus is plus.

LESSON 17. Adding bundle-numbers with units as 2 3s and 4 5s both next-to and on-top

The next day my 4-year-old baby brother woke me up early, bringing some snap-cubes.

Today you will learn how to add bundle-numbers with units as 2 3s and 4 5s both next-to and on-top, he said. Come, let us build them with snap cubes.

First, we add them next-to as 8s. It gives 2B 10 8s, but that is an overload, so we move 3s from the 5-stack to the 3-stack twice. Now we have the result, 3B 2 8s.

This we found by adding two areas, which our uncle calls integral calculus.

Now we predict the result on a calculator. Here, we enter the 4 5s plus 2 3s as '4x5 + 2x3' which gives 26 or 2.6 tens. To find out how many 8s that is we recount 26 in 8s as '26/8' which gives '3.more' 8s. To find the unbundled we pull-away the 2 8s from 26 as '26 – 2x8' which gives 2. So, the calculator predicts the result to be 3B 2 8s as it also is.

Next, we add 2 3s and 4 5s on-top as 5s. It gives 4B 6 5s but that is also an overload, so we move 2 from the 3-stack to the 5-stack. Now we have the result, 5B 1 5s.

We can also predict this on a calculator. Here, we enter the 4 5s plus 2 3s as '4x5 + 2x3' which gives 26 or 2.6 tens. To find out how many 5s that is we recount 26 in 5s as '26/5' which gives '5.more' 5s. To find the unbundled we pull-away the 5 5s from 26 as '26 – 5x5' which gives 1. So, the calculator predicts the result to be 5B 1 5s as it also is.

Likewise, if we add 2 3s and 4 5s on-top as 3s the result will be 8B2 3s.

Now we turn the question around and ask '2 3s and how many 4s give 4 7s?'.

So, first we build the 2 3s with red cubes. Then we finish the 4 7s with blue cubes. To find out how many 4s we have used we must first we pull-away the 2 3s before we can recount the rest in 4s. This our uncle calls differential calculus where subtraction comes before division. He says it is natural as reversed integration where multiplication comes before addition to creating the areas.

Again, we can predict the result on a calculator.

First, we enter the 4 7s with 2 3s pulled-away as '4x7 – 2x3' which gives 22 or 2.2 tens. To find out how many 4s that is we recount 22 in 4s as '22/4' which gives '5.more' 4s. To find the unbundled we pull away the 5 4s from 22 as '22 – 5x4' which gives 2. So, the calculator predicts the result to be 5B 2 4s as it also is.

You are tired?

OK, then we will wait until tomorrow to add per-numbers and fractions and squares.

LESSON 18. Learning how to add per-numbers

The next day my 4-year-old baby brother again woke me up early, bringing the same snap-cubes as yesterday.

● Today you will learn how to add per-numbers, he said. Come, we will use the same cubes as yesterday, the 2 3s and the 4 5s, but today they will mean something new. The 2 3s today means 3 days with 2 licorice sticks per day, and the 4 5s means 5 days with 4 licorice sticks per day. Now we would like to find the totals. Here 3 days and 5 days total 8 days, but 2 sticks/day and 4 sticks/day do not add directly to 6 sticks/day. First, we must multiply: 3 days with 2 sticks/day = 3*2 sticks= 6 sticks, and 5 days with 4sticks/day = 5*4sticks= 20sticks. In total I then get 6 sticks + 20 sticks = 26 sticks, but per 8 days, i.e., 26/8 sticks/days, or a little more than 3 sticks per day on average. But, the moment you multiply, you get areas. So, the per-numbers add by their areas, i.e., again by integral calculus.

● Now you will learn how to add fractions, which are per-numbers with like units, e.g., 2 days / 3 days = 2/3. So, now let 2/3 mean 3 days with 1/2 licorice stick per day, and let 4 5s mean 5 days with 1/4 licorice stick per day. Again, we would like to find the totals. And again, 3 days and 5 days total 8 days, but 1/2 stick/day and 1/4 stick/day do not add directly to 1/6 stick/day, or to 3/4 stick/day as our uncle says. Again, first we must multiply, 3 days with 1/2 stick/day = 3*1/2 sticks = 3/2 sticks, and 5 days with ¼ stick/day = 5*1/4 sticks= 5/4 sticks. In total I then get 3/2 sticks + 5/4 sticks per 8days. So, in 4*8 days I will get 4*3/2 + 4*5/4 sticks = 6 + 5 = 11 sticks, which is 11/32 sticks/day, or close to 1/3 stick each day on average. Instead, he gave me 3/4 stick each day. After a month I felt sorry for him: Uncle, you say that ½ + ¼ is ¾. But 1 of 2 sticks plus 1 of 4 sticks total 2 of 6 sticks, right? So, with a dozen sticks you would give me 3*3 = 9 sticks when I only should have 2*2 = 4 sticks. You spoil me, uncle.

● Now you will learn to reverse addition of per-numbers. We now ask, "2 days with 3 sticks/day + 4 days with how many sticks/day will total 6 days with 5 sticks/day?"

Again, we begin by multiplying: "2 days with 2*3 sticks + 4 days with how many stick/day will total 6 days with 6*5 sticks?"

Here, we first pull away the 2*3 sticks and recount the rest in 4s: (6 x 5 – 2 x 3) / 4 4s = 24 / 4 4s = 6 4s.

The answer is: "2 days with 3 sticks/day + 4 days with 6 sticks/day will total 6 days with 5 sticks/day?"

● So, reversed per-number addition means subtracting areas and then reshaping the rest. Our uncle calls this differential calculus. Here subtraction comes before division, which of course is the reverse of multiplication before addition as in integral calculus.

LESSON 19. Adding fractions of different totals and of the same total

The next day my 4-year-old baby brother again woke me up early, now bringing two chocolate bars each with 6 parts.

● Today you will learn how to add fractions of different totals and of the same total. We will split this chocolate bar so that I get 1 per 2, ½, and you get 2 per 3, 2/3.

With two bars that each has 6 parts that is no problem, I get ½ of 6, and you also get 2/3 of 6.

To find 1 per 2 of 6 I recount 6 in 2s as 6 = (6/2)*2 = 3 2s, so 3 times I get 1, that is 3 of 6.

To find 2 per 3 of 6 you recount 6 in 3s as 6 = (6/3)*3 = 2 3s, so 2 times you get 2, that is 4 of 6.

So together we get 3+4 = 7 of the 6+6 = 12 pieces when we have 2 bars, and 5 are left for tomorrow.

If we only have one bar, I can still get my 3 parts, but you can only get 3 of your 4 parts, sorry for you.

Of course you can still get 2/3 of what is left, i.e., 2/3 of 1/2 of 6, or 2/3 of 3, i.e., 2 of 3 if that makes you happier.

The important thing is that fractions need totals, always. Our uncle calls this the Bayes rule.

● Now, look at your fingers.

On your right hand you have 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs.

Now you bend the 2 outer fingers. So, 1/3 of the Ls are Bent, and ½ of the Rs are Bent.

Our uncle says that 1/3 + 1/2 = 5/6 so that 5 of 6 fingers should be Bent, but we can both see that only 2 of 5 fingers are Bent.

He thinks the total is the same and chooses this to be 2*3 = 6:

1/3 + 1/2 = 1/3*T + 1/2*T = 1/3*6 + ½*6 = 2 + 3 = 5 of 6 are Bent.

But we know that there are 2 different totals, 3 Ls and 2 Rs:

1/3 + 1/2 = 1/3*T + 1/2*T = 1/3*3 + 1/2*2 = 1 + 1 = 2 of 5 are Bent.

● We can show this on a cross table with the Ls as 1 Bent and 2 Unbent, and the R's as 1 Bent and 1 Unbent. From this total table we can make two per-number tables.

A horizontal table showing that among the Ls, 1/3 is Bent and 2/3 is Unbent. And, that among the Rs, 1/2 is Bent and 1/2 is Unbent.

A vertical table showing that among the Bent, 1/2 is from L, and 1/2 is from R. And, that among the Unbent, 2/3 is from L, and 1/3 is from R.

So, we can't say: "1/3 of the Ls is Bent, so 1/3 of the Bent are Ls." since ½ of the Bent are Ls.

Therefore, to go between fractions in a cross-table, we must pass the Totals first: NEVER EXCHANGE CATEGORIES.

● Our Uncle writes this as *p(A I B) * p(B) = p(B I A) * p(A)*

What he really means is: The number of fingers that are both Bent and part of L can be calculated in two different ways both going back to the totals:

Number of both Bent and L = fraction (Bent among Ls) * number of Ls = fraction (Ls among Bent) * number of Bent, or, 1 = 1/3 * 3 = 1/2 * 2.

Tired? OK, no more adding fractions in space. Tomorrow we will add fractions in time.

LESSON 20. Adding percentages

The next day my brother woke me up with a question:

"10 days at 10$ per day gives 100$, but 10 days at 10% per day gives 159%, says our uncle. Where does the extra 59% come from?"

● OK, let us add fractions in time to create growth.

100% + 10% = 110%

100$ + 10% = 100$ + 10% of 100$ = 100$ + 10$ = 110$

Next, to add 10% or 10 per 100 to 110$, we recount 110 in 100s:

110$ = (110/100)*100$ = 110% * 100$ = 1.10 * 100$

So, adding 10% means multiplying with 110% or 1.10

If we 2 times add 10%, we end with

1.10 * (1.10 * 100$) = (1.10^2) * 100$ = 121$ = 100$ + 2*10$ as expected + 1$ extra

We call 10% the single interest, r, and the 21% the total interest or compound interest, R.

If we 4 times add 10%, we end with

1.10^4 * 100$ = 146.4$ = 40$ as expected plus 6.4$ extra.

If we 10 times add 10%, we end with

1.10^10 * 100$ = 259$ = 100$ + 10*10$ as expected plus 59$ extra.

So, if we n times add the interest r, we get 1+ R = (1+r)^n = n*r + RR

where n*r is the expected interest and RR is the extra interest.

● The 100$ is doubled when R = 100%. Here

1+R = (1+r)^n gives 1+1 = (1+0.10)^n, or 1.10^n = 2,

Now, we recount 2 in 1.10s by using the log-bottom, log1.10(2) = 7.3. We test, 1.10^7.3 = 2.0.

So, adding 10% per year means adding 100% per 7.3 years, which is around 2,666 days.

● If we split the 100% interest in 2,666 parts to compounded daily we get

1+R = (1 + 1/2,666)^2,666 = 2.718 = 271.8% = 100% expected + 171.8% extra.

Our uncle says that this comes from an Euler e-number defined as

e = (1+1/n)^n for n large enough.

● What does he mean with a '8 4 3 rule' of compounding?

The first 8 years the money grows steadily, the next 4 years it accelerates, and the next 3 years it explodes.

Again, we use the formula 1+R = (1+r)^n, or R = (1+r)^n – 1.

R = 1.10^8 – 1   = 1.14 = 114% = 80% simple + 34% extra.

Here 34 = (34/80)*80 ≈ 43% of 80

R = 1.10^12 – 1 = 2.14 = 214% = 120% simple + 94% extra

Here 94 = (94/120)*120 ≈ 78% of 120

R = 1.10^15 – 1 = 3.18 = 318% = 150% simple + 168% extra.

Here 168 = (168/150)*150 ≈ 112% of 150

● Now, we subtract fractions in time to create decay.

100% – 10% = 90%.

100$ – 10% = 100$ – 10% of 100$ = 100$ – 10$ = 90$.

To subtract 10 per 100 from 90$, we must recount 90 in 100s:

90$ = (90/100)*100$ = 90% * 100$ = 0.9 * 100$

Subtracting 10% means multiplying with 90% or 0.90

If we 8 and 12 and 15 times subtract 10% we get

R = 0.9^8 – 1 = - 0.57 = -57% = -80% simple + 23% extra.

Here 23 = (34/57)*80 ≈ 43% of 80

R = 0.9^12 – 1 = -0.72 = -72% = -120% simple + 48% extra.

Here 94 = (94/120)*120 ≈ 78% of 120

R = 0.9^15 – 1 = -0.79 = -79% = -150% simple + 71% extra.

Here 34 = (268/150)*150 ≈ 179% of 150

So, the '8 4 3 rule' now is helping.

Growth by adding and multiplying is called linear and exponential.

LESSON 21. How to buy something you cannot afford now?

The next day my brother woke me up with a question: "How can our uncle buy a car if he has no money for it?"

Well, there are two ways to buy something you cannot afford now. You can create a saving to buy it in the future, or you can buy it now if you can borrow the money that then becomes a debt to pay off in the future.

In both cases you combine growth by adding with growth by multiplying.

Yesterday we saw that after 10 months with 10$ and 10%, 100 $ will grow to 200 $ at home or to 259% in a bank. But if we don't have the 100 $ we can borrow them and pay the debt back with10$ per month if the interest rate is 0%. But it is 10% per month, so we must pay more to settle the 259$ debt, but how much?

Now we will see what happens if we bring the 10$ to the bank instead to create a saving account that at the end of each month receives both 10% and 10$. We already know that adding 10% means multiplying with 110% or 1.10.

After month 1we have 1.10*0$ + 10$ = 10$

After month 2 we have 1.10*10$ +10$ = 21$

After month 3 we have 1.10*(1.10*10$ + 10$) + 10$ = 1.10^2 *10$ + 1.10*10$ + 10$ = 33.1$

And in this way, we continue to month 10.

Stop, said my brother, the 10$ only reduces the loan from 100$ to 0$. We must also pay the interest, which goes down when the loan goes down.

But if we say that the loan is half in the full period, we must also pay 10% of 50$ or 5$ extra per week.

But, how can we see if paying 15$ per week will finish the debt?

Do we really have to do all your ten calculations?

The textbook says so because it wants us to learn something called a geometric series. But instead, we now look at what really is happening when we save money.'

In the bank we set up two accounts, Left and Right.

Left receives one deposit, a/r, and its monthly interest, r*(a/r) = a.

This is at once transferred to Right that also receives its monthly interest.

So, Right is a saving, A, receiving both percentages and dollars.

In the end, Right also contains the total interest, R*(a/r).

So, A = R*a/r, or A/a = R/r, or A = a* R/r, or a = A * r/R,

Here 1+R = (1+r)^n, so with r = 10%, and n = 10,

1 + R = (1 + 10%)^10 = 2.594, and R = 2.594 – 1 = 1.594 = 159.4%.

With a = 15$ this gives the saving A = 15$ * (159.4%/10%) = 239$, which is close to the goal, 259$.

So, instead we ask A = a * (159.4%/10%) = 259$, or a * 15.94 = 259$, or a = 259$/15.94, or a = 16.2$

The original 100$ debt is settled with 259$, where we have paid 10*16.2$ = 162$ and the bank has paid the extra 97$.

We find this quicker by setting up a formula for when the saving and the debt are the same

A = a*R/r = D*(1+R), so a = D*(1+R)*r/R

So, here there are 3 ways to grow, by adding, by multiplying or combined as a saving. You will later learn the names linear and exponential and annuity growth for these.

LESSON 22. Pythagoras

In the middle of the night my little brother woke me up to see 4 paper sheets arranged after each other as a square "Can you see three invisible squares here?"

I could only see the two visible squares, an outer and an inner.

"Look at the invisible diagonals, they form an invisible square, right? What is outside that square?"

Four half sheets. "Right, which are the two sheets. So, if you ignore the sheets with letters on, can you then see two more invisible squares?"

I could not so my thoughts went back in time. My little brother is obsessed with squares since he discovered that 2 2s is a square as well as all other BundleBundles. Last week he built a 5x5 square with snap-cubes and said:

"Look how Bundle-Bundles are squares that only need two bundles and a corner to become the next Bundle-Bundle. See, 1BB2B1 is the next square. So, it is easy to find the square numbers

1, and 1 + 2*1 + 1 = 2 2s = 4, and 4 + 2*1 + 1 = 9, and 9 + 2*3 + 1 = 16, and

16 + 2*4 + 1 = 25, etc."

Our Uncle is very impressed with him "By seeing that a square needs two sides to become the next square he is close to prove that the derivative of x^2 is 2*x."

"Come on now, the visible square is part of a big square formed by the long side of the paper, and that leaves a small square on-top of the big square by the short side of the paper. So, squares can be added as squares. If you form a paper from the squares, their sum is the square of the diagonal. Right?"

Yes, it is called the Pythagoras rule.

"I see, but adding squares as squares shows how a square grows when eating new squares. You just make a square using the 'bottom-top' line to draw a circle."

Yes, you are right. Can we look at it tomorrow?

"No, look, how a tile has one diagonal, but one diagonal has many tiles as you can see when putting a circle around it. It shows how you can split a tour into many detours. Right?"

Yes, you are right again, but tomorrow, please?

(Our uncle will again be impressed with his splitting velocities and forces in different parts as they do in STEM.)

LESSON 23. The Re-Unite formula solves equations

Again, in the middle of the night, my little brother woke me up and showed me his right hand.

"How many fingers do I have here?" I saw all 5 fingers, so I said 5.

"No, try again" OK, 4 fingers and 1, or 3 fingers and 2?

"No, I have, ALL". OK.

He bent his thumb down.

"How many fingers do I have now?" I saw 4 fingers, so I said 4.

"No, I have ALL minus 1". OK?"

He bent his thumb up.

"How many fingers do I have now?" Again, I saw all 5 fingers so again I said 5.

"No, I have ALL, minus 1, plus 1".

He bent the thumb down and up several times.

"See: ALL, is ALL minus one, plus one. ALL , is ALL minus one, plus one." OK, and then what?

"We can write it down, see: A = (A – 1) + 1." OK

"And it also works with 2 fingers, see, and with 3, see"

He bent first 2 then 3 fingers down and up.

"See, it works for all bundles, so we can write: A = (A – B) + B". OK, and then what?

"Now I can solve equations as '5 = x + 2'.

Because, also 5 = (5 – 2) + 2, so x = 5 – 2, so I just move plus 2 across as minus 2. Look, I can show it with my fingers."

He bent two fingers up and down several times.

"And here I solve the equation '4 = x – 2'.

I bent 2 and ended with 4, what did I start with? I just bent 2 fingers up, and see that x is 4+2.

So, again I just move across and change the sign." Great, good night.

"No wait. Look again. Here are ALL and now I bend 2 and get A-2. But this time I bent them down, so here I now have 2 + (A-2).

So, to get ALL down I just added the difference, A-2. So, my number of down-fingers grow by adding the change that is the end-number minus the start-number, change = end – start = A – 2. So now I have a change equations A = 2+ (A-2).

Now, if I know my start number and my change numbers, then I can find the end number by adding all the change differences:

Here, the start number is 3, and the change differences 1, 2, 3, 2, 1 that add up to 9. So, the end number is 3 + 9 = 12.

By adding the change numbers, we can find all the middle numbers

3, and 3+1 = 4, and 4+2 = 6, and 6+3 = 9, and 9+2 = 11, and 11 + 1 = 12.

Or, if we know the middle numbers then we can find the change numbers as the differences

12 – 11 = 1, and 11 – 9 = 2, and 9 – 6 = 3, and 6 – 4 = 2, and 4 – 3 = 1,

Now we add all these differences and get 12 – 11 + 11 – 9 + 9 – 7 + 6 – 4 + 4 – 3.

Here we see that when adding many differences, all the middle terms disappear, so we only have one difference between the end- and the start-number: That is smart, right?" Right, but tomorrow, please.

(Our uncle will again be impressed with his adding differences since basically my baby brother has discovered the core of calculus, where what we add is rewritten as differences that add as only one difference. But he will not like the move across method to solve equations.)

LESSON 24. Helping with coin flipping

Today, my little brother didn't wake me, but he came in the afternoon with a coin.

"Please help me with coin flipping, I lose too many peanuts."

Well, tell me all about it.

"In the beginning it went well. We only flipped one coin with an A and B side. And with only two outcomes the winning chance is 1 of 2 so I would win 1 per 2 times. Not each other time, but in the long run I was able to keep my peanuts since I got 2 in per 1 that I gave out."

OK, so what is the problem?

"The problems came when we flipped two coins. Here there are three outcomes with A coming up 0, 1 or 2 times. I was not allowed to bet on the middle outcome, 1. But still with a winning chance at 1 of 3, I would get 3 in per 1 out. Only this time, after 20 games, I lost 5 peanuts. What is wrong?"

You should ask for 4 peanuts instead of 3.

"Why 4 when there are only three outcomes? A comes 2 times, once, or not?"

Well, flipping two coins at the same time is the same as flipping one coin twice. And here you can see there are four outcomes, AA, AB, BA, and BB. So, if you bet on AA the winning chance is 1 of 4. Here you should win 1 per 4 times and get 4 in per 1 out. You only got 3 in, so you risk losing 1 per 4 games, or 5 per 20 games.

"I see. And if I could bet on 1 where the winning chance is 2 of 4 then I should get 4 in per 2 out, or 2 in per 1 out, to keep my peanuts."

Precisely. "But, in an outcome tree the still are four outcomes?"

Yes, but not if we unite AB and BA as 2A.

"Ok, so instead of 4 branches I get 2 streets and 2 avenues ending in 0A, or 1A, or 2A?"

Precisely.

"Fine. And, if we flip 3 coins, we will not get 8 branches but 3 streets and 3 avenues ending in 0A, or 1A, or 2A, or 3A? And with 4 coins we 4 streets and 4 avenues ending in 0-5A?"

Yes. "But then, what is my winning chance if I we flip 3 coins?"

Well, at the road map we find 1 tour to 0A, and 3 tours to 1A, and 3 tours to 2A, and 1 tour to 3A, which gives 1+3+3+1 or 8 different tours. So, if you bet on 0A then your winning chance is 1 of 8 and you should get 8 in per 1 out. And if you bet on 1A then your winning chance is 3 of 8 and you should get 8 in per 3 out. A calculator says that 3/8 is 0.375 between ½ and 1/3. So to get a profit they will only give you 2 in per 1 out.

"Hey, the numbers 1 & 1, and 1 & 2 & 1, and 1 & 3 & 3 & 1, reminds my of the Pascall numbers from when we added interest, so the next numbers perhaps will be 1 & 4 & 6 & 4 & 1 that total 16 so that the winning chance for the outcomes 0A, 1A, 2A, 3A, and 4A will be 1/16, 4/16, 6/16, 4/16, and 1/16?"

Yes. Have you noticed the totals: 2, 4, 8, 16

"Yes, they are the Bundles, BBs, BBBs and BBBBs when we bundle-count in 2s."

Precisely, and you can also use C-numbers where C(4,1) tells how many 4road trips have 1 left turn.

"Great, see you later."

LESSON 25. Understanding Times Tables

Today, my little brother brought some friends that would like to understand Times Tables.

So we asked: How to understand that 6*7 = 4ten2?

Answer: Use a BundleBundleBoard, a BBBoard.

Here 6*7 is de-modeled (brought from essence into existence) as 6 7s that we can both see and feel on the BBBoard.

● Method 01. We let the fingers wander up the board touching the ½Bs while saying

"½B2 or 7, 1B4 or 14, 1B ½B 6 or 1B 11 or 2B1 or 2ten1, 2B8 or 2ten8, 2B ½B 1B or 3B ½B or 3B5 or 3ten 5, 3B 1B 2 or 4B2 or 4ten2."

● Method 02. We see that 7 = ½B2, so 6*7 = 6 * ½B2 = 3B 12 = 4B2 = 4ten2

● Method 03.

6*7 = (½B + 1) * (½B + 2)

= 2B ½B (in the lower left corner) + 1*½B (over) + 2*½B (next-to) + 2 (in the right corner)

= 3B + 1B + 2 = 4B2 = 4ten2.

So, while pointing or feeling we say "1, 2, 3, 4 Bundles and 2, so 4B2, or 4ten2.

With fingers, we show 6 as 1 finger up to the left and 7 as 2 fingers up to the right (thus representing the ½Bs). Then we add and multiply the up-fibers 1 and 2 into 3 and 2 while saying "5, +3 is 8 ½bundles 2, or 4B2, or 4ten2".

● Method 04: (The 'less-method' cutting away unneeded, or the Algebra method showing that minus times minus must be plus).

6*7 = (B – 4) * (B – 3)

= 10B (all) – 3B(next-to)) – 4B(on-top) + 4 3s (added since pulled away twice)

= (10 – 3 – 4)B 12 = 3B12 = 4B2 = 4ten 2.

But (10 – 3 – 4) = (5 – 3) + (5 – 4) = 2 fingers + 1 finger

With fingers, we show 6 as 1 finger up and 4 down to the left, and 7 as 2 fingers up and 3 down to the right. The up fingers we add as before and the down fingers we multiply. This gives 1+2 = 3 and 3*4 = 12. So, the result is 3B12 or 4B2 or 4ten2.

● Method 05. (Connected vessels). Askling "6*7 is what?" means recounting 6 7s in tens. On a BBBoard we see two vessels next to each other, 6 7s to the left and 0 3s to the right. We then see that moving the top 7s to the right is not enough since then we have 5 7s to the left and 2B1 3 to the right. Moving one more 7s to the right will do the job since now we have 4 7s to the left and 4B2 3s to the right. So, the answer is 4B2 tens or 4ten2.

LESSON 26. How to add letters

Today, my little brother brought some friends that would like to know how to add letters.

"What is $a + b$, and what is $a + a*b$?"

OK. Let us begin by adding 2 3s and 4 5s. They can't add directly with unlike units, so first recounting must produce like units, 3s or 5s or tens. So, in '$a + b$', they both must be recounted in the same unit.

"But they don't have a unit?"

Well, all numbers have 1s as a unit, 2 = 2 1s = 2*1, 3 = 3 1s = 3*1, and a = a 1s = a*1.

"So, $a*b$ is really a b's with b as the unit?"

Yes, b is the unit, or a is the unit if we commute the letters and write $a*b = b*a = b$ a's. This is called the commutative law for multiplication. We see it if turning 2 3s over to 3 2s. It also applies for addition where 2+3 = 3+2, but not for subtraction or division where 4 – 2 and 2 – 4 , and 4/2 and 2/4 are not the same.

"OK, so $a + a*b$ is $a*1 + a*b$, and then what?"

Good, we now can see the units, and by the commutative law we can write the total T as

T = $a + a*b = a*1 + a*b = 1*a + b*a$ = 1 a's + b a's = (1+b) a's = (1+ b) * a.

So, we may take a common unit or factor outside a bracket. Or we can do the opposite and write

T = $a * (1+ b) = a*1 + a*b = a + a*b$

Here we take a common outside unit inside a bracket and distribute it to all numbers inside. This is called the distributive law.

"OK, what about $a*c$ + a^2*b?"

Again, we write out all factors and choose the common factors as the unit to move outside the bracket.

T = $a*c + a^{2*}b = a*c + a*a*b = c*a + a*b*a = (c + a*b) * a = (c + a*b)$ a's.

"Then, how about 2*a + 6*c?"

On a BBBoard we see that 2 already is a pair and that 6 can be split in 3 pairs, 6 = 3*2 = 3 2s.

So where 2 is an 'unfolded' prime unit, 6 is a 'folded' unit hiding a prime unit, 6 = 3 2s. So again, we write down all the factors to find the common unit to take outside the bracket:

T = $2*a + 6*c = 2*a + 3*2*c = a*2 + 3*c*2 = (a + 3*c) * 2 = (a + 3*c)$ 2s.

If we turn it around and distribute 2 on both inside numbers, we get

T = $2 * (a + 3*c) = 2*a + 2*(3*c) = 2*a + (2*3)*c = 2*a + 6*c$.

With 2 * (3 * c) = (2 * 3) * c = 6 * c, we meet the associative law when moving the bracket from associating 3 and c to associating 2 and 3.

"Tell us about 2 + 3 * 4, is that 5 * 4 = 20 or 2 + 12 = 14?"

Again, we must first show all the units

T = 2 + 3 * 4 = 2*1 + 3*4 = 2 1s + 3 4s = 0B2 tens + 1B2 tens = 1B4 tens = 1ten4 = 14

So, with multiplication having priority over addition, we also can say that

T = 2 * 1 + 3 * 4 = (2 * 1) + (3 * 4) = 2 + 12 = 14, or simply T = 2 + 3 * 4 = 2 + (3 * 4) = 2 + 12 = 14

Or, to get a common unit, we can write 4 with a prime unit as 4 = 2*2. Then we have

T = 2 * 1 + 3 * 4 = 1 * 2 + 3 * (2 * 2) = 1 * 2 + (3 * 2) * 2 = 1 * 2 + 6 * 2 = (1 + 6) * 2 = 7 * 2 = 14.

LESSON 27. Parabolas as lines bending down

Today my little brother woke me up early bringing some sheets to show me.

"Yes, now I know how to travel along lines, but, oh, it is so boring. Some go up, some go down. Some are steep, some are not. And they each have an angle the we can find from its per-number and the tangent button: 1per 1 gives 45, and 2/1 gives 63.4, and 2/3 gives 33.7 degrees.

So yesterday I decided to bend the lines. You know how I love the BundleBundle squares where 1BB 2B 1 gives the next square.

So I began with this boring line y = 2*x, then I pulled it downwards, first with ½* x^2, then with 1/3 * x^2, and then with ¼* x^2, so that they all bent, but less and less.

But see what I found out.

The bent lines have touchdown in 4 and 6 and 8.

And they top at 2 and 3 and 4. Can you explain that?"

OK, let us begin with your first bent line, which is called a parabola by the way,

y = 2*x – ½ * x^2

First we find the common unit to place outside a bracket

y = 2*x – ½*x * x = (2 – ½*x) * x

Now you se that y is zero when x = 0 and when

2 – ½*x = 0, or ½*x = 2 or x = 4

So, you take 2, the midpoint between 0 and 4, and find that here

y = 2*2 – ½ * 2^2 = 4 – 2 = 2, OK?

"Fine, but what with the rest?"

OK, are you ready for some letter calculations?

"How can you ask?"

Let us look at the parabola y = b*x – a*x^2 = (b – a*x) * x that is zero for x = 0 and x = b/a, OK?

"Wait, b – a*x = 0, so a*x = b, so x = b/a. Yes, OK, go on."

The you take, ½*b/a, as the midpoint between 0 and b/a, OK?

And then you find the y-number here

y = (b – a*x) * x

y = (b – a*½*b/a) * ½*b/a

Y = (b - ½*b) * ½*b/a

Y = (½*b) * ½ * b/a

Y = 1/4* b^2/a, OK?

"Yes, go on."

You have b = 2, so you have y = ¼ * 2^2/a = ¼ * 4/a = 1/a.

So, with yours, a = ½, and 1/3, and ¼, you get the tops in 2 and 3 and 4.

"Wauv, can the Down and Top point be related?"

Yes TOP = b/4 * b/a = b/4 * DOWN

"OK but to top in 8 instead of 4 what do I do?"

You just change units to c's by multiplying with c

y = (b*x – a*x^2)*c, and with y = 8, b = 2 and a = ¼, and x = 4, you get

8 = (2*4 – ¼*4^2)*c, or 8 = (8 – 1*4)*c, or 8 = 4*c, or 8/4 = c, or c = 2

So, y = (2*x – ¼*x^2)*2, or y = 4*x – ½*x^2, that with x = 4 gives y = 4*4 – ½*4^2 = 16 – 8 = 8.

And, y = 4*x – ½*x^2, is the parabola that comes from pulling down the line y = 4*x with ½*x^2.

"Ok, but a parabola is a bending line so its steepness will decrease?

In the beginning the steepness is 4/1 at x = 0, but what is it in x = 2?"

Well, just around x = 2 the parabola is almost linear, we call it locally linear, so here we can calculate the per-number to be 2/1, which gives 63.4 degrees where the angle was 76.0 degrees at x = 0.

"Fine, but cant we just calculate the steepness?"

Lesson 28, Parabolas as lines bending up

Today my little brother again showed me some sheets, saying:

The go-up line, y = 2*x, I yesterday pulled down, with ½*x^2, with 1/3*x^2, and then with ¼*x^2.

The go-down line, y = 10 - 2*x, I today pulled up, in the same way.

The three bent lines turn where x is 2, and 3, and 4, and where y = 8, and 7, and 6.

Did you know that with ½ as the bend-number, and turning in x = 2 with y = 8, the parabola formula can be written as, y = ½ * (x – 2)^2 + 8, instead of as, y = 10 - 2*x + ½ * x^2? Smart, so,

y = B * (x – T)^2 + H is the parabola that has B as bend-number and turns in x = T with y = H.

This makes solving square equations easy:

B * (x – T)^2 + H = y gives (x – T)^2 = (y – H)/B, and x = T + or - √((y – H)/B), e.g.,

1/3 * (x – 3)^2 + 7 = 10 gives x = 3 + or - √((10 – 7)/(1/3)), so, x = 3 + or - √9, so, x = 0 or x = 6.

Also, I really like to do letter-calculations as you have taught me. Let us use this to find the connection between the turn-number and the bend-number:

So instead of, y = 10 - 2*x + ½ * x^2, we now look at

y = C - B*x + A*x^2, or with u = y – C

u = -B*x + A*x^2 = (-B + A*x) * x

Here, u is zero for x = 0 and x = B/A.

Then we take x as, ½*B/A, the midpoint between 0 and B/A. Here, the u-number is

u = -B*(1/2*B/A) + A*(1/2*B/A)^2

u = -½* B^2/A + ¼* B^2/A = -1/4*B^2/A

So, y – C = -1/4*B^2/A. we change units to 4*A:

4*A*y = -B^2 + 4*A*C = -D, where D = B^2 – 4*A*C

So, the parabola y = C - B*x + A*x^2 turns in x = B/(2*A), and y = -D/(4*A)

With A = ½, and B = 2, and C = 10, we have D = B^2 – 4*A*C = 2^2 – 4*½*10 = 4 – 20 = -16.

So, our parabola turns when x = 2/(2*½) = 2 and y = - -16/(4*½) = 16/2 = 8. Just as observed.

Likewise with the other two parabolas.

But there is also another way to find the turn-point, since here the steepness is zero.

You didn't show me how to calculate the steepness yesterday, so I found out a method myself.

From my BundleBundle formula, 1BB2B1, I see that a BB-square basically grows with 2B to become the next square. So, I guess, a x*x-square also basically grows with 2*x to become the next square. While a stack as 3*x only grows with 3 when x grows with 1.

So, with, y = C - B*x + A*x^2, y's growth then is, 0 – B + 2*A*x = -B + 2*A*x.

This is zero when x = ½*B/A.

Now I will find the bent-number, A, that makes my parabola kiss the floor where y = 0.

Here, y = 10 – 2*x + A*x^2, so y's growth is 0 – 2 + 2*A*x, which is zero at x = 1/A.

Here, y = 10 – 2 * 1/A + A * (1/A)^2 = 10 – 2/A + 1/A = 10 – 1/A = 0 for A = 1/10.

So, the parabola, y = 10 – 2*x + 1/10*x^2, kisses the floor where x = 1/(1/10) = 10, and y = 0.

I really like parabolas; they are so easy. Tomorrow I will try to bend parabolas also.

LESSON 29. Double-parabolas as lines bending down and up

Today my little brother said:

The other day I bent the line, y = 2*x, to parabolas first with ½*x^2. They all went through the floor.

So now I will counter-bend the line, so it stays above the floor as a double-parabola bending down and up.

So, I added some x^3 to get the counter-bending also. After some trials, I settled with

y = 2*x – ½ * x^2 + (1/30)* x^3.

This is a double parabola. First y goes up, then y turns in about x = 3. Then y goes down until y turns again in about x = 7. Finally, y goes up again. To find precisely where y turns, I will use y's growth formula.

To find out how x^3 grows I use the formula, 1BBB3BB3B1, that gives the next BBB BundleBundleBundle. This is a cube that grows with its 3 squared sides if we neglect the 3 lines and the upper corner. So, I guess the growth formula for x^3 is 3* x^2.

Therefore, y's growth is 2 – 2*½*x + 3* 1/30* x^2, or if we write y' instead of y's growth formula

y' = 2 – 1*x + 1/10 * x^2,

This parabola shows how the growth, y', grows. To find where y' turns we look at y''s growth formula, which we shortly write as

y'' = 0 – 1 + 2*(1/10)*x = -1 + 2/10*x, that is zero for 2/10*x = 1, or x = 10/2 = 5, which is where the bending turns from down to up or from up to down. In or case the bending turns from down to up.

To find where y itself turns we must find where its growth, y', is zero.

We will use our own smart turning-point formula. We already know that y' turns in x = 5, and here y' is 2 – 1*5 + 1/10*5^2 = 2 – 5 + 2.5 = -0.5. So, with then bend-number 1/10 we get this formula

y' = 2 – 1*x + 1/10 * x^2 = 1/10 * (x – 5)^2 – 0.5.

We see that y' is zero for 1/10 * (x – 5)^2 – 0.5 = 0, or (x – 5)^2 = 10*0.5 = 5, or x = 5 + or - √5, or x = 7.2 and x = 2.8.

We see that while both y' s growth, y', and its growth, y'', can both be zero in its turning points, y itself cannot be zero.

So, instead of 1/30, we try with two weaker counter-bending numbers 1/32 and 1/33 to see if here y can be zero. Heer we chose 1/32*x as a common factor to bracket out

y = 2*x – ½ * x^2 + (1/32) * x^3 = (64 – 16 * x + x^2) * (1/32) * x.

To find out where 64 – 16 *x + x^2 is zero we fits find its turning point x = -B/(2*A) = 16/(2*1) = 8.

Here, y = 64 – 16*8 + 8^2 = 64 – 128 + 64 = 0.

So, with bending-number +1, the formula my be written as

Y = 1*(x – 8)^2 + 0 = (x – 8)^2

So, the y-formula may now be written as

Y = (1/32) * x * (x – 8)^2

Here we see that y is zero in x = 0, and in x = 8.

In the same way we can see that there are three zeros in the double-parabola

y = 2*x – ½ * x^2 + (1/33)* x^3 = (66 – 16.5 * x + x^2) * (1/33) * x.

Here, 66 – 16.5 * x + x^2 have two zeros, for x = 6.8 and for x = 9.7

LESSON 30. Parabola Dart is a Candy Game.

"Parabola Dart means candy for breakfast for a week, for I win each time." Said my little brother this morning showing me a basket full of candy.

"What is Parabola dart?" I asked, and my brother began to explain:

● On our ten-by-ten BundleBundleBoard where we must hit y = 7 in the end from y = 0 in the start.

First, we travel by lines, then we travel by parabolas, and finally we travel by double parabolas.

You have three tries, and if you miss 7 you lose, and your stake goes to the winner who has used the fewest tries.

And I win each time because I know how to predict the result with parabola calculations.

● First, we must hit 7 with a line by entering its angle.

Here I use the line's per-number that is 7 per ten or 0.7 per 1.

Then the tan-button tells me that the angle is 35 degrees since tan(35) = 0.7

So, I enter 35 and get 'hole in one'.

The others typically aim too high and too low before getting closer the third time.

● Next, we must hit 7 with a parabola by entering its bending-number. We are told that the start angle is 63 degrees.

As BBM'ers we know that 63 degrees comes from a line where the per-number is 2 per 1 since tan(63) = 2.0.

So, we know that y = 7 for x = 10, and that the parabola has the formula

y = 2 * x + a * x^2, so

7 = 2 * 10 + a * 10^2, or

7 = 20 + a * 100, or

a = (7 – 20)/100 = -0.13

So, I enter -0,13 as the bending-number for the parabola, and again I get 'hole in one'.

● Finally, we must hit 7 with a double parabola by entering its bending-number. We are told that the start angle is 63 degrees and that the parabola has -0.5 as its bending-number.

Again, we know that 63 degrees comes from a line where the per-number is 2 per 1

since tan(63) = 2.0.

So, we know that y = 7 for x = 10, and that the double parabola has the formula

y = 2 * x – 0.5 * x^2 + a * x^3, so

7 = 2 * 10 – 0.5 * 10^2 + a * 10^3, or

7 = 20 – 50 + a * 1000, or

7 = -30 + a * 1000, or

a = (7 + 30)/1000 = 0,037

So, I enter 0,037 as the bending-number for the double parabola, and again I get 'hole in one'.

● Then they changed the units to 2s horizontally and 3s vertically so we should hit

y = 21 where x = 20.

But as BBM'ers we are used to using units, so the calculations are the same.

With lines, the per-number now is 21/20 = 1.05 giving 46 degrees with the tan-button.

With parabolas, the formula is, y = 2 * x + a * x^2, with y = 21 for x = 20.

Now, 21 = 2 * 20 + a * 20^2, gives a = (21 – 40)/400 = -0.048

With double parabolas, the formula is, y = 2 * x – 0.5 * x^2 + a * x^3, with y = 21 for x = 20

Now, 21 = 2 * 20 – 0.5 * 20^2 + a * 20^3, gives a = (21 + 160)/8000 = 0.023

● So you see, candy and candy as long as we want.

Solving Quadratic Equations the Vertex way, (x – P)^2 = (y – Q) * 1/a

A parabola is a bended line with Vertex (P,Q) & bending-number, a, and a formula,

y = a*(x – P)^2 + Q, giving x = P ± √((y – Q)/a)

We calculate the bracket to compare with the standard form:

y = a* (x^2 -2*P*x + P^2) + Q

y = a*x^2 - 2*a*P*x + (a*P^2 + Q)

y = a*x^2 + b*x + c

We now see that

P = - b/(2*a), since b = -2*a*P, and

c = a*P^2 + Q = a*(-b/(2*a))^2 + Q = b^2/(4*a) + Q, so

Q = c - b^2/(4*a)

= (4*a*c – b^2)/(4*a)

= -D/(4*A) for D= b^2 – 4*a*c

LESSON 31. The Algebra Square shows how to reunite unlike and like unit- and per-numbers.

Counting ten fingers in 3s we get T = 1BB0B1 3s = 1*B^2 + 0*B + 1. My uncle calls this an example of a general bundle-formula called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration.

With units we see there can be only four ways to unite numbers:

addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers.

If we go backwards, we split a total in unlike or like unit-numbers or per-numbers. Here subtraction and division split a total into unlike and like unit-numbers.

Differentiation splits a total into unlike per-numbers. And finally, a total is split into like factors by the factor-finding root and the factor-counting logarithm.

My uncle calls this 'the Algebra square' where the Arabic word Algebra means to reunite.

Calculations unite/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n *T – n = a*	T = a * n *T/n = a*
Per-numbers m/s, $/100$ = %	T = ∫ f dx *dT/dx = f*	T = a^b *b√T = a loga(T) = b*

LESSON 32. The three genres of number language tales, fact and fiction and fake.

Once we know how to count and recount totals, and how to unite and split the four number-types, unlike and like unit-numbers and per-numbers, we can actively use this number-language to produce inside tales about outside totals existing in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then and what-then models, or room, rate, and risk models.

Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas.

Typically, they model the past and the present. They need to be checked for correctness and units.

Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas.

Typically, they model the future. They need to be supplied with scenarios building on alternative assumptions.

Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas.

Typically, they add without units or hide alternatives. Here, number stories need to be replaced by word stories.

LESSON 33. Showing your work with formula tables

Unknown number	c = ?	T = a + b*c	From a T-Formula to a C-Formula
Known numbers	a = 2 b = 3 T = 14	T = a + (b*c) T – a = b*c (T – a)/b = c (14 – 2)/3 = c 4 = c	From multi to single calculation with hidden bracket + moves over as the opposite, - * Moves over as the opposite, / Bracket around any calculation Inserting the numbers The solution is calculated
The test	Test	14 = 2+3*4 14 = 14 ☺	The solution is tested because we changed a T-formula to a c-formula

With opposite signs, the unknown number is moved first:

c = ?	T = a – c
	T + c = a c = a – T

c = ?	T = a/c
	T * c = a c = a/T

LESSON 34. A history of Mathematics I

“Please, tell me the history of Mathematics now, NOW", said my little brother.

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means ’earth-measuring’ in Greek. Algebra means ’reuniting’ in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces?

Originally, human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus trade was only possible with those highlanders that had silver in their mountains. The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire.

The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized this findings by claiming: All is numbers. This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don’t know Geometry.

However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation. The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g; and the distance s and the time t are connected by the formula: s = ½*g*t^2.

However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middlemen. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

LESSON 35. A history of Mathematics II

"Tell me more about the history of Mathematics, please, please, and now, not tomorrow" said my younger brother after midnight.

OK. The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon.

But how does the moon move?

The church said 'among the stars'.

Newton objected: No, the moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why does the moon move like that?

The church said: OK, but it follows the metaphysical Lord's will that is unpredictable.

Newton objected: No, it follows the physical will of a force called gravity that is predictable since it obeys a formula.

The church then said: OK, and Aristotle tells us the a force upholds order.

Newton objected: No, a force changes order.

The church said: OK, then we need Arabic algebra for the calculations.

Newton objected: No, Algebra only deals with level calculations, and we need to do change-calculations, which is so new that I have to invent it myself and call it differential calculus, since changes are found as differences.

Once in India, England exchanged silver for cotton to be planted in their North-American colonies bought from Spain who had no interest in land without silver.

By replacing silk and pepper trade from the Far East with production of cotton in the US, England created a triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton. The profit was used for investment buying stock and establishing an industrial production.

Fighting for colonies led to the second world war, leading to the creation of the computer and operation research.

Later the invention of the chip let to pocket calculators making tables unneeded, and to CAS and Artificial Intelligence, AI.