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ALLAN.TARP@GMAIL.COM
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POSTER PRESENTATION (PP)

Poster Presentation is designed to facilitate the exchange of ideas among participants. This format provides an interactive dynamic where attendees can engage directly with presenters by asking questions and discussing the work in a more informal setting. Poster Presentation is an excellent opportunity for both presenters and attendees to network, gain feedback, and inspire new collaborations. Accepted presenters will 90 minutes for presenting and discussing their research.

WORKING GROUPS (WG)

A Working Group (WG) is designed to foster in-depth discussion, collaboration, and the exchange of ideas among researchers within the mathematics education community. Unlike other formats that focus primarily on the presentation of individual research findings, WG encourages collective engagement with a common research topic. The goal of these sessions is to generate new insights, start joint research activities, and build lasting collaborations that can extend beyond the conference itself. WG session can address both emerging and well-established research topics within mathematics education. However, the focus is on topics where the research is evolving, and there is potential for new contributions. The session should include a clear goal, supported by a structured strategy to engage participants in meaningful collaboration. Opportunities for participant contributions are central to the success of WG, allowing for shared materials, collaborative work on texts, or focused discussions on well-defined questions.

SPECIAL SHARING GROUPS (SSG)

We are excited to introduce the Special Sharing Group (SSG) as a distinctive feature of the upcoming EARCOME 9 conference. The SSG provides a dynamic and collaborative platform designed to bridge the gap between research and practice in mathematics education. These sessions will foster meaningful discussion and exchange among educators, researchers, and practitioners, focusing on innovative practices, critical insights, and the latest developments in the field.

TOPIC STUDY GROUPS

A Topic Study Group (TSG) is designed to bring together participants who share an interest in a particular topic related to mathematics education. Participants whose papers are accepted for TSGs will have 10-15 minutes for their oral presentations, although the presentation time may vary depending on the number of papers. Each TSG will be facilitated and organized by the following invited chairs.

NO.01 THE ESSENCE OF MATHEMATICS EDUCATION IN CURRICULUM AND MATERIALS

NO.02 THE ESSENCE OF MATHEMATICS EDUCATION IN CLASSROOM PRACTICE

NO.03 THE ESSENCE OF MATHEMATICS EDUCATION IN ASSESSMENT AND EVALUATION

NO.04 THE ESSENCE OF MATHEMATICS TEACHER EDUCATION

NO.05 THE ESSENCE OF MATHEMATICS EDUCATION IN LEARNING AND COGNITION

NO.06 THE ESSENCE OF MATHEMATICS EDUCATION IN THE USE OF DIGITAL TECHNOLOGY

NO.07 THE ESSENCE OF MATHEMATICS EDUCATION IN AFFECTIVE AND EMOTIONAL ASPECTS

NO.08 THE ESSENCE OF MATHEMATICS EDUCATION WITH EQUITY AND CULTURE

NO.09 THE ESSENCE OF MATHEMATICS EDUCATION IN UNDERGRADUATE LEVEL

NO.10 THE ESSENCE OF STE(A)M EDUCATION

INVITED LECTURES (IL)

Speakers

- Jaepil Han & Jinfa Cai, University of Delaware, USA
Teaching Mathematics Through Problem Posing: A Systematic Review of Problem-Posing-Based Learning Research
- Zhu Yan, East China Normal University, China
The Impact of Self-Regulated Learning on Mathematics Performance: Insights from East Asian Students in PISA 2022
- Rully Charitas Indra Prahmana, Universitas Ahmad Dahlan, Indonesia
Reinvention Mathematics Contents Through the Indonesian Cultural Contexts With or Without Technology: A Story From Indonesia
- Mitsuru Kawazoe, Osaka Metropolitan University, Japan
Teaching Mathematics Through Applications: A Case of Mathematics Education for Non-STEM Undergraduate Students
- JinHo Kim, Daegu National University of Education, Korea
Creating Mathematics Classrooms Where All Students Can Succeed and Enjoy Learning
- Allan Tarp, MATHeCADEMY.net, Denmark
Mathematics Is Fun When Decolonized by Bundle-Numbers With Units on a Bundle-Bundle-Board
- Gabriele Kaiser, University of Hamburg, Germany
Recent Discourse in Research on the Competence of Mathematics Teachers
- Caroline Yoon, University of Auckland, New Zealand
East Asian Narratives of Mathematics in an Arts-Based Research Project
- Oi-Lam Ng, The Chinese University of Hong Kong, Hong Kong
Computational Thinking as a Catalyst for Change: Reflecting on What, Why, How, and for Whom We Teach Mathematics in the Digital Age
- Padmanabhan Seshaiyer, George Mason University, USA
Transforming Mathematics Education Practices through Innovative Pedagogy and Data-driven Pathways
- Jennifer M. Suh, George Mason University, USA
Empowering Students as Mathematicians Through Community-Based Math Modeling
- Ho Kyoung Ko, Ajou University, Korea
Integrating AI and Digital Technologies in STEM+I Mathematics Education: Opportunities and Challenges

CONTENTS

PP. CHILDREN'S OWN BUNDLE-NUMBERS WITH UNITS MAY REACH THE UNITED NATIONS DEVELOPMENT NUMERACY GOAL

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

Counting Many with bundles, children deserve a bundle-number with units curriculum

"No, that is not four, that is two twos". Said a 3year old child when asked "How many years next time?"; and when seeing four fingers held together two by two. This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time. The number 'two' thus exists both in space and in time. In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Either horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle. In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units a counting-numbers are operators to be multiplied with units to become totals that can be added if the units are the same. On my hand, a collapsed V-sign shows that 1 1s + 1 1s = 1 2s, which together with a V-sign's 2 1s total 1 2s + 2 1s = 1 4s, and not 3 3s as expected if 1+1 = 2.

A01. Bundle-counting with units and using snap-cubes or a ten-by-ten BundleBundleBoard, 2 3s is 2 bundles with 3s per bundle. So, the per-number 3s exists in space and the counting-number 2 in time. The Algebra square reunites unlike and like counting- & per-numbers (fig. 1). Polynomials use bundle-counting with units. 43 = 4*B + 3*1 = 4B3 tens, and 543 = 5BB4B3 tens. Bundle-numbers falsify '1+1 = 2' with 2 V-signs showing that 1 1s + 1 1s = 1 2s and 2 1s + 1 2s = 1 4s, and not 3 3s.

A02. Flexible Bundle-counting in space. Space-count five and ten fingers in 2s, 3s, 4s and 5s. 5 = 1B3 = 3B-1 = 2B1 = 1BB0B1 2s, and Ten = 2BB0B2 = 1BBB0BB1BB0 2s. And T = 38 = 3B8 = 2B18 = 4B-2. T = 35+46 =3B5+4B6 = 7B11 = 8B1. T = 6*28 = 6*2B8 = 12B48 = 16B8 = 168. And T = 4507 = 4BBB 5BB 0B 7, T = 4*B^3 + 5*B^2 + 0*B + 7*1. Place value and carrying is unneeded.

A03. Add and subtract 1digit numbers counted in half-bundles. T = 6+7 = ½B1 + ½B2 = 1B3 = 13. T = 4+7 = ½B-1 + ½B2 = 1B1 = 11. T = 3+4 = ½B-2 + ½B-1 = 1B-3 = 7. And T = 8-6 = ½B3 - ½B1 = 3-1 = 2. T = 6-4 = ½B1 - ½B-1 = 1 - -1 = 2 (so, - - = +). T = 6-8 = ½B1 - ½B3 = 1-3 = -2.

A04. Time-counting fingers in ½B, "1,2,3,4,5,6" no! "0B1, 0B2, 0B3, 0B4, 0B5 or ½B0, ½B1". Time-count from 88 to 100: "8B8, 8B9, 8Bten or 9B0, …, 9B9, 9Bten or tenB0 or 1BB0B0".

A05. Digits are icons. 4 strokes as a 4-icon: I I I I → IIII → 4. And 5 as a 5-icon: I I I I I → IIIII →5.

A06. Operations are icons also. Push-away and -back to lift to stack, (division & multiplication), 6/2 means 'from 6 push-away 2s to lift', so 6 = 3x2 = (6/2)x2, T = (T/B)xB (the recount-formula). Pull-away and -back (minus and plus) to get decimals, fractions and negatives. 7 = 3B1 = 3½B = 4B-1 2s.

A07. Recounting between icon and tens. “? 5s gives 40”: u*5 = 40 = (40/5)*5, so u = 40/5, i.e., “To Opposite Side with Opposite Sign”. 6 7s = ? 8s and 6 7s = ? tens leads to division and multiplication tables where 6 7s = 6*7 = (B-4)*(B-3) = From BB, pull-away 3B & 4B and pull-back the 4*3 pulled-away twice = 3B12 = 4B2 = 42. So (B-4)*(B-3) = BB - 3B - 4B + 4*3. Here, minus * minus is +.

A08. Recounting physical units gives per-numbers as 2$/5kg. 20kg = (20/5)*5kg = (20/5)*2$ = 8$. Meter = (meter/second)*second = speed*second. Fractions with like units: 2$/5$ = 2/5. Trigonometry in a stack: height = (height/base)*base = tan(Angle)*base. Pi = n*tan(180/n) for n high enough.

A09. Bundle-bundles are squares. 3 3s= 1BB 3s= 1BB2B1 2s= 1BB-2B1 4s. So, 1BB2B1 = next BB.

A10. Squaring stacks. T = 6 4s = 1BB where B=√(6*4). Guess 1: ‘(6-1) (4+1)s’ or ‘5 5s’, since ½(6-4) = 1. The empty 1-corner needs two ‘t 4s’ stacks, and t*4 = ½ gives t = 1/8. Guess 2: ‘4.9 4.9s’.

A11. Solving quadratics. A (u+3) square has two squares and stacks: (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6*u + 9. If u^2 + 6*u + 8 = 0, all disappears but 1, so, (u + 3)^2 = 1, so u = -4 or u = -2.

A12. Adding next-to and on-top or reversed. 2 3s + 4 5s = ? 8s. Here integral calculus adds areas, and recounting change units. 2 3s + ? 5s = 3 8s. Here ? = (T2-T1)/5 = ΔT/5 roots differential calculus.

A13. Adding per-numbers & fractions. 2kg at 3$/kg+ 4kg at 5$/kg=(2+4)kg at (3*2+5*4)/(2+4) $/kg. And, 2 apples with ½ red + 3 apples with 2/3 red = (2+4) apples of which (2*½+3*2/3)/(2+3) red. Integral calculus adds piecewise or locally constant per-numbers.

A14. The Algebra square. The ‘Algebra Square’ reunites unlike and like unit- & per-numbers

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n *T – n = a*	T = a * n *T/n = a*
Per-numbers m/s, $/100$ = %	T = ∫ f dx *dT/dx = f*	T = a^b $\sqrt[b]{T}=a$ *loga(T) = b*

Figure 1. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

A15. Fact, fiction & fake models. Fact ‘since-then’ models quantify and predict predictable quantities by using factual numbers and formulas. Typically modeling the past and the present, they need to be checked for truth and units. Fiction ‘if-then’ models quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically modeling the future, they need to be supplied with scenarios building on alternative assumptions. Fake ‘what-then’ models quantify and predict unpredictable qualities by using fake numbers and formulas or by adding without units (Tarp, 2001).

A16. ‘Existence-based’ math (Tarp, 2018) will allow a communicative turn in the number-language as in the word-language in the 1970s (Widdowson, 1978). Using children’s own bundle-numbers with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre- and in-service teacher training at MATHeCADEMY.net.

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WG. Replacing STEAM with STEEM to also Include Economics

Allan Tarp, MATHeCADEMY.net, Denmark, Allan.Tarp@gmail.com.
Yujin Lee, Kangwon, National University, YLEE@kangwon.ac.kr.

From STEM over STEAM to STEEM

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of predictable physical quantities, and to model unpredictable quantities by scenarios. Statistics 'post-dicts' unpredictable quantities by setting up probabilities for future behavior, using fact or fiction numbers as median and fractals or average and deviation. Including economics in STEM opens the door to statistics also. Art may be an appetizer, but not a main course since to play a core role in STEM, geometry and algebra should be together always and never apart. Art is a sugar coating making the pill go down but does not make the pill more digestible. STEM thus may be extended to STEAM to make it more appealing and motivating, but extending STEM to STEEM will increase the understanding of the nature of numbering and calculating to meet the fourth of the UN sustainable development goals saying that within 2030 all youth and most adults should possess numeracy. Which will enable a communicative turn in the number-language as the one that took place within the word-language around 1970.

Economics gives a fundamental understanding of numbers and calculations in primary school

The basic meanings of geometry and algebra show that they are both rooted in economics. In Greek, geometry means to measure earth, and in Arabic, algebra means to reunite numbers, so they have a common root in the basic economic question "How to divide the earth and what it produces?" As a hunter-gatherer you need not tell the different degrees of many apart but as a farmer you do since here you produce to a market to survive and need to be numerate to answer the question "How many here?". This immediately leads to the answer "That depends on the unit." Economics thus begin at once by reusing the number-names when using bundling to count.

The romans unsystematically gave names to the bundles 5s, 10s, 50s, 100s, 500s and 1000s. This worked well for administrative addition and subtraction jobs but not for multiplication. So, when German silver reopened the trade between India and Renaissance Italy, Hindu-Arabic numbers named only the unbundled, the bundles, the bundle of bundles (BB or B^2), the bundle-bundle-bundle (BBB or B^3), etc. Typically, ten was used as the bundle-size, but also dozens and scores, 12s and 20s.

At a market you sell goods in bundles with different units, e.g., 2 3s. But the buyer may want to have 5s or trade 4 per 5 or pay 4$ per 5. So, changing units becomes a core job: '2 3s = ? 5s', and '6 7s = ? tens', and '3 tens = ?6s'. Likewise, when changing the units for length, weight, volume, and currency, And, when changing from the quantity to the price. Here, Renaissance Italy used 'regula detri', the rule of three. Asking "With the per-number 2$ per 3kg, what is the price for 9kg?", first they arranged the three numbers with alternating units: '9kg, 2$, 3kg'. Then they found the answer by multiplying and dividing: 9*2/3 = 6$. Today we use proportionality and say 9kg = (9/3)*3kg = (9/3)*2$ = 6$ when using the core linear recount-formula T = (T/B)*B, coming from recounting 8 in 2s as 8 = (8/2)*2.

Before school, children use bundle-numbers with units as 2 3s and 4 5s thus telling apart counting numbers in time as 2 and 4 from bundle-numbers in space as 3s and 5s. The school does not do so and insists that 1+1 = 2, which the children question by using an open and a closed V-sign to show that 2 1s and 1 2s add to 1 4s and not to 3 3s as the school says. Then they point out that the three core unit-change questions lead to a division table, a multiplication table, and to solving equations by recounting. And that adding 2 3s and 4 5s next-to as 8s is adding areas found by calculus. And that recounting the height in the base in 4 5s is trigonometry giving $\pi = n$*tan(180/n) for n high enough. They thus learn core math by counting and recounting before adding when beginning with economics.

Macroeconomics and microeconomics in middle school and high school

Later, macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheets for simulations using different parameters.

And, microeconomics describes equilibriums in individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable costs, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determine the equilibrium price. Market surveys lead to statistics, as does insurance. In the households, spending comes from balancing income and transferals with saving and tax. In a bank, income comes from simple and compound interest, from installment plans as well as risk taking. On the stock market, courses fluctuate. Governments must consider quadratic Laffer-curves describing a negative return to a tax-raise. To avoid units, factories use variations of Cobb-Douglas power elasticity production functions for modeling.

The working group program

In the working group we will experience how basic economic questions in primary school lead to a somewhat different mathematics that uses bundle-numbers with units, which need to change units before being traded. This allows core mathematics as linearity and calculus to grow from the two questions '2 3s is how many 5s?', and '2 3s + 4 5s total how many 8s?' This roots the proportionality recount-formula T = (T/B)*B, as well as calculus when adding globally and locally constant per-numbers. Finally, examples from the other STEM areas will be considered under the light of bundle-numbers with units, and the difference between fact, fiction and fake models will be discussed.

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SSG: CAN A DECOLONIZED MATHEMATICS SECURE NUMERACY FOR ALL?

Allan Tarp, MATHeCADEMY.net, Denmark, Allan.Tarp@gmail.com.
Yujin Lee, Kangwon National University, YLEE@kangwon.ac.kr.

This Special Sharing Group will discuss 'Quality Education', the fourth of the 17 UN Sustainable Development Goals, which has as a goal target to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". Looking at the relationship between numeracy and mathematics, a core question is: Which to teach and learn first?

Introduction

"That is not four. That is two twos." Said a 3year-old when asked "How many years next time?" and seeing 4 fingers 2 by 2. Which indicates that children have their own number-language before they are asked to shift to the school's version. The child sees what exists, bundles of twos in space serving as units when counted in time. And as in the word-language, the child's number-language also uses a full sentence with an outside existing subject, a linking verb, and an inside predicate.

The school thus could help children to further develop their own number-language that uses two-dimensional bundle-numbers with units where multiplication always holds by simply changing the unit, e.g., from 4s to tens where 3 x 4 = 12 states that 3 4s may be recounted in tens as 1.2 tens.

So, by what right and how ethical is it when the school imposes upon the children its own one-dimensional non-unit numbers where addition without units only holds inside the school but seldom outside the school where '2+1 = 3' typically is falsified, e.g., by 2 days + 1 week = 9 days?

To separate reliable 'multiplication-math' from unreliable 'addition-math' the latter should maybe be called 'mathematism' (Tarp, 2018), true inside but seldom outside school. But then, why teach addition of non-unit numbers inside schools when outside students need to add numbers with units?

We therefore could ask: To impose unreliable addition of one-dimensional non-unit numbers upon students that use multiplication in their two-dimensional unit numbers, isn't that an example of "a colonization of the life world by the system", the key concept in the sociology of Jürgen Habermas?

Then we could ask if sociological imagination should use demodeling to bring inside concepts back once more to their outside roots to decolonize mathematics and its education so it may meet the fourth of the 17 UN Sustainable Development Goals that has as a goal target to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy".

Decolonization will not be easy as seen by different definitions of 'numerate'. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers." In their common history, England once colonized America. So, the difference may hide a hidden agenda where existence is colonized by a chosen essence instead of preceding it. The English uses the passive term ‘competent’ where the American uses the active term ‘work’. The word ‘competent’ is a predicate, a non-action word, I cannot ‘competent’ something, I can only be judged as competent by someone who is competent. In contrast, ‘work’ is an action word, a verb, since with my body and mind I can work on something and test the result to see if it works.

Also, there is a difference between the words ‘mathematics’ and ‘numbers.’ Again, mathematics is a non-action word, I cannot ‘mathematics’ or even ‘math’ a thing. In contrast, ‘number’ is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

One example of a decolonized mathematics education that respects the children’s bundle-numbers with units may be found in the article “Mastering Many by counting, re-counting and double-counting before adding on-top and next-to.” The article shows that a ‘counting before adding’ approach leads to the same concepts as a traditional approach but with different identities, and in a different order. Counting in 3s leads to 9 as a bundle-bundle, a B^2, which leads on to squares, square roots, and quadratics. Counting transforms the operations into icons where division and multiplication become a broom and a lift that pushes-away bundles to be stacked as shown when recounting 8 in 2s as 8 = (8/2) x 2, or with T and B for Total and Bundle, T = (T/B) x B, that creates per-numbers when recounting in physical units, $ = ($/kg) x kg. Subtraction becomes a rope that pulls-away the stack to find the unbundled that, placed on-top of the stack as part of an extra bundle, become decimals, fractions, or negatives, e.g., 9 = 4B1 = 4½ = 5B-1 2s. Finally, addition becomes a cross showing the two ways to add stacks, on-top using the linearity of recounting to make the units like, or next-to creating integral calculus by adding areas, which is also used when adding per-numbers needed to be multiplied to areas before adding. All this provides an ‘Algebra Square’ showing how to unite the four types of existing numbers: multiplication and addition unite like and unlike unit-numbers, and power and integration unite like and unlike per-numbers. And how to split totals with the opposite operations: division and subtraction, together with root or logarithm and differentiation.

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers	T = a + n	T = a * n
m, s, kg, $	*T – n = a*	*T/n = a*
Per-numbers	T = ∫ f dx	T = a^b
m/s, $/100$ = %	*dT/dx = f*	$\sqrt[b]{T}=a$ *loga(T) = b*

Figure 1. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

Reuniting like and unlike unit- and per-numbers is “ability to understand and work with numbers” to produce quantitative tales, reports, and literature; and to discuss to which genre they belong, fact or fiction or fake. Which will allow a communicative turn in the number-language as the one that took place in foreign language education in the 1970s allowing all to use the English language without first knowing its abstract grammar. Which again may create a world where numeracy is no longer a privilege for an elite that colonize the number-language with unreliable mathematism.

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TSG01. A DECOLONIZED CURRICULUM AND CHILDREN'S BUNDLE-NUMBERS WITH UNITS MAY REACH THE UN DEVELOPMENT NUMERACY GOAL

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from an essence-based to an existence-based curriculum may allow reaching the UN Development Goal and may provide new ways for students to interact with curriculum and materials.*

Keywords: Early childhood mathematics, numeracy, special education, curriculum, arithmetic

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world where humans see and name it differently. In contrast, mathematics does not do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions.

To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

Grand Theory looks at Mathematics Education

Within philosophy, Existentialism holds that 'existence precedes essence' (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. 'Many' thus should be seen ontologically in itself, instead of epistemologically, how some may perceive and verbalize it.

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Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that '1+1 = 2'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). As to differences, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to test with Design Research and building on the observation that when asked "How many years next time?" a 3-year-old child will say "That is not 4, that is two 2s" if seeing the 4 fingers held 2 by 2.

Counting Many with Bundles, Children Deserve a Bundle-Number Curriculum

A01. Bundle-counting with units and snap-cubes or a ten-by-ten BundleBundleBoard, two 3s is two bundles with 3s per bundle. The per-number 3s exists in space, the counting-number two in time. The Algebra Square reunites unlike and like counting- & per-numbers (table 1). Polynomials use bundle-counting with units. 43 = 4*B+3*1 = 4B3 tens, and 543 = 5BB4B3 tens. Bundle-numbers falsify '1+1 = 2' with two V-signs where one 1s + one 1s = one 2s and two 1s + one 2s = one 4s, and not three 3s.

A02. Flexible Bundle-counting in space. Space-count five and ten fingers in 2s, 3s, 4s and 5s. 5 = 1B3 = 3B-1 = 2B1 = 1BB0B1 2s, and Ten = 2BB0B2 = 1BBB0BB1BB0 2s. And T = 38 = 3B8 = 2B18 = 4B-2. T = 35+46 =3B5+4B6 = 7B11 = 8B1. T = 6*28 = 6*2B8 = 12B48 = 16B8 = 168. And T = 4507 = 4BBB 5BB 0B 7, T = 4*B^3 + 5*B^2 + 0*B + 7*1. Place value and carrying is unneeded.

A03. Add and subtract 1digit numbers counted in half-bundles. T = 6+7 = ½B1 + ½B2 = 1B3 = 13. T = 4+7 = ½B-1 + ½B2 = 1B1 = 11. T = 3+4 = ½B-2 + ½B-1 = 1B-3 = 7. And T = 8-6 = ½B3 - ½B1 = 3-1 = 2. T = 6-4 = ½B1 - ½B-1 = 1 - -1 = 2 (so, - - = +). T = 6-8 = ½B1 - ½B3 = 1-3 = -2

A04. Time-counting fingers in ½B, "1,2,3,4,5,6" no!, "0B1, 0B2, 0B3, 0B4, 0B5 or ½B0, ½B1". Time-count from 88 to 100: "8B8, 8B9, 8Bten or 9B0, …, 9B9, 9Bten or tenB0 or 1BB0B0".

A05. Digits are icons. 4 strokes as a 4-icon: I I I I → IIII → 4. And 5 as a 5-icon: I I I I I → IIIII →5.

A06. Operations are icons also. Push-away and -back to lift to stack, (division & multiplication), 6/2 means 'from 6 push-away 2s to lift', so 6 = 3x2 = (6/2)x2, T = (T/B)xB (the recount-formula). Pull-away and -back (minus and plus) to get decimals, fractions and negatives. 7 = 3B1 = 3½B = 4B-1 2s.

A07. Recounting between icon and tens. "? 5s gives 40": u*5 = 40 = (40/5)*5, so u = 40/5, i.e., "To Opposite Side with Opposite Sign". 6 7s = ? 8s and 6 7s = ? tens leads to division and multiplication tables where 6 7s = 6*7 = (B-4)*(B-3) = From BB, pull-away 3B & 4B and pull-back the 4*3 pulled-away twice = 3B12 = 4B2 = 42. So (B-4)*(B-3) = BB - 3B - 4B + 4*3. Here, minus * minus is +.

A08. Recounting physical units gives per-numbers as 2$/5kg. 20kg = (20/5)*5kg = (20/5)*2$ = 8$. Meter = (meter/second)*second = speed*second. Fractions with like units: 2$/5$ = 2/5. Trigonometry in a stack: height = (height/base)*base = tan(Angle)*base. Pi = n*tan(180/n) for n high enough.

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A11. Solving quadratics. A (u+3) square has two squares and stacks: (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6*u + 9. If u^2 + 6*u + 8 = 0, all disappears but 1, so, (u + 3)^2 = 1, so u = -4 or u = -2.

A12. Adding next-to and on-top or reversed. 2 3s + 4 5s = ? 8s. Here integral calculus adds areas, and recounting change units. 2 3s + ? 5s = 3 8s. Here ? = (T2-T1)/5 = ΔT/5 roots differential calculus.

A13. Adding per-numbers & fractions. 2kg at 3$/kg+ 4kg at 5$/kg=(2+4)kg at (3*2+5*4)/(2+4) $/kg. And, 2 apples with ½ red + 3 apples with 2/3 red = (2+4) apples of which (2*½+3*2/3)/(2+3) red. Integral calculus adds piecewise or locally constant per-numbers.

A14. The Algebra square. The 'Algebra Square' reunites unlike and like unit- & per-numbers

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n *T – n = a*	T = a * n *T/n = a*
Per-numbers m/s, $/100$ = %	T = ∫ f dx *dT/dx = f*	T = a^b $\sqrt[b]{T}=a$ *loga(T) = b*

Table 1. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

A15. Fact, fiction & fake models. Fact 'since-then' models quantify and predict predictable quantities by using factual numbers and formulas. Typically modeling the past and the present, they need to be checked for truth and units. Fiction 'if-then' models quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically modeling the future, they need to be supplied with scenarios building on alternative assumptions. Fake 'what-then' models quantify and predict unpredictable qualities by using fake numbers and formulas or by adding without units (Tarp, 2001).

Conclusion, Recommendation and Discussion with a Reviewer

"By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". We may reach this UN Development Goal by replacing an essence-based curriculum with an existence-based curriculum answering the question 'how many?' by counting and recounting totals before adding them. Digits then occur as icons with as many strokes as they represent, thus becoming units when counting totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads, which makes place values and carrying unneeded. Bundling bundles also lead to squares and square roots; and to power as the first of the operations. The operations are icons also, but with different meanings and opposite order. Division now means counting, iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Thus, both proportionality and trigonometry occur in year one. Once counted and recounted, totals may add on-top, or next-to by areas as with integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own flexible bundle-number with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre-service and in-service teacher training asking the subject on a BundleBundleBoard instead of the instructor as exemplified on the MATHeCADEMY.net website. **Reviewer**: *Students will find it easier to understand the relationship between units and quantities. The calculation process becomes more flexible, without relying too heavily on traditional algorithms. But I suggest adding case studies and experimental data to make the argument more convincing.* **Answer**: Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order from addition to power where only few hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. With counting preceding adding, they will meet addition in part 12 and meet the core of math directly, calculus and linearity. What will happen if two students exchange classes? How ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return far ahead to normal education. Ethical testing is for teacher education and home education. And a composer should be allowed to compose and leave it to others to play and evaluate the music.

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TSG03. In BundleNumber-Math you just ask the BundleBundle Board

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal and exploring innovative assessment models and practices that move beyond traditional methods.*

Keywords: Early childhood mathematics, numeracy, special education, curriculum, arithmetic

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

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A14. The Algebra square. The 'Algebra Square' reunites unlike and like unit- & per-numbers

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Conclusion, Recommendation and Discussion with a Reviewer

"By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". We may reach this UN Development Goal by replacing an essence-based curriculum with an existence-based curriculum answering the question 'how many?' by counting and recounting totals before adding them. Here, digits and operations both become icons representing concrete things and actions. A BundleBundleBoard makes visible the children's own 2D bundle-numbers with units, just waiting to be recounted to teach the children about division and multiplication tables and solving linear and quadratic equations, as well as how to add stacks next-to by areas or on-top with like units. Thus, both proportionality and trigonometry occur in primary school, as does the Algebra Square showing how to reunite unlike and like unit- and per-numbers. Later, it becomes a coordinate system allowing also change formulas to be visible. The BBBoard may thus be used for self-instruction and for self-assessment, allowing the instructor to encourage the students to use their numerate number-language to make number-tales about things and actions within STEEM, Science, Technology, Engineering, Economics and Math. Thus, an existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own flexible bundle-number with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre-service and in-service teacher training asking the subject on a BundleBundleBoard instead of the instructor as exemplified on the MATHeCADEMY.net website.

Reviewer: *Are there empirical studies such as, e.g., teaching experiments examining the development of learners' number sense and mathematical reasoning under different number representation systems. What are the implications of the bundle number approach for assessment and evaluation?*

Answer: Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order: addition, subtraction, multiplication, division, power. And only few will hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. And, since counting and recounting precedes adding, they will not meet addition until they in part 12 meet the core of math directly, calculus and linearity. What will happen if a class B student changes to class A, or the other way around? In short, how ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return to normal education far ahead. Ethical testing is for teacher education and home education. As to evaluation, line-numbers are in the students' heads and can only be evaluated through oral answers. Whereas the Bundle-numbers are on a BundleBundleBoard where the students' 'ability to understand and work with numbers' can be observed both in actions, individual reporting and group discussions.

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TSG05. TEACHERS SEE 1 NUMBER WHERE KIDS SEE 3 NUMBERINGS IN 507, WHO IS NUMBER-BLIND?

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

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Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world observed by humans. In contrast, mathematics does not, it is institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions. To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

Grand Theory looks at Mathematics Education

Within philosophy, Existentialism holds that 'existence precedes essence' (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. 'Many' thus should be seen onto-logically in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget (1969) sees learning as adapting to outside existence, whereas a Vygotsky (1986) sees learning as adapting to inside institutionalized socially constructed essence.

Within sociology, a structure-agent debate discusses if institutions should be obeyed or negotiated between peers. Here, a Weberian viewpoint (1930) asks if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. As to the goal, Bauman (1990) suggests that, by institutionalizing mastery of Math as the means to reach mastery of Many, 'essence-math' has created a 'goal displacement' making the means a goal instead. Then, with the end goal, mastery of Many or existence-math, essence-math may be a means, but are there other means also?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that '1+1 = 2'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). And, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to be tested.

Children use 2D Bundle-numbers with Units, Teachers use 1D Line-numbers without Units

When asked "How many years next time?" a 3-year-old child will say "That is not 4, that is 2 2s" if seeing the 4 fingers held 2 by 2. This statement will change math education forever since, as educated, the essence, 4, is all we see. But as uneducated, the child sees what exists, bundles of 2s in space, and 2 of them when counted in time. Counting in tens instead of in 2s, we write 47 and say 'forty-seven' instead of '4 bundles with tens per bundle and 7 unbundled' to respect what exists and call it one number despite it is two numberings. 2 thus exists both in space and in time. In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle. In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit. And since we count bundles and singles, the unit should be included in the counting sequence: 0B1, 0B2, …, 0B9, 1B0. So, outside in the real world, 2 is a 'ghost-number'. What exists are 0B2 in time and 2s in space.

What kind of knowledge is mathematics when integrating two different phenomena, 0B2 in time and 2s in space, into one phenomenon, 2, defined as 1+1 = 2? Knowledge may be created individually (Piaget, 1969) and collectively (Grounded theory, 1967) both respecting that learning means differentiating a phenomenon into sub-phenomena with different properties. By doing the opposite, mathematics become a self-referring anti-science with inside descriptions of itself and not of the outside world where its foundation, 1+1=2, is falsified by children seeing that with an open and a closed V-sign on a hand, adding the 2 1s and 1 2s gives 1 4s, and not the 3 3s, as mathematics claims.Likewise, the school teaches both '2+3 = 5' and '2x3 = 6'. Here 2 3s always recounts as 6 1s, but 2weeks + 3days is 17days. So, even if both hold inside the school, outside 'multiplication holds, but addition folds.' Adding numbers without units may be called 'mathematism', true inside but seldom outside the school, whereas mathematics that add numbers with units may be called 'Many-math', using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units is called 'calculus' and 'proportionality', the core of mathematics. And here they occur in the first lessons where normally they enter very late.

To get a more precise definition of space-numbers as 2s and time-numbers as 0B2 we observe the following when using the time-number sequence '0B1, 0B2, …, 0B5' to count the fingers on a hand: After in time saying '0B1' when pulling up one finger we now in space have 1 1s. And, after in time saying '0B2' when pulling up one more finger we now in space have 2 1s that may bundle as 1 2s. Likewise with 0B3, 0B4, and 0B5. Then, after in time saying '0B6' when pulling up one more finger we now in space have 6 1s that may bundle as 1 6s, or that may bundle as ½B1 with ½B as 1 5s. We may repeat the same on a ten-by-ten Bundle-Bundle board that contains all the first Bundle-numbers.

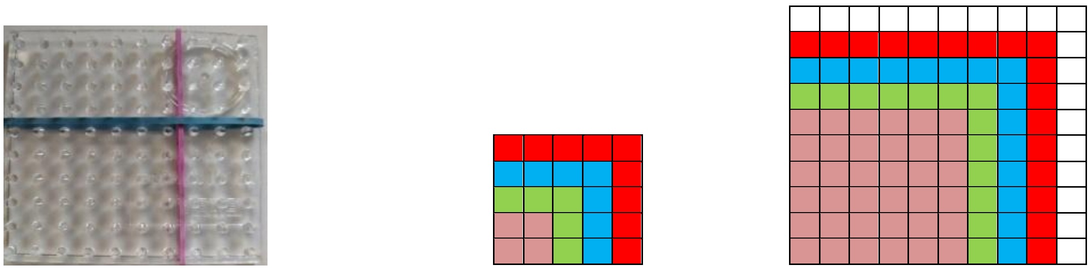

Figure 1. A BundleBundleBoard with 6 7s; and squares with 1BB2B1 or 1BB-2B1 as the neighbor

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 B 5 + 2 B 7	6 B 5 – 2 B 7	7 x 4 B 8	33 B 6 /7
8 B12 9 B 2	4 B-2 3 B 8	28 B 56 33 B 6	28 B 56 /7 4 B 8
92	38	336	48

Figure 2. Bundles and over- and underload make place values and carrying unneeded

Counting in half-bundles later eases addition, subtraction and multiplication: 6+7 = ½B1 + ½B2 = 1B3 = 13; and 9-7 = ½B4 - ½B2 = 2; and 6*7 = 6* ½B2 = 3B12 = 4B2 = 42.

To see that also bundles may be bundled we count five fingers in 2s: Five = 0B5 = 1B3 = 2B1 = 3B-1 = 1BB0B1 2s. Holding two hands together we see that ten = 2BB0B2 = 1BBB0BB1B0 = 1010 2s. And that ten =1B0 = 10, hundred is 1BB0B0 = 100, and thousand = 1BBB0BB0B0 = 1000 tens.

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that we have 8 = (8/2)x2, or T = (T/B)xB with T and B for Total and Bundle. This linear 'proportionality' recounting-formula to shift units now occurs in the first lesson when we bundle-count with units. And, it solves equations where 'u x 2 = 8' is asking 'How many 2s in 8?', which is simply answered by 'u = 8/2' since 8 is recounted in 2s as above, thus moving 'to opposite side with opposite sign'.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, 8 = 2B2 3s, or as a fraction when also counted in 3s as 2 = (2/3) x 3, 8 = 2 2/3 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, 8 = 3B-1 3s = 1BB0B-1 3s since the 3 bundles of 3s as one bundle of bundles.

Conclusion, Recommendations and Discussion with a Reviewer

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equations by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas all carried out on a Bundle-Bundle Board. So, with Many-math's 'counting before adding' we have learned most mathematics almost before we begin. This will fulfill the UN Development numeracy goal. And it performs miracles in special education classes where the losers suddenly become users understanding and using numbers to count and calculate things and events in space and time. We therefore should stop teaching mathematics as 'mathematism' with its ghost-numbers replacing units with a place value system where carrying is needed when adding without units, and built on the rule that '1+1 = 2. Instead, we should use a 'counting before adding' curriculum to make children develop their innate number sense to allow a communicative turn in the number-language as the one that took place in the word-language around 1970 (Widdowson, 1970), so that the children can produce end enjoy number tales with its three genres, fact, fiction and fake (Tarp, 2001). **Reviewer**: Say more about how the ideas can contribute to our understanding of teaching numeracy. **Answer**: Numeracy may be understood as having the ability to understand and work with numbers. Everyday numbers carry units that often change. So, it is only natural that the children use and develop the Bundle-numbers with units they bring to school ready to be recounted to change the units. Existence-math is based upon Existentialism, the root of French post-structural thinking where Foucault's skepticism towards institutions and Derrida deconstruction of centrism is at the root of postmodern sociocultural thinking. Here we respect a social group that has had no voice until lately (Tarp 2024, 2025): children which Bundle-number language is colonized by institutionalized essence. And a composer should be allowed to compose and leave it to others to play and evaluate the music.

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TSG06. POCKET CALCULATORS IN GRADE ONE TO PREDICT DIVISION TABLES

Allan Tarp, MATHeCADEMY.net, Denmark, Allan.Tarp@gmail.com.

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal and finally attain equity and access to mathematics education for youth and adults.*

Keywords: Early childhood mathematics, numeracy, calculator, curriculum, arithmetic

Background, the 4th United Nations Sustainable Development Goal

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Finally, Many exists in the outside world where humans see and name it differently. In contrast, mathematics does not do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions. To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

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Grade one in a Decolonized Future

The teacher: Welcome, I am your teacher in math, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So …

Showing a V-sign a child says: Mister teacher, here is one 1s in space, and here is also one 1s. Now we count them in time to see how many 1s we have by saying 'one, two'. So, we have two 1s. But only until we add them as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. And together with this neighbor V-sign the total is one 2s plus two 1s which is one 4s, and not three 3s. And, if I two times show you three 1s I have shown you six 1s. So, the counting-numbers two and three can be multiplied, but they cannot add.

So, instead of adding line-numbers without units, please help us add the bundle-numbers with units we bring to school, as 2 3s and 4 5s, that we can add next-to as eights, or on-top as 3s or 5s. If we add them next-to, we add tiles, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says it is taught the first year at college, but we need it here to keep and develop our bundle-numbers with units instead of being colonized with your line-numbers without units. We know that you want to bundle in tens, and in ten-tens, and in ten-ten-tens, but we like to bundle also in 2s, in 3s, in 4s, in half-tens, etc. We know that you have not been taught this, and that the textbook doesn't teach it. But don't worry, we will teach you what we found out in preschool. Or better, instead of you colonizing our ways let us find out together what math may grow from our bundle-numbers with units. My uncle is a philosopher, and he uses the word 'existentialism' form letting existence come before essence.

With sticks we see that 5 1s may be bundled as 1 5s that may be rearranged as one icon with 5 sticks. The other digits may also be seen as icons with the number of sticks they represent, where zero is a looking glass finding nothing. We don't need an icon for ten since here the total is 1B0 if we count in tens. And the calculations are icons also. If we bundle-count 8 in 2s, division is a 'push-away' icon for a broom so that 8/2 means 'from 8 push-away 2s'. Now a calculator can predict the result, 8/2 = 4. Stacking the 4 2s, multiplication is an icon for 'push-back-to-lift' predicting the result, 4x2 = 8. That 8 contains 8/2 of 2s is written as a recount formula '8 = (8/2) x 2', or 'T = (T/B) x B' with T for Total and B for Bundle. Bundle-counting 7 in 2s, we need a rope called subtraction to pull-away the three 2s to find the unbundled 1 to be placed on-top of the stack as a decimal, a less-number, or a fraction when counted in 2s also as 1 = (1/2)x2. So, we have: 7 = 3B1 = 4B-1 = 3½ 2s. A calculator can predict this. On the display we see, first [7/2 3.more], then [7–3x2 1]. This predicts that 7 may be recounted in 2s as 3B1 2s. Likewise we can predict that recounting 30 in 5s by entering 30/5. This allows us to solve the equation 'u*5 = 30' asking 'How many 5s in 30?' by u = 30/5, moving 5 'to opposite side with opposite sign'. Finally, to recount 6 7s in tens we multiply. Or use that 7 may be recounted in half-bundles as ½B2, so 6*7 = 6*½B2 = 3B12 = 4B2 tens = 42.

We know that your numbers without units are on a ruler, but our bundle-numbers with units are on a ten-by-ten Bundle-Bundle-Board, a BBBoard, that we can use for recounting and adding stacks. Recounting between icons, and from icons to tens, we call a division and a multiplication table. It is fun seeing a calculator predict the result when we recount nine 1s in 2s, 3s, etc., on the BBBoard. After recounting in a different unit, we can also use the BBBoard to add stacks on-top or next-to.

Enter	Read	Predict	Unit
8/2	4	8 = 4B0	2s
8/3	2.more		
8 – 2x3	2	8 = 2B2	3s
8/4	2	8 = 2B0	4s
8/5	1		
8 – 1x5	3	8 = 1B3	5s

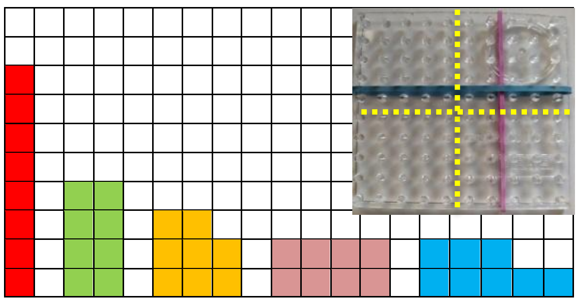

*Figure 1. A calculator as a division table recounting 8 1s in some other units 2s, 3s, 4s and 5s. And a BundleBundleBoard as a times table recounting in tens, 6*7 = 6*½B2 = 3B12 = 4B2 tens = 42*

Conclusion, Recommendation and Discussion with a Reviewer

"By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". We may reach this UN Development Goal by replacing an essence-based curriculum with an existence-based curriculum answering the question 'how many?' by counting and recounting totals before adding them (Tarp, 2018, 2024, 2025). Digits then occur as icons with as many strokes as they represent, thus becoming units when counting totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads, which makes place values and carrying unneeded. The operations are icons also, but with different meanings and opposite order. Division now means counting iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Calculators may give an inside prediction of an outside recounting on a BBBoard. Thus, both proportionality and trigonometry occur in year one. Once counted and recounted, totals may add on-top, or next-to by areas as with integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978) and use the predicting power of mathematics that also have three genres, fact, fiction and fake (Tarp, 2001). Using children's own bundle-numbers with units thus is a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation. **Reviewers**: *The paper would benefit from clarifying the methodological approach and providing concrete examples or empirical support to strengthen the argument. This paper offers an original and potentially transformative view on early mathematics education. If connecting theory to practice, and offering concrete recommendations, this paper could make a significant contribution to the field.* **Answer**: The paper's methodological approach is that a composer should write but not play a score. Its philosophy is Existentialism holding that existence precedes essence. As to empirical support we may have an ethical flat-or-round-earth situation where we can't have both paradigms, either 1+1 = 2 or it depends on the unit. And, by its very nature, multiplicity is multiplicative thus always occurring with units. Assume class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order from addition to power where only few will ever hear about calculus. To compare, class B has an existence-math curriculum and 2D Bundle-numbers with units having the opposite order. Here counting and recounting precedes adding, so addition first occurs as the core of mathematics, calculus and linearity. What will happen if two students change classes? How ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return far ahead to normal education? Ethical testing is for teacher education and home education. Mathematics exists in a high and low level version, so does digital technology also offering digital calculators to all, which will be accepted if all youth and most adults achieve numeracy where a calculator plays the important role as a number-predictor, and where all its nine buttons from the Algebra Square (Tarp, 2018) have been used to count and add in time and space.

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TSG10. FROM STEM OVER STEAM TO STEEM BUILT ON ECONOMICS

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

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Keywords: Early childhood mathematics, numeracy, arithmetic, STEM, calculus

Background, the 4th United Nations Sustainable Development Goal

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Finally, Many exists in the outside world where humans see and name it differently. In contrast, mathematics does not do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions. To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that ‘existence precedes essence’ (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. ‘Many’ thus should be seen ontologically in itself, instead of epistemologically, how some may perceive and verbalize it. Within psychology, Piaget (1969) sees learning as adapting to outside existence, whereas a Vygotsky (1986) sees learning as adapting to inside institutionalized socially constructed essence.

Within sociology, a structure-agent debate discusses if institutions should be obeyed or negotiated between peers. Here, a Weberian viewpoint (1930) asks if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an ‘iron cage’. As to the goal, Bauman (1990) suggests that, by institutionalizing mastery of Math as the means to reach mastery of Many, ‘essence-math’ has created a ‘goal displacement’ making the means a goal instead. Then, with the end goal, mastery of Many or existence-math, essence-math may be a means, but are there other means also?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that ‘1+1 = 2’. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference sees mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math then could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). As to differences, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to be tested with Design Research and building on the observation that when asked “How many years next time?” a 3-year-old child will say “That is not 4, that is 2 2s” if seeing the 4 fingers held 2 by 2.

From STEM over STEAM to STEEM

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of predictable physical quantities, and to model unpredictable quantities by scenarios. Statistics ‘post-dicts’ unpredictable quantities by setting up probabilities for future behavior, using fact or fiction numbers as median and fractals or average and deviation. Including economics in STEM opens the door to statistics also. Art may be an appetizer, but not a main course since to play a core role in STEM, geometry and algebra should be together always and never apart. Art is a sugar coating making the pill go down but does not make the pill more digestible. STEM thus may be extended to STEAM to make it more appealing, but extending STEM to STEEM may increase the ability to understand and work with numbers.

Economics gives a Basic Understanding of Numbers and Calculations in Primary School

The basic meanings of geometry and algebra show that they are both rooted in economics. In Greek, geometry means to measure earth, and in Arabic, algebra means to reunite numbers, so they have a common root in the basic economic question “How to divide the earth and what it produces?” A hunter-gatherer need not tell the different degrees of many apart but a farmer does since here you produce to a market to survive and need to be numerate to answer the question “How many here?”. This immediately leads to the answer “That depends on the unit.” Economics thus begin at once by reusing the number-names when using bundling to count.

The romans unsystematically gave names to the bundles 5s, 10s, 50s, 100s, 500s and 1000s. This worked well for administrative addition and subtraction jobs but not for multiplication. So, when German silver reopened the trade between India and Renaissance Italy, Hindu-Arabic numbers named only the unbundled, the bundles, the bundle of bundles (BB or B^2), the bundle-bundle-bundle (BBB or B^3), etc. Typically, ten was used as the bundle-size, but also dozens and scores, 12s and 20s.

At a market you sell goods in bundles with different units, e.g., 2 3s. But the buyer may want to have 5s or trade 4 per 5 or pay 4$ per 5. So, changing units becomes a core job: ‘2 3s = ? 5s’, and ‘6 7s = ? tens’, and ‘3 tens = ?6s’. Likewise, when changing the units for length, weight, volume, and currency, And, when changing from the quantity to the price. Here, Renaissance Italy used ‘regula detri’, the rule of three. Asking “With the per-number 2$ per 3kg, what is the price for 9kg?”, first they arranged the numbers with alternating units: ‘$, 9kg, 2$, 3kg’. Then they found the answer by multiplying and dividing: 9*2/3 = 6$. Today we use proportionality and say 9kg = (9/3)*3kg = (9/3)*2$ = 6$ when using the core linear recount-formula T = (T/B)*B, coming from recounting 8 in 2s as 8 = (8/2)*2.

Before school, children use bundle-numbers with units as 2 3s and 4 5s thus telling apart counting numbers in time as 2 and 4 from bundle-numbers in space as 3s and 5s. The school does not do so and insists that 1+1 = 2, which the children question by using an open and a closed V-sign to show that 2 1s and 1 2s add to 1 4s and not to 3 3s as the school says. Then they point out that the three core unit-change questions lead to a division table, a multiplication table, and to solving equations by recounting. And adding 2 3s and 4 5s next-to as 8s means adding areas found by calculus. And that recounting the height in the base in 4 5s is trigonometry giving $\pi = n$*tan(180/n) for n high enough. They thus learn core math by counting and recounting before adding when beginning with economics.

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers	T = a + n	T = a * n
m, s, kg, $	*T – n = a*	*T/n = a*
Per-numbers	T = ∫ f dx	T = a^b
m/s, $/100$ = %	*dT/dx = f*	$\sqrt[b]{T}=a$ *loga(T) = b*

Table 1. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

Macroeconomics and Microeconomics in Middle School and High School

Later, macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheets for simulations using different parameters. And, microeconomics describes equilibriums in individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable costs, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determine the equilibrium price. Market surveys lead to statistics, as does insurance. In the households, spending balance income and transferals with saving and tax. In a bank, income comes from simple and compound interest, from installment plans as well as risk taking. On the stock market, courses fluctuate. Governments consider quadratic Laffer-curves describing a negative return to a tax-raise. And factories use variations of Cobb-Douglas power elasticity production functions for modeling.

Conclusion, Recommendation and Discussion with a Reviewer

We may reach the UN Development Goal by replacing an essence-based curriculum with an existence-based curriculum answering the question 'how many?' by counting and recounting totals before adding them (Tarp, 2018). Digits are icons with as many strokes as they represent, thus being units when counting totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads, which makes place values and carrying unneeded. The operations are icons also, but with different meanings and opposite order. Division now means counting iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Once counted and recounted, totals may add on-top, or next-to by areas as with integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own bundle-number with units is a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation. **Reviewer**: Including classroom observations, experimental results, or working illustrations of the proposed changes in the curriculum would strengthen the paper immensely. **Answer**: Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order from addition to power where only few hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. With counting preceding adding, they will not meet addition until they meet the core of math directly, calculus and linearity. What will happen if two students exchange classes? In short, how ethical is it to test essence-math against existence-math in an ordinary school? Or even in special education where the students will return far ahead to normal education. Ethical testing is for teacher education and home education. And a composer should be allowed to compose and leave it to others to play and evaluate the music.

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IL. MATH IS FUN WITH BUNDLE-NUMBERS ON A BUNDLE-BUNDLE-BOARD

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

Looking at four fingers held together two by two, we see four fingers, the essence. But, before school, children see what exists, bundles of twos in space, and two of them when counted in time. So, we may ask how mathematics may be taught to children if using their own two-dimensional bundle-numbers with units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by working with existence instead of listening to essence. Here we use the two core concepts of philosophical Existentialism holding that existence precedes essence. This will mean that counting precedes adding to bring outside totals inside by counting to be recounted and added. In this 'existence-math' approach, the mathematical essence is re-rooted as abstractions of outside existing examples instead of being defined as examples of inside abstractions. Hundred is a bundle-bundle, as four when recounted in 2s on a ten-by-ten BundleBundleBoard. The operations are icons created by counting processes and bundle-units make place values and carrying unneeded. Once counted, calculus and linearity are needed for next-to and on-top addition.

Keywords: mathematics education, arithmetic, early childhood, numeracy, existentialism

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as "ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all." Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' something. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world as multiplicity in space and repetition in time. In contrast, mathematics does not do so, its concepts are institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions.

It thus seems that with the English definition of numeracy the assessment must be carried out by persons seen as experts on the predicates competent and mathematics. In contrast, the American definition of numeracy allows laymen to judge themselves the actions carried out when working with numbering and numbers. Also, in their common history, England once colonized America, so we may wonder if the two different views are the views of a former colonizer and a former colonized.

To understand these differences and to enlighten and discuss the core of education formulated as ‘teach learners something’, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs and to discuss the ‘something’. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs and in the textbook-teacher-learner interaction. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs. Here, discussing may be a better word than enlightening since alle three areas contain conflicting theories.

Grand Theory looks at Mathematics Education

Within philosophy, a discussion between existentialism (Sartre 2007) and essentialism about the precedence of existence or essence has taken place since philosophy began in ancient Greece (Russell 1945). Here the ‘knowing’ sophists argued that to practice democracy we must tell nature from choice to avoid being patronized by choice masked as nature. In other words, we must be able to tell outside existence from its many chosen inside essences and especially the ones that have been institutionalized as the tradition and thus colonizes its outside existence (Habermas, 1981). Against this, the ‘better knowing’ philo-sophists argued that choice is nonexistent since everything physical is only imperfect examples of metaphysical essence as illustrated in Plato’s Cave allegory, and that essence is only accessible to philosophers educated at the Plato Academy.

This disagreement between sophists and philosophers about nature versus choice, and existence versus essence, runs through history. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. In the counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe’s line-organized office-preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the French Enlightenment republic’s post-structuralism inspired by Heidegger (1962) who argued that “P is Q” is a statement judging an outside existence, P, with an inside constructed essence-predicate that may be a preconceived prejudice, e.g., gossip (Gerede), and which therefore should be met with skepticism and be deconstructed. With ‘essence’ coming from Latin ‘esse’ meaning ‘being’, Heidegger gave four answers to the question “What is ‘is’?” pointing up, down, over, or nowhere: is an example of, is for example, is like, or is period. In mathematics, a function thus may be an example of a subset in a setproduct where first-component identity implies second-component identity, or a name for calculation containing both specified and unspecified numbers, or a number-language sentence including a subject and a verb and a predicate as in a word-language sentence, or simply what it is, a stand-by calculation awaiting a number, e.g., ‘3 * x’ in contrast to ‘3 * 5’.

Psychology (Skemp, 1971) has a controversy within constructivism where Vygotsky (1986) sees education as adapting to the institutionalized essence-regime by building ladders down from it to the learners’ proximal learning zones. Piaget (1969) replaces this top-down view with a bottom-up view inspired by American Grounded Theory (Glaser et al, 1967) allowing categories to emerge from concrete experiences. Siding with Piaget, Ausubel says that “The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly” (Ausubel, 1968, p. vi). Also siding with Piaget, Luhmann (1995) sees the brain as a closed self-referring psychic system inside a world of complexity from which it learns by itself reducing the complexity presented to it by another closed self-referring social system called education with its own reductions that it cannot export. Education thus means disturbing the students’ systems with outside complexity or hiding complexity if instead wanting uneducated students not able to reduce this, e.g., hiding that numbers occur differently in time and space. This conflict has sociological consequences. Presented top-down from above as examples of inside abstractions, concepts become hard to learn, which forces many learners to stop learning what is meaningless to them and to accept patronization by those who accept such meaninglessness. In contrast, bottom-up concepts grounded from below in the outside world are natural to learn for children through meeting and acting on the concrete examples that exemplify and root the concepts; and for teenagers since knowing the subject in a sentence creates automatic gossip-like learning.

Within Sociology, reviving the ancient Greek sophist skepticism towards patronization masking choice as nature, the Enlightenment created two democracies, one in North America still having its first republic, and one in France now having its fifth republic. In Germany, Weber (1930) was the first to theorize the increasing social goal-oriented rationalization that disenchants the world and create an ‘iron cage’ if carried to wide. Later, wanting to establish a third German Enlightenment democracy based upon communicative action and the convincing force of the better argument, Habermas (1981) accepts the role of science in rationalizing society, but warns against its instrumentalism spreading from the system to the life world to colonize it.

North American showed skepticism towards rationalist philosophy by developing American Pragmatism (Menand, 1997) leading to Symbolic Interactionism (Blumer, 1998) and Grounded Theory (Glaser et al, 1967).

Here, Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now” (p. 16).

Living together involves deciding upon which tasks to do individually and which to do collectively by creating rational institutions, “in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79).” However, Bauman warns against a possible ‘goal displacement’ where “The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).”

An example of a goal displacement may be institutionalizing a knowledge-regime (Foucault 1995) saying ‘The goal of mathematics education is to master mathematics”. By this self-reference, such a goal is meaningless. To master math cannot be the goal of math education, but it can be a means to reach the real end goal, to master Many. Which of course should be reached by different means if a goal displacement has changed institutionalized mathematics from a means to a goal itself colonizing any further road to mastery of Many and numeracy.

The debate on patronization beginning in ancient Greece between the philosophers and the sophists is still with us today between socialist top-down critical theorists and skeptical bottom-up existentialist theorists that are inspired by the French post-structuralist thinker Foucault saying

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

So, seeing mathematics education as an institution (Freudenthal, 1973), a Weberian viewpoint would ask if the set-concept is a rationalization of Many gone too far thus leaving Many disenchanted and leaving the learners in an iron cage. And a Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. So, as an institution, mathematics is a means, hence the word ‘mathematics’ should go from goal descriptions. If not, there is a risk that mathematics education will practice ‘the banality of evil’ (Arendt, 1963). So, sociology would recommend replacing the truth regime saying ‘the goal of teaching math is to learn mastery of math’ by the real end goal, mastery of Many, e.g., by uncovering and developing the existing mastery of Many children create through adaptation to Many before school.

In short, it may be time to show skepticism towards institutionalized mathematics and use difference research (Tarp, 2018) to look for differences that may make a difference.

We therefore ask the Cinderella question: If mastery of Many is the end goal, essence-math may be a means for some but not for all as indicated by the more than fifty years since the first International Congress on Mathematical Education was held in 1969, but are there different means also, e.g., an existence-math, that will make the prince dance? And if so, what is its content and what kind of mathematics does the prince learn during the dance?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that ‘1+1 = 2’. In contrast, a different existence-math occurs when asking a 3-year-old child “How many years next time?”. The answer is four with four fingers shown. But seeing the four fingers together two by two, the child reacts: “That is not four, that is two 2s”.

This observation shows that numbers occur both as bundle-numbers in space, 2s, and counting-numbers in time, two, that need to be together since asking, “What is 1+1?” leads to asking back “One what + one what?”. And asking, “What is 1s + 1s?” leads to asking back “How many 1s + how many 1s?”. And asking, “What is 1 + 1s” leads to asking back “What??”

So now we will look for a different existence-math curriculum built on the bundle-numbers with units as 2 3s and 4 5s that children create before school when adapting to nature’s multiplicity in space and repetition in time? To gather data about the possible content of an existence-math curriculum we now use sociological imagination to visit a classroom in a not so far decolonized future.

Grade one Class one in a Decolonized Future

The teacher: Welcome children, I am your math teacher, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So …

Showing a V-sign a child says: Mister teacher, here is one 1s in space, and here is also one 1s. Now we count them in time to see how many 1s we have by saying ‘one, two’. So, we have two 1s. But only until we add them as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. And together with this neighbor V-sign the total is one 2s plus two 1s which is one 4s, and not three 3s as you will say.

And, if I show you three 1s two times then I have shown you six 1s. Counting-numbers as two and three can be multiplied, but they cannot be added without units since two 1s + one 2s = 2 * 1s + 1 * 2s = 1 * 4s. So, we see that counting-numbers in time as 1 and 2 are different from bundle-numbers in space as 1s and 2s.

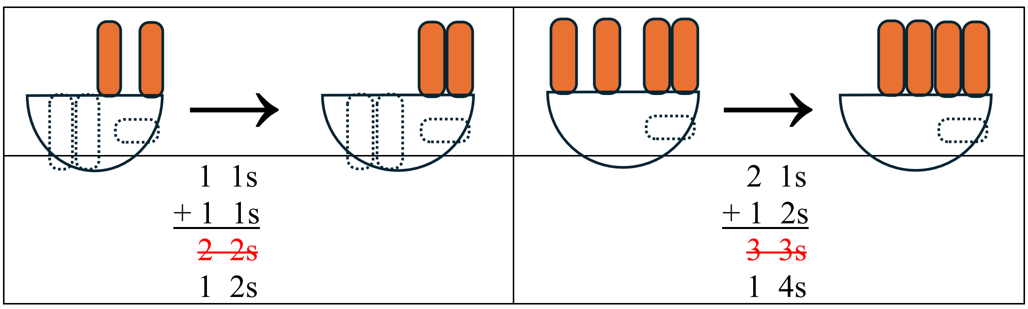


Figure 1. A closed and open V-sign show that 1+1 = 1; and that 1s+2s = 4s.

You would too if you ask a 3-year-old “How many years next time?” The child will say ‘four’ and show four fingers. Then, if you show the child the four fingers held together two by two, the child will protest: “That is not four, that is …”

Do you know what the child will say? You can ask other adults or students in primary or secondary school. They will all say ‘four’ because they all see the essence that they have been adapting to ever since they started in school where they meet the culture that they are loyal to. And, they don’t mind having their brain colonized by whatever essence the culture has chosen to follow, which my uncle who is a philosopher calls an essence regime. But, before meeting the school’s culture and its essence, children adapt to the nature where they grow up and here they meet and adapt to existence.

So, when we come to school, we don’t want to know what 1+1 as you would like to teach us. We at once ask “one what plus one what?” And 1+1 certainly cannot be 2 as the collapsing V-sign shows. Instead, we show you two 2s on one hand and one 3s on the other hand and ask you how to add them. Then you say “We cannot add if the units are different, but don’t worry for in math there are no units. And what you show me is four and three that add to seven.” Then we must ask you what unit you use. If you use 7 as a unit, we show you 0B 4 and 0B 3 7s that add to 0B 7 7s or 1B 0 7s. And if you use ten as the unit, we show you 0B 4 and 0B 3 tens that add to 0B 7 tens. My uncle says that, by its nature, multiplicity is multiplicative so there will always be units. And if the school uses ten as unit without telling us, it practices ten-centrism that shows disrespect to local units.

But there are many other units than tens. Looking into my right hand I see two fingers to the left and three fingers to the right. Then I bend one left finger and two right fingers. Now, 1 of 2 of the left are bent, but only 1 of 3 of the bent are left. So, you cannot just disrespect the units.

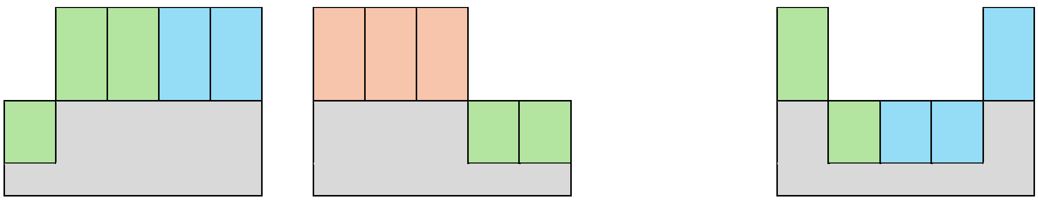


Figure 2. Two 2s in the left hand and one 3s in the right hand. And 1 left and 2 right bent.

So, let us now add two 2s and one 3s with respect to the units. We can see with our fingers that we can add them next-to as 5s and get one bundle and two 5s, 1B 2 5s. Or we can add them on-top as 3s or as 2s. If we add two 2s and one 3s as 3s we get two bundles and one 3s, 2B 1 3s. And if we add two 2s and one 3s as 2s we get 3 bundles and one 2s, 3B 1 2s.

But since we bundle in 2s, then the two 2s is one bundle of 2s or one Bundle of Bundles, 1 BB. So, the three bundles now are one bundle of bundles and one bundle and one unbundled, 1BB 1 B 1 2s.

II II III → IIIIIII   II II III → III III I   II II III → II II II I → II•II II I

We can also show it with snap cubes on a table or on a ten-by-ten BundleBundle Board, a BBBoard.

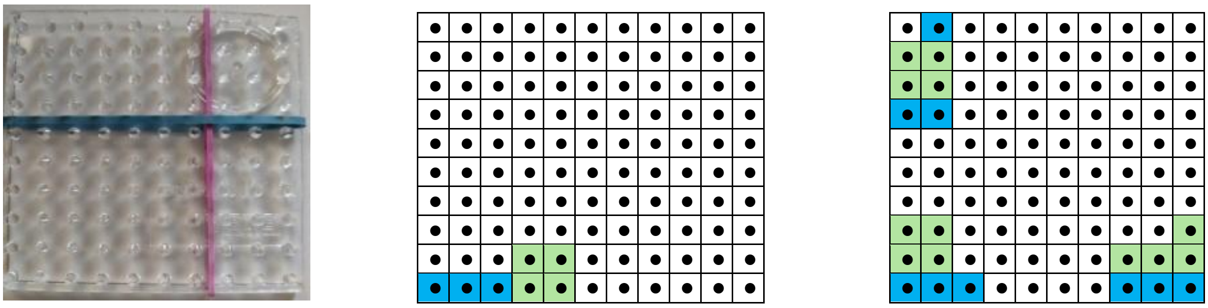


Figure 3. BundleBundle Boards with six 7s; and one 3s and two 2s added next-to as 5s or on-top

Therefore, mister teacher, please forget to add your line numbers without units. Instead, please help us add the bundle-numbers with units we bring to school, as two 3s and four 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on the BBBoard. If we add them next-to we add tiles, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says this is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school, instead of being colonized with your line numbers without units.

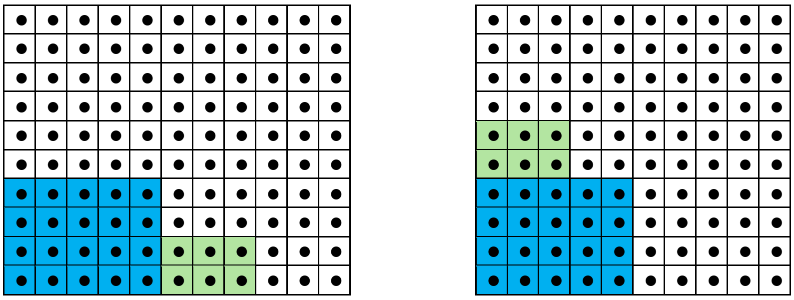


Figure 4. BundleBundle Boards with four 5s and three 2s added next-to as 8s or on-top as 5s

We know that you want to bundle in tens, and in ten-tens, and in ten-ten tens, but we like to bundle also in 2s, in 3s, in 4s, and in half-tens that we call half-bundles. You only write 47 and say '4ten7' when you should write 4B 7 and say '4 bundles with ten per bundle and 7 unbundled ones'.

You say that 8 divided by 2 is 8 split in 2 parts. We agree that 1 8s split in 2 gives 2 4s, but we also say that 1 8s split in 2s gives 4 2s. So, to us, 8 divided by 2 mostly means 8 counted in 2s.

Figure 5. 1 8s split in 2 gives 2 4s, and 1 8s split in 2s gives 4 2s.

You cannot split 9 in 2 parts, but we can count 9 in 2s as 4 bundles and 1 unbundled that becomes a decimal, 9 = 4B 1 2s. Or, with negative less-numbers in space or taken-away numbers in time, we get 5 bundles less 1 in space, and 5 bundles with 1 taken away in time, so here 9 = 5B -1 2s.

Figure 6. 1 9s split in 2s gives 9 = 4B1 2s, or 9 = 5B-1 2s

We know that you have not been taught this, and that the textbook doesn't teach it. But don't worry, we will teach you the way we learned to master Many in preschool.

Or better, instead of you colonizing our way let us find out together what math may grow from our bundle-numbers with units. My uncle uses the word existentialism when we let existence come before essence. So let us call our math for existence-math and your math for essence-math.

And, in existence-math, counting the totals must come before we later add them.

F01. Counting Totals in Bundles and Bundles-of-Bundles

Now, let us begin with bundle-counting the fingers on our hands. You only see the essence, five, but we see all the ways the five fingers may exist.

• Bundle-counted in 5s, a total of five fingers may exist as five 1s, T = 0B 5 5s, or as one bundle of 5s, T = 1B 0 5s.

I I I I I, or IIIII

Bundle-counted in 5s, a total of ten fingers may exist as one bundle and five 1s, T = 1B 5 5s, or as two bundles, T = 2B 0 5s or as two half-bundles tens, T = 2 ½B 0 tens.

IIIII I I I I I, or IIIII IIIII

• Bundle-counted in 4s, a total of five fingers may exist as five 1s, T = 0B 5 4s, or as one bundle and one 1s, T = 1B 1 4s, or as two bundles less three, T = 2B -3 4s. Here, we use the words overload for 0B 5, and underload for 2B -3.

I I I I I, or IIII I, or IIII I~~III~~

Bundle-counted in 4s, a total of ten fingers may exist as one bundle and six 1s, T = 1B 6 4s, or as two bundles and two ones, T = 2B 2 4s, or as three bundles less two ones, T = 3B -2 4s,

IIII I I I I I I , or IIII IIII II , or IIII IIII II~~II~~

• Bundle-counted in 3s, a total of five fingers may exist as five 1s, T = 0B 5 3s, or as one bundle and 2 unbundled, T = 1B 2 3s, or as two bundles less 1, T = 2B -1 3s.

I I I I I , or III I I , or III II~~I~~

Bundle-counted in 3s, a total of ten fingers may exist as one bundle and seven 1s, T = 1B 7 3s, or as two bundles and four 1s, T = 2B 4 3s, or as three bundles and one 1s, T = 3B 1 3s, or as four bundles less two 1s, T = 4B -2 3s, or as one bundle of bundles, and no bundles, and one 1s, T = 1BB 0B 1, since three bundles with 3 per bundle is one bundle of 3-bundles, or one BundleBundle of 3s, 1BB 3s.

III I I I I I I I , or III III I I I I , or III III III I , or III III III I~~II~~, or III•III•III I

Here, power shows how many 1s the is in one 3-BundleBundle, 3^2 = 9. And, the logarithm shows how many times a 3-Bundle has been bundled, log3(9) = 2.

• Bundle-counted in 2s, a total of five fingers may exist as five 1s, T = 0B 5 2s, or as one bundle and 3 unbundled, T = 1B 3 2s, or as two bundles and 1 unbundled, T = 2B 1 2s , or as three bundles less 1, T = 3B -1 2s , or as one bundle of bundles, and no bundles, and one 1s, T = 1BB 0B 1, since two bundles with 2 per bundle is one bundle of 2-bundles, or one BundleBundle of 2s, 1BB 2s.

I I I I I , or II I I I , or II II I , or II II II̶ , or II•II I

Bundle-counted in 2s, a total of ten fingers may exist as one bundle and eight 1s, T = 1B 8 2s, or as two bundles and six 1s, T = 2B 6 2s, or as three bundles and four 1s, T = 3B 4 2s, or as four bundles and two 1s, T = 4B 2 2s, or as five bundles and no 1s, T = 5B 0 2s, or as six bundles less two 1s, T = 6B -2 2s,

II I I I I I I I I , or II II I I I I I I , or II II II I I I I , or II II II II II, or II II II II II , or II II II II II II̶

Here, five 2s may also be bundled in 2s. First as 1 BundleBundle 3Bundles 0 2s, T = 1BB 3B 0 2s, then as 2 BundleBundle 1Bundles 0 2s, T = 2BB 1B 0 2s, and then as 1BundleBundleBundle 0BundleBundle 1Bundle 0 2s, T = 1BBB 0 BB 1B 0 2s.

II II II II II , II•II II II II , II•II II•II II , II•II II•II II

Again, power shows how many 1s there are in a 2-BB, 2^2 = 4, and in a 2-BBB, 2^3 = 8. And, the logarithm shows how many times a 2-Bundle has been bundled, log2(4) = 2, and log2(8) = 3.

• Bundle-counted in tens, a total of five fingers may exist as five 1s, T = 0B 5 tens, or as one half-bundle, T = ½B 0 tens. And, a total of ten fingers will exist as one bundle and no unbundled, T = 1B 0 tens, or as two half-bundles, T = 2½B 0 tens. And, a total of twelve fingers (we include the two arms as fingers) will exist as one bundle and two unbundled, T = 1B 2 tens, or as two half-bundles and two unbundled, T = 2½B 2 tens.

Did you know that eleven and twelve come from the Vikings that counted 'one left' and 'twe left'? And that twenty comes from the Viking word for two tens, 'twe ti".

You use the words ten, hundred, and thousand for B, BB, BBB, but you don't have a special word for BBBB. You just call it ten thousand, but in China they call it one Wan, maybe because an army of hundred hundreds will win.

If we count in tens, forty-seven is 4B 7, and 547 is 5BB 4B 7.

T = 4tens & 7 = 4B 7 tens = 47 T = 5 hundreds & 4 tens 7 = 5BB 4B 7 tens = 547

So, 547 is not one number but three numberings. You use a place value system, but, with bundles as the units, we do not need that.

Here we have counted totals in space, but we also use bundles as the unit when we count totals in time. If we count our five fingers in 3s we cannot say ‘1, 2, 3’, and so on since 1 is not 1 3s. Instead, it is 0 bundle 1 3s, so we count: 0B 1, 0B 2, 0B 3 or 1B 0, 1B 1, 1B 2 or 2B-1 3s. Or we may count: 1B -2, 1B -1, 1B 0, 2B -2, 2B -1 3s

F02. Digits and Operations are Icons for Degrees of Many and how we Act with them

With sticks we see that five 1s may be bundled as one 5s that may be rearranged as one 5-icon with five sticks. The other digits may also be seen as icons with the number of sticks they represent, where zero is a looking glass finding nothing. We don’t need an icon for ten since here the total is 1B 0 if we count in tens.

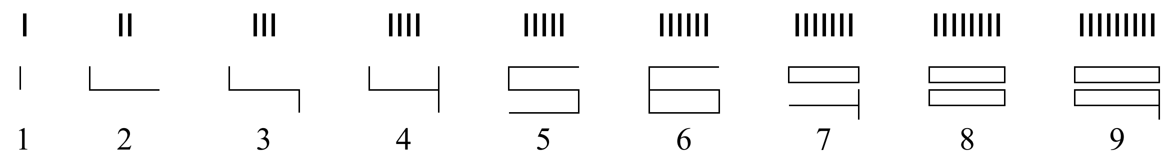


Figure 7. The bundles may be rearranged as icons with as many sticks or strokes as they represent

The calculations are icons also.

When counting eight fingers in 2s, division is a ‘push-away’ icon for a broom so that 8/2 means ‘from 8 push-away 2s’. And the calculator can predict the result, 8/2 = 4.

I I I I I I I I → I I I I I I II → I I I I II II → I I II II II → II II II II

If we use snap cubes instead, after pushing-away bundles we can ‘push-back-to-stack’ the four 2s, which makes multiplication a ‘lift icon’ predicting the result of ‘4 times stacking 2s’, 4 x 2 = 8.

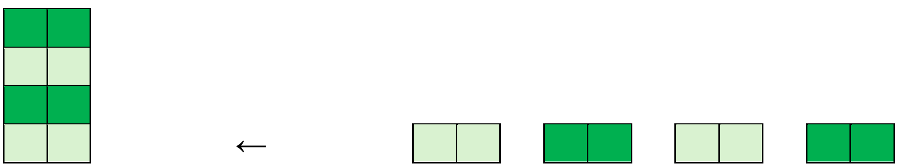


Figure 8. After counting 8 in 2s by pushing-away 2s, the four 2s are pushed-back-to-stack as 4x2

Since 8/2 = 4, 8 = 4 x 2 may be rewritten as ‘8 = (8/2) x 2’ saying that 8 contains 8/2 of 2s. With T and B for Total and Bundle we get a ‘recount formula’ that changes units.

T = (T/B) x B or T = (T/B) * B.

My uncle calls it a very important formula for proportionality or linearity. Well, let’s look at minus.

How many fingers do I have here on my right hand? You say five, but we say ‘ALL’.

Now I bend my thumb down. How many fingers now? You say four, but we say ‘ALL minus 1.

Now I bend my finger up. How many fingers now? You say five, but we say ‘ALL minus 1 plus 1’.

You learn it by bending the finger down and up several times while saying “ALL, is ALL minus one, plus one. ALL, is ALL minus one, plus one. We now write it down as ‘A = (A – 1) + 1’.

Her we have shortened All to A, and we use a rope as an icon for pulling away, which is cancelled by a stroke when we pull back. So here we meet minus and plus as icons also.

Bending two or three fingers down and up we see that it also works with 2 fingers, and with 3 fingers, and with other bundles as well so we can now write down a ‘reunite formula’,

A = (A – B) + B.

F03. Recounting and Reuniting Solve Equations by Reversing Reversed Calculations

The recount formula, T = (T/B) * B, can solve equations as ‘*u* * 2 = 8’ asking ‘How many 2s in 8?’ to which the answer, *u*, of course is found by recounting 8 in 2s as 8 = (8/2) * 2, giving *u* = 8/2.

My uncle says this is correct since 8/2 is defined as the number *u* that multiplied with 2 gives eight. And, on our hands we see that 8 contains the 4 2s that we pushed away when we counted 8 in 2s.

This we find by moving the known number across from the left side’s uniting process that can’t be done because of the unknown number, *u*, to the opposite right side’s splitting process.

Forward		Reversed	Forward		Reversed
u * 2	=	8	*u* + 2	=	5
* 2	↑ ↓	/2	+ 2	↑ ↓	– 2
u	=	8/2	*u*	=	5 – 2

*Figure 9. Solving the equations u * 2 = 8 and u + 2 = 5: ‘To opposite side with opposite sign’*

My uncle has warned us against a ‘same to both sides’ lever-method you might want to teach us. We don’t know what he is talking about, we just solve equations as ‘3 * *u* + 2 = 14’ with a song

3 * *u* + 2	= 14	Equations are the best we know; they’re solved by isolation.
(3 * *u*) + 2	= 14	But first the bracket must be placed, around multiplication.
3 * *u*	= 14 – 2	We change the sign and take away, so only *u* itself will stay.
u	= (14 – 2)/3	We just keep on moving, we never give up.
u	= 4	So feed us equations, we don’t want to stop.

Figure 10. Solving equations with the ‘to opposite side with opposite sign’ method

The reunite formula can solve equations as ‘5 = *u* + 2’ where *u* is the unknown number. We just bend two fingers and say 5 = (5 – 2) + 2, so *u* = 5 – 2. So, we just move plus 2 across as minus 2. My uncle says this is correct since 5 – 2 is defined as the number *u* that added to two gives five.

Here I have ALL my fingers on the right hand. Now I bend two fingers down and get ALL-2. Which I now bent down also, so down I now have 2 + (ALL – 2) which is ALL. So, to get ALL down instead of up, I just added the difference, ALL – 2.

So, my number of down-fingers changed by adding the end-number minus the start-number, which is called the change of A, or ΔA if we use a triangle for change. So here, ΔA = end – start = A – 2.

Now we have a change formula, A = 2 + (A – 2) = 2 + ΔA, that works with all bundles:

A = B + ΔA.

Now, if I know my start number and my change numbers, then I can find the end number by adding all the change-differences: Here, the start number is 3, and the change differences are 1, 2, 3, 2, 1 that add up to 9. So, the end number is, 3 + 9 = 12.

By instead adding the change numbers one by one, we can find all the middle numbers

3, and 3+1 = 4, and 4+2 = 6, and 6+3 = 9, and 9+2 = 11, and 11 + 1 = 12.

Or, if we know the middle numbers then we can find the change numbers as the differences

12 – 11 = 1, and 11 – 9 = 2, and 9 – 6 = 3, and 6 – 4 = 2, and 4 – 3 = 1,

Now we add all these differences and get (12 – 11) + (11 – 9) + (9 – 6) + (6 – 4) + (4 – 3) = 12 – 3.

T		**ΔT = T2 – T1**	**Sum of ΔT**	**Sum of ΔT = T6 – T1**
T1 =	**3**			
T2 =	**4**	4 – 3 = 1	1	
T3 =	**6**	6 – 4 = 2	1 + 2 = 3	
T4 =	**9**	9 – 6 = 3	3 + 3 = 6	
T5 =	**11**	11 – 9 = 2	6 + 2 = 8	
T6 =	**12**	12 – 11 = 1	8 + 1 = 9 =	**12 - 3**

Figure 11. Add Differences: (T6 – T5) + (T5 – T4) + (T4 – T3) + (T3 – T2) + (T2 – T1) = T6 – T1

Here we see that when adding many differences, all the middle terms disappear, so we only have to add one difference between the end-number and the start-number. This we will later use to solve difference equations in calculus.

F04. Recounting from Icons to Tens makes us Understand Times Tables better

Solving equations is an example of recounting from tens to icon-units. We will now look at the opposite, how to recount from icon-units to ten-units when asking, e.g., 6 7s is how many tens?

On a BundleBundleBoard we see 6 * 7 as 6 7s that we can both see and feel. We see that 6 also occurs as ½B 1 tens and as 1B -4 tens, and that 7 also occurs as ½B 2 tens and as 1B -3 tens.

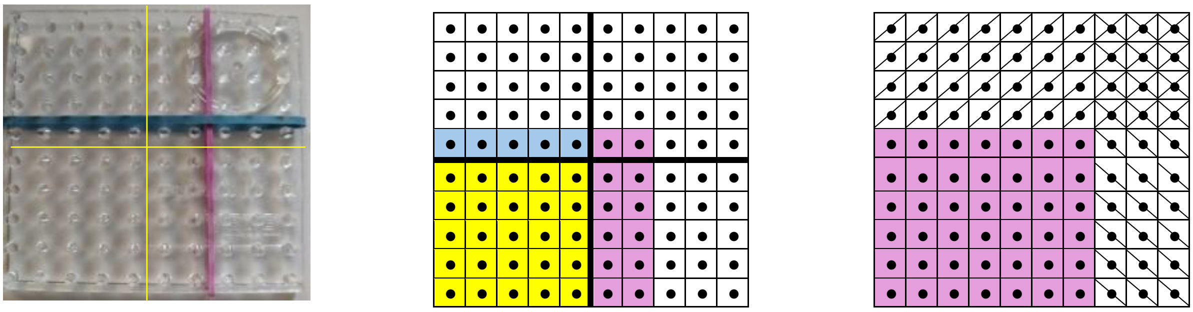


*Figure 12. On our ten-by-ten BundleBundle Board we see 6*7 as 6 7s; and 6 as ½B 1 tens and as 1B -4 tens, and 7 as ½B 2 tens and as 1B -3 tens.*

● Method 01. Here, we let the fingers wander up the board touching the half-bundles while saying

"½B 2 or 7, 1B 4 or 1ten4 or 14, 1B ½B 6 or 1B 11 or 2B 1 or 2ten1 or 21, 2B 8 or 2ten8 or 28, 2B ½B 1B or 3B ½B or 3B5 or 3ten 5 or 35, 3B 1B 2 or 4B 2 or 4ten2 or 42."

● Method 02. We see that 7 = ½B 2, so 6 * 7 = 6 * ½B 2 = 3B 12 = 4B 2 = 4ten2 = 42

● Method 03. We see that 6 * 7 is (½B + 1) * (½B + 2), which is

= 2B ½B (in the lower left corner) + 1 * ½B (over) + 2 * ½B (next-to) + 2 (in the right corner)

= 3B + 1B + 2 = 4B2 = 4ten2 = 42.

So, while pointing or feeling we say "1, 2, 3, 4 Bundles and 2, so 4B2, or 4ten2.

With fingers, we show 6 as 1 finger up to the left, and 7 as 2 fingers up to the right. We don't show the half-bundles. Then we add and multiply the up-fingers 1 and 2 into 3 and 2. Finally. We say "5, +3 is 8 ½bundles and 2, so 4B2, so 4ten2, so 42".

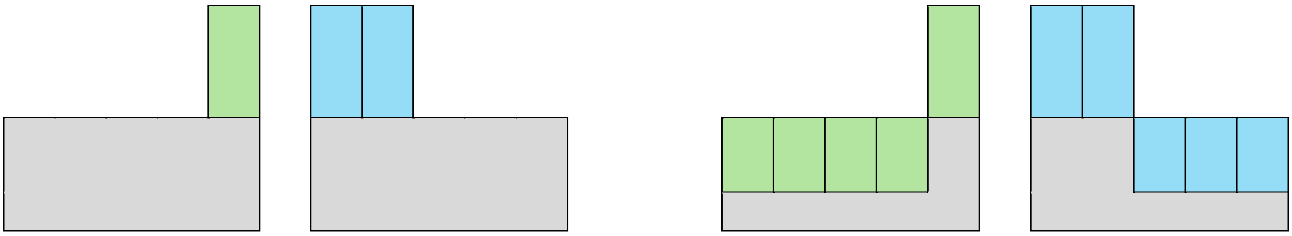


*Figure 13. Six sevens or 6*7 recounted in tens by adding and multiplying fingers.*

● Method 04. We call this the 'less-method' since we pull-away the unneeded. My uncle calls it the Algebra method showing that minus times minus must be plus, and showing a FOIL method, First, Outside, Inside, Last.

6 * 7 = (B – 4) * (B – 3) = 10B (all) – 3B(next-to) – 4B(on-top) + 4 3s (added since pulled away twice) = (10 – 3 – 4)B 12 = 3B 12 = 4B 2 = 4ten 2 = 42. But (10 – 3 – 4) = (5 – 3) + (5 – 4) = 2 + 1 fingers where 7 is 3 fingers and 6 is 4 fingers down. So, with fingers, we show 6 as 1 finger up to the left, and 7 as 2 fingers up to the right. The up fingers we add as before and the down fingers we multiply. This gives 1+2 = 3 and 3 * 4 = 12. So, the result is 3B 12 or 4B 2 or 4ten 2 or 42.

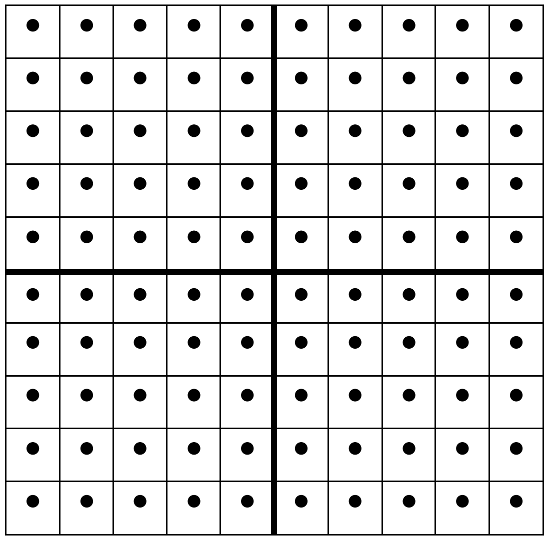
TT1, the lower table from 2-5. Use fingers, arms and toes. The 2-table 2 * 2 = 2 2s = 4 3 * 2 = 3 2s = 6 = ½B1 4 * 2 = 4 2s = 8 = ½B3 = 2 * ½B-1 = B-2 = 8 5 * 2 = 2 5s = 2 * ½B = 1B0 = 10. The 3-table 2 * 3 = 2 3s = 6 = ½B1 3 * 3 = 3 3s = 9 = ½B4 = 1B-1 4 * 3 = 2 * ½B + 2 * arm = 1B2 = 12 (twelve = 'two left' in Viking language) 5 * 3 = 3 5s = 3 * ½B = 1B5 = 15. The 4-table 2 * 4 = 2 4s = 2 * ½B1 = 1B-2 = 8 3 * 4 = 3 4s = 2 * ½B + 2 * arm = 1B2 = 12 4 * 4 = 4 4s = 4 * ½B-1 = 2B-4 = 1B6 = 16 = fingers and toes less the outer ones 5 * 4 = 4 5s = 4 * ½B = 2B0 = 20 (twenty = 'two tens' in Viking language)	
TT2, the middle table Look at a BBBoard The full 5-table 2 * 5 = 2 5s = 2 * ½B0 = 1B0 = 10 3 * 5 = 3 5s = 3 * ½B0 = 1B5 = 15 4 * 5 = 4 5s = 4 * ½B0 = 2B0 = 20 5 * 5 = 5 5s = 5 * ½B0 = 2B5 = 25 6 * 5 = 6 5s = 6 * ½B0 = 3B0 = 30 7 * 5 = 7 5s = 7 * ½B0 = 3B5 = 35 8 * 5 = 8 5s = 8 * ½B0 = 4B0 = 40 9 * 5 = 9 5s = 9 * ½B0 = 4B5 = 45 ten * 5 = 5 tens = 5B0 = 50. The 6-table 2 * 6 = 2 * ½B1 = 1B2 = 12 3 * 6 = 3 * ½B1 = 1B(5+3) = 1B8 = 18 4 * 6 = 4 * ½B1 = 2B4 = 24 5 * 6 = 6 5s = 6 * ½B0 = 3B0 = 30 The 7-table Etc.	The 6-table 2 * 6 = 2 * ½B1 = 1B2 = 12 TT3, the upper table The full 6-table 6 * 6 = 6 6s = 6 * ½B1 = 3B6 = 36 7 * 6 = 6 7s = 6 * ½B2 = 3B12 = 4B2 = 42 8 * 6 = 8 6s = 8 * ½B1 = 4B8 = 48 9 * 6 = 6 9s = 6 * ½B4 = 3B24 = 5B4 ten * 6 = 6 tens = 6B0 = 60. Or, 6 * 6 = (B-4) * (B-4) = BB-8B+16 = 2B16 = 3B6 = 36, etc. The full 7-table 6 * 7 = 6 7s = 6 * ½B2 = 3B12 = 4B2 = 42 7 * 7 = 7 7s = 7 * ½B2 = 3B(5+14) = 3B19 = 4B9 = 49 8 * 7 = 7 7s = 8 * ½B2 = 4B16 = 5B6 = 56 9 * 7 = 9 7s = 9 * ½B2 = 4B(5+18) = 4B23 = 6B3 = 49 ten * 7 = 7 tens = 7B0 = 70. Or, 6 * 7 = (B-4) * (B-3) = BB-7B+12 = 3B12 = 4B2 = 42, etc. The full 8-table Etc.

Figure 14. To learn the times table we use our finger to travel on a BundleBundleBoard

● Method 05. (Connected vessels). On our BBBoard we see two vessels next to each other, 6 7s to the left and 0 3s to the right. We then move the top 7s to the right. It is not enough since now we have 5 7s to the left and only 2B1 3 to the right. Moving one more 7s to the right will do the job since now we have 4 7s to the left and 4B2 3s to the right. So, the answer is 4B2 tens or 4ten2 or 42.

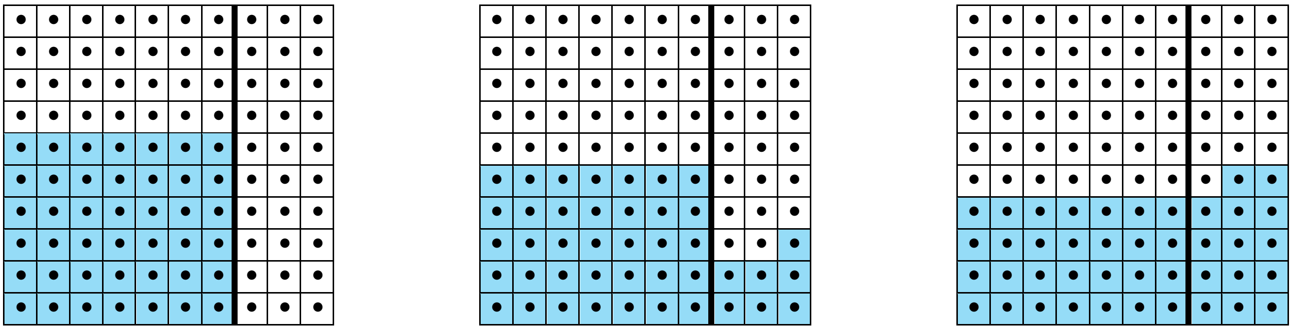


Figure 15. Six 7s recounted to tens by leveling two vessels. The width goes up, the height goes down

F05. Changing from Icons to Icons

Asking “Two 3s is how many 4s?’ you see an equation, ‘2 * 3 = *u* * 4’, that you use ‘do the same on both sides to solve as *u* = (2 * 3) * 1/4 = 6/4. We see two 3s recounted in 4s carried out on a BBBoard and tested or predicted by a calculator where entering, 2 * 3/4. The answer, 1.more’, means ‘1 bundle and some unbundled’ that are found be pulling away the stack, 1 4s, from the 2 3s, which gives 2 unbundled. So, the calculator predicts that 2 3s may be recounted as 1B2 4s, which is also observed on a BBBoard.

Enter	Read	Predict
2 * 3/4	1.more	2 * 3 = 1B 2 4s
2 * 3 – 1 * 4	2	

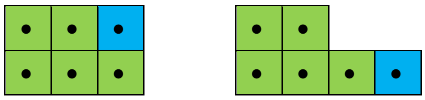


Figure 16. A calculator predicts the result of recounting 2 3s in 4s as 1B2 4s

F06. BundleBundles are squares

On our Bundle-Bundle-Board we see that all the bundle-bundles are squares. And we see that two 2s need two more 2s and one corner to become three 3s, which again need two more 3s and one corner to become four 4s, which again need two more 4s and one corner to become five 5s. So, going up we see that 1BB 3s = 1BB 2B 1 2s = 4 + 4 + 1 = 9, and 1BB 4s = 1BB 2B 1 3s = 9 + 6 + 1 = 16, and 1BB 5s = 1BB 2B 1 4s = 16 + 8 + 1 = 25. So, 1, 4, 9, 16, and 25 are the first squares.

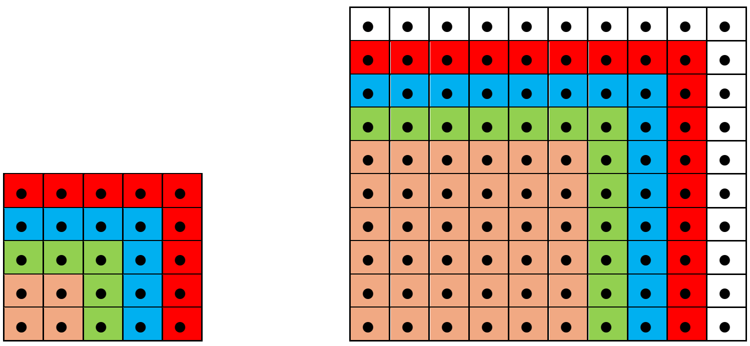


Figure 17. BundleBundles as squares where 1BB 2B 1 or 1BB -2B 1 gives the next or previous

Going down form ten instead, we see that. 1 BB 9s = 1BB -2B 1 tens = 100 – 20 + 1 = 81 where 1 is added since it is pulled away twice. And that 1 BB 8s = 1BB -4B 4 tens = 100 – 40 + 4 = 64, and that 1 BB 7s = 1BB -6B 9 tens = 100 – 60 + 9 = 49, and that 1 BB 6s = 1BB -8B 16 tens = 100 – 80 + 16 = 36. The first 9 squares thus are 1, 4, 9, 16, 25, 36, 49, 64, and 81

We see that, of course, 1 and 9 have the same last digit, as has 2 and 8, and 3 and 7, and 4 and 6.

F07. Solving Quadratic Equations

Splitting a BundleBundleBoard in two squares and two stacks solves quadratic equations.

If the sides are split in an unknown number *u* plus 3 then the board contains a *u*-square and a 3-square and two 3 * *u* stacks, so if *u* squared + 6 * *u* + 8 is zero, then the (*u* + 3) square only contains 9 – 8 = 1 which is a 1-square.

This means that *u* can be -2 or -4 since both (-2 + 3) and (-4 + 3) give 1 when squared.

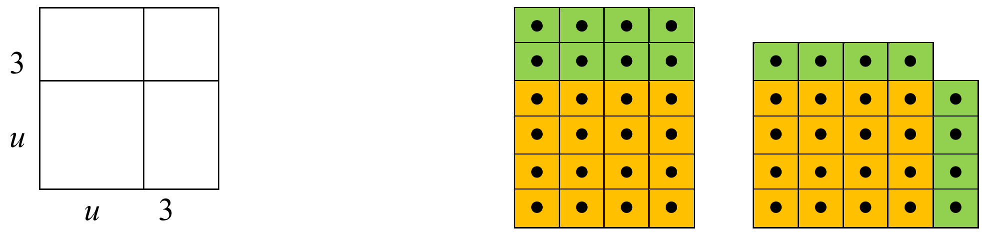


*Figure 18. A u+3 square solves the equation u^2 + 6*u +8 = 0. A 6 4s stack is being squared*

F08. Squaring Stacks

A rectangular stack as 6 4s may be changed into a square by moving half the surplus from the top to the side to make it close to the so-called square root that is the side of the wanted square.

Here the top surplus is 6 – 4 = 2 4s, so we move half of 2 4s from the top to the side. But that is

too much since we also need to fill the top right corner with a 4 * t stack from the side and t * 4 stack from the top. This gives 2 * 4 * t = 1, or 8 * t = 1 = (1/8) * 8, so t = 1/8, so T = (5 – 1/8) * (5 – 1/8) = 4.88 * 4.88. Guessing 4.88 as the square root is close to the calculator, $\sqrt{6 * 4}$ = 4.89….

F09. Using Per-Numbers when Trading

We see that in shops we can buy apples with money. If the sign says 4$ per 5kg then we have the per-number 4$ / 5kg, or 4$ for each 5kg, or 4/5 $/kg.

So, if we want to have 20kg we simply recount them in 5s to see how many times we must pay 4$:

20kg = (20/5) * 5kg = (20/5) * 4$ = 16$.

Likewise, 20$ may be recounted in 4s to see how many times we get 5kg.

20$ = (20/4) * 4$ = (20/4) * 5kg = 25kg

F10. Per-Numbers become Fractions with like Units

If my share of a 5$ bet is 2$ then from a gain I will get 2$ per 5$ or 2$/5$ or 2/5. So, we see that per-numbers become fractions with like units. To find 2/5 of 30$ I simply recount 30 in 5s to see how many times I get 2$: 30$ = (30/5) * 5$ gives me (30/5) * 2$ = 12$.

The fraction 2/5 can occur with other units as (2 3s)/ (5 3s) = (2 * 3)/(5 * 3) = 6/15. Likewise with the fraction 8/20 = (2 * 4)/(5 * 4) = (2 4s)/(5 4s) = 2/5.

My uncle says that this is called to expand or reduce a fraction which we will understand if we later learn about equivalence classes in set products. We don't know what he is talking about so we asked him why we can't understand fractions as special per-numbers. That is because per-numbers do not exist in mathematics, he said. That we really don't understand since per-numbers are the same as bundle-numbers. So maybe they also have no bundle-numbers in mathematics? But then, whys shall we learn it, because without bundle-numbers there are no units?

F11. Mutual Recounting in a stack gives Trigonometry

A stack has two internal lines called diagonals. One slopes up, the other down.

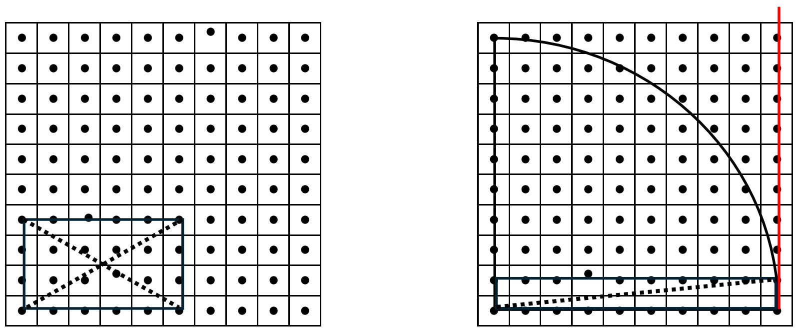


Figure 19. A BBBoard with two internal diagonals and a circle split in small triangles

To find out how steep it is we can recount the up-number in the out-number: up = (up/out) * out. Here the per-number, up/out, is called the line's slope or tangent (Angle) number. So, with a 3 per 5 slope, a total of 20 out-meters out will give (20/5) * 5out = (20/5) * 3up = 12 up-meters.

In a circle with radius 1, half the length is called pi. In the beginning the circle follows its tangent, the line that shows its direction if it was not to hold a constant distance to its center. So here the circle and its tangent are almost of the same length. So, pi = n * (tan(180/n)) for n large. And my calculator shows that 1000 * tangent (180/1000) = 3.1416. So, pi must be close to 3.1416.

F12. Triangles on a BundleBundleBoard

Now I will look at totals with other forms than rectangles to see how much they fill. My uncle says that we should use the word area instead of fill-number, so we will do that.

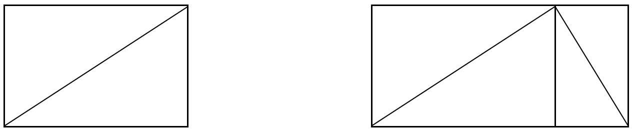


Figure 20. A triangle has half the area of the surrounding stack

First, we see that a diagonal splits a stack in two right triangles with half the area to each. Then we see that any triangle can be split into two right triangles surrounded by a stack. So again, the triangle has half the area of the surrounding stack. Now we look at triangles on a BBBoard.

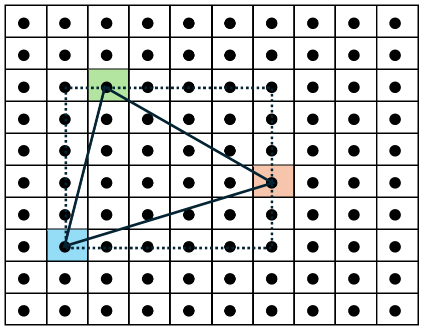


Figure 21. A calculator predicts the result of recounting 2 3s in 4s as 1B2 4s,

We place pearl A on the 2out and 3up (2,3) dot in the upper right corner of a 3 2s stack. Pearl B we place on the (3,8)-dot, and pearl C on the (7,5)-dot. Now we want to find the length and the angles and the area in the triangle ABC that is surrounded by a 5 5s stack.

The AB line is diagonal in a 5 1s stack with a 5/1 diagonal angle. On a calculator, the tangent button tells that this is 79 degrees, so there are 90 – 79 = 11 degrees to the other angle. We find the length of the diagonal as $\sqrt{26}$ = 5.1 by adding the squares of the stack's sides, 5^2 + 1^2 = 26. My uncle call this the Pythagoras rule. And the area we find as half of 5 * 1, i.e., 2½.

The BC line is diagonal in a 3 4s stack with a 3/4 diagonal angle, or 37 degrees with the tangent button, and 90 – 37 = 53 degrees to the other angle. We find the length of the diagonal as $\sqrt{25}$ = 5. And we find the area as half of 3 * 4, i.e., 6. The AC line is diagonal in a 2 5s stack with a 2/5 diagonal angle, or 22 degrees with the tangent button, and 90 – 22 = 68 degrees to the other angle. We find the length of the diagonal as $\sqrt{29}$ = 5.4. And we find the area as half of 2 * 5, i.e., 5.

To find the area of the ABC triangle, we begin with the full area of the surrounding 5 5s stack where 5 * 5 = 25. Then we pull away the three outer right triangles and get 25 – 5 – 6 – 2½ = 11½.

To find the angles in the triangle ABC we begin with 90 degrees for A and 180 degrees for B and C. Then we pull away the two neighbor angles in the outer right triangles and get: A = 90 – 22 – 11 = 57 degrees, and B = 180 – 79 – 37 = 64 degrees, and C = 180 – 68 – 53 = 59 degrees. Then, we test the results by adding the three angles: 57 + 64 + 59 = 180.

Finally, we draw this triangle on a squared paper to test the answers by measuring instead of calculating.

F13. Trips on a BBBoard

Now I will look at my BBBoard, not as a space-board with fixed forms, but as a time-board where I can change place with different trips. I will travel on a line to see when it meets other lines.

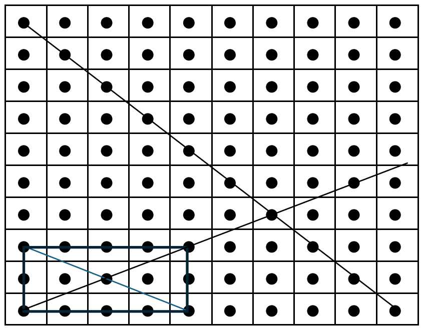


Figure 22. A BBBoard as a time-board where a trip from (0,0) through (4,2) meet another line

On a space-board, the first dot is number 1, but on a time-board it is number 0 since I have not yet changed place. Now, a 2 4s stack will be called a 4x2 box, where 4 is the run- or out-number, and 2 is the rise- or up-number. In a 4x2 box, the two inside diagonals slope up 2 per 4, 2/4, or down -2 per 4, -2/4. Its angle can be found on a calculator to 26.6 degrees up or down.

If we begin at the (0,0)-dot, a 2 per 4, 2/4, trip will end at the (4,2)-dot. We now continue in the same direction to an unknown (x,y)-dot. The angle hasn't changed so we have that

$y/x = 2/4$, or $y = 2/4 * x$, or $y = ½ * x$.

This formula is called the line's equation, and it can find one number if the other is known. If the x-number is 9, then the y-number is

$y = ½ * x = ½ * 9 = 4.5$.

And if the y-number is 4, then the x-number is found in the equation $4 = ½ * x$ giving $x = 4 * 2 = 8$.

Another line goes from the (0,9)-dot to the (9,0)-dot. Inside its 9x9 box the diagonal slopes -9/9 or -1/1. So, after x steps y have decreased to $y = 9 – x$. The lines then meet where $y = ½ * x = 9 – x$. Here, we change the unit by recounting x in 2s as $x = (x/2) * 2 = 2 * u$ with $u = x/2$.

Now, $½ * 2 * u = 9 – 2 * u$, or $u = 9 – 2 * u$, or $u + 2 * u = 9$, or $3 * u = 9$, or $u = 9/3 = 3$.

This gives $u = x/2 = 3$, or $x = 2 * 3 = 6$. Here, $y = 9 – 6 = 3$, so the two lines meet at the (6,3)-dot.

F14. Adding On-top or Next-to

Once counted and recounted, totals may be added on-top or next-to. To add 2 3s and 4 5s on-top as 5s, we must first recount 2 3s to 1B1 5s to get the sum is 5B1 5s. To add them as 3s, we must first recount 4 5 to 6B2 3s to get the sum 8B2 3s. Adding 2 3s and 4 5s next-to as 8s we get 3B2 8s by adding the areas, which my uncle calls integral calculus where we multiply before we add.

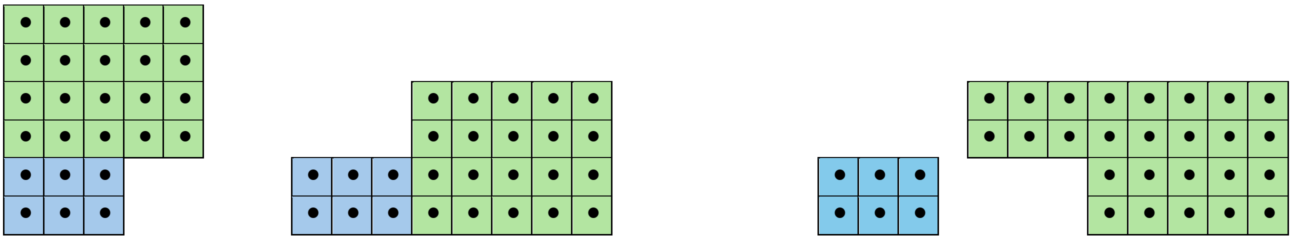


Figure 23. Two 3s and four 5s added on-top or next-to. And the reverse question, 2 3s + ? 5s = 4 8s

Now we ask ‘2 3s and how many 5s total 4 8s?’. So, we pull away the 2 3s and we recount the rest in 5s. So subtraction comes before division, which my uncle calls differential calculus used to calculate changes as per-numbers. And here the n-square grows with 2 * n almost as we saw before.

F15. Adding Squares

To add squares, we take four copies of a 2 3s stack and place them as a 5-square. Inside this square we see two stacks and a 2-square and a 3-square. But we also see a square formed by four diagonals that is surrounded by four half stacks.

So, the two side squares add as the diagonal square. The bottom-top line shows the side of the new square when a 3-square is added to a 4-square.

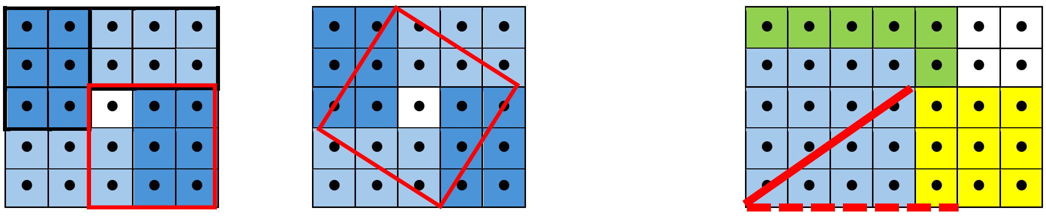


Figure 24. Two squares below and one inside two stacks. Two squares adding as a square

F16. Adding Per-numbers

In a shop we add both kg-numbers and per-numbers in mixtures as 2kg at 3$/kg plus 4kg at 5$/kg.

We can add the unit-numbers 2 and 4 directly to 6 kg, but before we add the per-numbers they must be multiplied to unit-numbers also, so they add as 2 * 3$ and 4 * 5$, which is the same as adding 2 3s and 4 5s by using integral calculus.

My uncle says that we later will learn to use integral calculus to add piecewise or locally constant per-numbers as the area under their per-number curve, which is we find by writing the areas as differences so that all the middle terms cancel out when added.

F17. Adding Fractions

Fractions also occur in mixtures as 2 apples with 1/2 red and 3 apples with 2/3 reds. Again, we add the unit numbers directly to 5 apples, whereas the per-numbers must be multiplied first to ½ * 2 and 2/3 * 3 that add to 3. So here the sum is 5 apples with 3/5 reds. My uncle says that you will try to teach us that 1/2 plus 2/3 is 7/6, but I think he jokes because there cannot be 7 red among 6 apples.

F18. Adding one-digit Numbers

We see that the textbook wants to teach us that 6+8 is 14. But 6 and 8 cannot exist without units.

If we count in 6s, then 6+8 = 1B0 + 1B2 = 2B2 6s or 1B4 tens. If we count in 8s, then 6+8 = 1B-2+ 1B0 = 2B-2 8s or 1B6 8s or 1B4 tens. If we count in tens we can use half-bundles, ½B, so here 6+8 = 1 ½B1 + 1 ½B3 = 1B4 tens. On a BBBoard we place the 6 and 8 on-top instead of next-to see the sum, and to see that 8-6 = 2, and that 6-8 = -2.

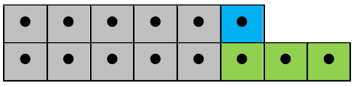


Figure 25. A BBBoard with 6 as 1 ½B 1 on-top of 8 as 1 ½B 3 to see how they add and subtract

F19. Using Underloads and Overloads in Calculations

We don't need that since bundle-numbers are flexible with both overloads and underloads.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B*12 9 *B* 2	4 *B*-2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 26. With Bundles, carrying and place values are unneeded when using under- and overload

On a BBBoard, the two-digit numbers 23 and 47 can be multiplied as 2B 3 and 4B 7 to 10BB 8B 1. And as 10BB 8B 1 we can divide 1081 by 23 as 2B 3 to become 5B -3, or 4B 7 tens.

	2B	3	
↓	8BB	12B	4B
↓	14B	21	7
8BB	26B	21	
10BB	**8B**	**1**	

	2B	3	
↑	10BB	15B	**? 5B**
↑	-6B	-9	**? -3**
10BB	9B	-9	
10BB	8B	1	

*Figure 27. On a BBBoard, 23 * 47 = 1081, and 1081/23 = 4B7 = 47*

F20. Adding Letters, what is *a* + *a* * *b*?

To add, numbers must have like units, so how do we add letters that stand for unknown numbers? Well, all numbers have 1s as a unit, 2 = 2 1s = 2 * 1, and *a* = *a* 1s = *a* * 1. And, *a* * *b* is really *a* *b*'s with *b* as the unit, or with *a* is the unit if we exchange the letters and write *a* * *b* = *b* * *a* = *b* *a*'s.

My uncle calls this the commutative law for multiplication. We see it if turning 2 3s over to 3 2s. It also applies for addition where 2 + 3 = 3 +2, but not for subtraction or division where 4 – 2 and 2 – 4 , and 4/2 and 2/4 are not the same.

So, now we can add a and $a * b$ to a total T

T = $a + a * b = a * 1 + a * b = 1 * a + b * a = 1$ a's + b a's = $(1 + b)$ a's = $(1 + b) * a$.

So, we may take a common unit or factor outside a bracket. Or we can do the opposite and write

T = $a * (1 + b) = a * 1 + a * b = a + a * b$

Here we take a common outside unit inside a bracket and distribute it to all numbers inside. This my uncle calls the distributive law.

To add, $a * c + a$^2 $* b$, we choose the common factors as the unit to move outside the bracket.

T = $a * c + a$^2 $* b = a * c + a * a * b = c * a + a * b * a = (c + a * b) * a = (c + a * b)$ a's.

F21. The Algebra Square Reunites Unlike and Like Unit- and Per-numbers

Counting ten fingers in 3s we get T = 1BB 0B 1 3s = 1 * B^2 + 0 * B + 1. My uncle calls this an example of a general bundle-formula called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration.

With units we thus see there can be only four ways to unite numbers: addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers.

Going back, subtraction and division split a total into unlike and like unit-numbers. Differentiation splits a total into unlike per-numbers. And finally, a total is split into like factors by the factor-finding root and the factor-counting logarithm. My uncle calls this 'the Algebra square' where the Arabic word Algebra means to reunite.

Calculations unite/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + b $T - b = a$	T = a * b $\frac{T}{b} = a$
Per-numbers m/s, $/100$ = %	T = ∫ f dx $\frac{dT}{dx} = f$	T = a^b $\sqrt[b]{T} = a$ $\log_a T = b$

Figure 28. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

F22. Fact, Fiction and Fake, the three Genres of our Number Language Tales

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers and about totaling and totals in time and space. My uncle calls this modeling. And he says that as in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models.

Fact models talk about the past and present and only need to have the units checked. Fiction models talk about the future and need to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that ‘2+3 = 5’ always despite 2weeks + 3days = 17days, thus transforming mathematics to ‘mathematism’.

My uncle says that if we are allowed to keep and develop our natural bundle-numbers with units then we will all be numerate as asked for by the UN Sustainable Development Goal 4 about quality education. So, allowed to keep and develop our natural intelligence we don’t need any AI. All you must do is to use our Bundle-numbers with units. And no adding the first year, please, only counting and recounting so we really get to know what we are numbering and later calculating. Please, let us Grip to Grasp, at least the first three years? Please, please it is so fun. You will not?

F23. Children’s Declaration of Independence

Well, if you will not use our Bundle-numbers with units we will protect our own number-language and innate number-sense by slightly reformulate the American Declaration of Independence:

“We, the children, declare unanimously that when it becomes necessary for us to dissolve the educational bands which have connected us with you, and to assume among the separate and equal station to which our human nature entitle us, a decent respect requires that we above have declared the causes which impel us to the separation of an education that prohibits this.”

Discussion

We asked the Cinderella question: If mastery of Many is the end goal, essence-math may be a means, but are there differnt means also, e.g., an existence-math, that will make the prince dance? And if so, what is its content and what kind of mathematics does the prince learn during the dance?

We used difference-research building on the observation that when asked “How many years next time?” a 3-year-old child will say “That is not 4, that is 2 2s” if seeing the 4 fingers held 2 by 2. Adapted to the culture, adults only see the essence, four. But still adapting to nature, preschool children only see what exists, bundles of 2s in space and two of them when counted in time. To gather data, we used sociological imagination to visit a grade 1 class in a near decolonized future.

Here we found one content of an existence-math curriculum building on the children bundle-numbers with units as two 3s and four 5s. And we also found that this curriculum allows students to meet the core of mathematics, linearity and calculus, right away when adding their bundle number on-top and next-to as areas.

But first they met power and logarithm when bundle-counting in bundles and bundle-of-bundles. Then they met division, multiplication, subtraction, and addition in that order when using push-away and push-back followed by pull-away and pull-back to recount a total in a specific bundle size with the unbundled placed on-top as decimals, fractions or negative numbers.

Next, they met equations and basic algebra when recounting between tens and icon-units. And with BundleBundle as squares, they also met quadratic equations.

Then, when recounting between physical units, they met per-numbers and fractions, as well as trigonometry that leads on to coordinate geometry before plane geometry. And, since they must be multiplied to unit-numbers before adding, per-numbers and fractions also add next-to as areas thus also using calculus with addition and reversed addition. Adding many differences as one difference here offers a simple way to integrate and to solve difference equations.

So, very soon they created the ‘Algebra Square’ showing how to master Many by uniting and splitting totals into the four kinds of numbers in the world, unlike and like unit-numbers and per-numbers. And with this number-language to number the world for getting numbers to use in forward or reversed calculations, they now can produce quantitative tales having the same genres as has the word language, fact, fiction and fake (Tarp, 2001).

On this direct way to mastery of Many they have learned the core of classical mathematics as well as its essential part so that the rest may be taught as footnotes to those who may find it interesting. At the same time, we can now recommend a curriculum to directly reach mastery of Many as a means to reach the UN Sustainable Development goal that youth and adults all shall be numerate.

A BBM Bundle-Bundle Math Curriculum

By its very nature, multiplicity is multiplicative, not additive. Students therefore should be allowed to keep and develop their innate number sense and their bundle-numbers with units as two 3s where the counting-number in time, two, and the bundle-number in space, 3s, must be treated together.

When counted in 2s, 5fingers can occur as I I I I I, 0B 5, or as II I I I, 1B 3, or as II II I, 2B 1, or as II II I~~I~~, 3B -1 2s. But since we bundle in 2s, two bundles is one bundle of bundles, so five can also be IIII I, 1BB 0B 1 2s, which makes ten be 2BB 0B 2, or 1BBB 0BB 1B 0 2s. Here power shows how many 1s there are in a BBB of 2s, 2^3 = 8. And a logarithm shows how many times bundles have been bundled, $\text{llog}_2\ 8 = 3$.

Counting 8 in 2s means from 8 push-away 2s, iconized by a broom as 8/2 so the calculator can predict the result, 8/2 = 4. Here we must tell counting, 8s /2 = 4 2s, from splitting, 8s/2s = 2 4s.

The opposite to ‘push-away’ is ‘push-back-to-stack’ iconized by a lift called multiplication, 4 x 2, where the calculator predicts that 4 times stacking 2s gives 8 1s.

Combined push-away and push-back create a ‘re-count formula’ used to change units all over STEEM including economy, 8 = (8/2) * 2, or with T and B for Total and Bundle, T = (T/B * B).

Pulling-away is iconized by a rope called subtraction, and pulling-back a cross called addition.

Combined pull-away and pull-back create a ‘re-unite formula’ T = (T – B) + B.

The two formulas solve equations, which is another word for reversed calculation, by moving ‘to opposite side with opposite sign’.

The equation $u * 2 = 8$ is solved by $u = 8/2$ since 8 recounts in 2s as $8 = (8/2) * 2$.

The equation $u + 2 = 8$ is solved by $u = 8 – 2$ since 8 reunites with 2 as $8 = (8 – 2) + 2$.

The equation, $2^u = 8$, is solved by $u = \log_2 8$ since 8 as 2folded gives 8 = 2^3, and $\log_2 8 = 3$.

The equation, $u^2 = 8$, is solved by $u = \sqrt{8}$ since, by definition, a $\sqrt{8}$ -square has area 8.

Recounting from tens to icons gives equations as $u * 5 = 20$, solved by $u = 20/5$ recounting 20 in 5s.

Recounting from icons to tens gives times tables as 6 7s = ? tens, found by multiplication where 6 * 7 = 6 * ½B 2 = 3B 12 = 4B 2 = 42.

Recounting between physical units gives per-numbers as 2$/3meter that become fractions with like units, 2$/3$ = 2/3, and that bridge the units by recounting:

8$ = (8/2) * 2$ = (8/2) * 3m = 12m.

Recounting the sides in a stack gives trigonometry: up = (up/out) * out. Here the per-number, tangent(Angle) = up/out, finds the angle of the diagonal. This allows the length of a half circle to be found as $\pi = n * \tan(180/n)$, for n large.

Two 3s and four 5s add on-top after recounting has made the units like, i or next-to as areas, i.e., as integral calculus where adding many differences as one difference eases integration.

Fractions and per-numbers also add as areas with integral calculus since they must be multiplied to unit numbers before adding.

So, with 2m at 3$/m and 4m at 5$/m we get (2+4)m and (2*3 + 4*5)$.

The Algebra Square shows how to master Many by reuniting and splitting a total into the four kinds of numbers in the world, unlike and like unit-numbers and per-numbers.

A number-language to number the world for getting numbers to forward or reverse calculations, may produce quantitative tales with the same genres as in the word-language, fact, fiction and fake.

How Different is the Difference?

Digits now are no longer symbols like letters, but icons with as many sticks as they represent. 3 now is called ‘1B0 3s’ or ‘0B3 tens’. Ten, eleven and twelve now are also called ‘one-bundle-zero’, ‘one-bundle-one’, and ‘one-bundle two’. And hundred and thousand are called ‘bundle-bundle’ and ‘bundle-bundle-bundle’.

Multidigit numbers no longer occur without units, 23 = 2B3). This makes the place value system unneeded. Calculations with overloads and underloads give bundle-numbers with units flexibility that makes carrying and borrowing unneeded, e.g., 46 + 37 = 4B 6 + 3B 7 = 7B 13 = 8B 3 = 83.

Addition now depends on the units so 2 + 3 is not 5 by necessity. 2weeks + 3weeks = 5weeks, but 2weeks + 3days = 17days. So, without a unit, 3 does not exist, only with a bundle-unit like, e.g., 0B3 tens, or 1B 0 3s, or 1B 1 2s, or 1B -1 4s, or 1B -2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., '2 + 3' = (1B 0 + 1B 1) 2s = 2B 1 2s, or '2+3' = (1B -1 + 1B 0) 3s = 2B -1 3s = 1B 2 3s. Likewise with subtraction '9 – 6' = (1B 3 – 1B 0) 6s = 0B 3 6s = '3', or '9 – 6' = (1B0 – 1B-3) 9s = 0B --3 9s = 0B3 9s, showing that minus times minus must be plus.

Also, addition now is not well-defined since two 3s and four 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and 6 * 8 is not 48 by necessity. Instead, 6 * 8 exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or in tens: 6 8s is 5B 3 9s, and 4B 8 tens. Division like 8/2 has two different meanings, 8s / 2s gives 4 2s, and 8s / 2 = 2 4s.

Solving equations now is different. The equation 'u * 2 = 8' asks "How many 2s in 8?" which is found by recounting 8 in 2s as 8 = (8/2) * 2. The solution, u = 8/2, now is found by 'moving to opposite side with opposite sign', which follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, if u * 2 = 8 then u = 8/2.

Equations thus no longer are seen as two equivalent numbers-names that remain equivalent if the same operation is performed to both. So, the balancing method now is unneeded with its transformations using the communicative, associative, and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements to addition and multiplication. And we no longer use the neutralizing 'do the same to both sides' lever method to solve the equation 2 * x = 8 by saying:

$$2 * x = 8 \leftrightarrow (2 * x) * ½ = 8 * ½ \leftrightarrow (x * 2) * ½ = 4 \leftrightarrow x * (2 * ½) = 4 \leftrightarrow x * 1 = 4 \leftrightarrow x = 4$$

The multiple calculation 2 + 3 * 4 no longer is 14 by definition or by the 'PEMDAS' rule. With units, 2 + 3 * 4 exits as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation '2 * a + 3 * a = (2+3) * a' no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer 'go over one'. Instead, a per-number like 4$/5kg, links the two units by recounting: 16$ = (16/4) * 4$ = (16/4) * 5kg = 20 kg, and 30kg = (30/5) * 5kg = (30/5) * 4$ = 24$.

Fractions no longer are numbers by themselves, instead they are per-numbers with like units, 3meter/4meter = ¾, 3 meter/100meter = 3/100 = 3%. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., 1/2 + 2/3 = (1+2)/(2+3) = 3/5 in this case, and not 7 red of 6 apples.

Trigonometry no longer must wait until after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal. Good news for STEM (Tarp, 2021).

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait until secondary school since Bundle-Bundles are squares that lead directly to the question ‘how to square a rectangle’ that provides a double split square containing the three parts of a quadratic equation.

The simplicity of the Algebra Square will no longer be hidden. And no longer will models be seen as mere approximations but as tales with three genres, fact and fiction and fake.

And no longer will 8 competences be needed in mathematics education, only the two natural competences, to count and add in time and space (Tarp, 2017).

Teacher education in existence-math could begin by demodeling the concepts in essence-math (Tarp, 2020) and by watching some MrAlTarp YouTibe videos, or join the self-instructing online education on the MAHTeCADEMY.net, or read some articles (Tarp, 2016-2025).

So finally, the number-language may have its communicative turn to be learned by telling tales about things and actions in space and time, just as the word-language had around the 1970’s as described in H.G. Widdowson’s book ‘Teaching Language as Communication’.

	Essence-math, mathematism	Existence-math, Many-math
Digits	Symbols	Icons
345	Place value system	T = 3BB 4B 5, BB = B^2, BBB = B^3
Operations	Functions, order: + – x / ^	Icons, opposite order: ^ / x – +
3 + 4	3 + 4 = 7	Meaningless without units
3 * 4	3 * 4 = 12	3 * 4 = 3 4s, may be recounted to 1.2 tens
9 = ? 2s	Meaningless, only ten-counting	9 = 3B3 = 5B-2 = 4B1 = 4½ 2s
8 = ? 2s	Meaningless, only ten-counting	8 = (8/2) * 2, T = (T/B) * B, proportionality
2 * u = 8	2 * u = 8 then (2 * u) * ½ = 8 * ½ then (u * 2) * ½ = 4, then u * (2 * ½) = 4 then u * 1 = 4 then u = 4	2 * u = 8 = (8/2) * 2, so u = 8/2 = 4
6 * 7 = ?	eh 44? eh 52? eh 42? OK	6 * 7 = 6 * ½B 2 = 3B 12 = 4B 2 = 42 6 * 7 = (B-4) * (B-3) = (10-4-3) * B + 4 * 3 = 3B12 = 4B2 = 42

4 kg =5 $, 6 kg = ?	1 kg = 5/4 $, 6 kg = 6 * 5/4 $	6 kg = (6/4) * 4 kg = (6/4) * 5 $
1/2 + 2/3 = ?	1/2 + 2/3 = 3/6 + 4/6 = 7/6	1/2 * 2 + 2/3 * 3 = 3/5 * 5
3 + 4 * 5 = ?	7 + 5 = 12, or, 3 + 20 = 27 ??	2 * 3 + 4 * 5 = 3B 2 8s (next-to) or 5B 1 5s, or 8B 2 3s (on-top)
6 + 9 = ?	6 + 9 = 7, 8, 9, , 10, 11, 12, , 13, 14, 15.	2B 3 6s or 2B -3 9s or ½B 1 + ½B 4 = 1B 5 tens = 15
9 – 4	9 – 4 = 8, 7, , 6, 5.	1B 0 – 0B 4 9s = 0B 9 – 0B 4 = 0B 5 9s = 5 2B 1 – 1B 0 4s = 1B 1 4s = 5 1 ½B 4 – (1 ½B -1) tens = 4 - - 1 = 4 + 1 = 5
4 – 9	Impossible	1B0 – 2B 1 4s = 1B0 – 1B 5 4s = 0B -5 = -5
Tangent = ?	tan = sine/cosine	raise = (raise/run) * run, tan = raise/run

Figure 29. Overview of the differences between Essence- math and Existence-math

Will the Difference make a Difference, Reactions to a BBM BundleBundle Math Curriculum

At the 9th ICMI-East Asia Regional Conference on Mathematics Education (EARCOME 9) in 2025 in Korea, the Special Sharing Groups accepted my paper "Can a Decolonized Mathematics Secure Numeracy for All?" with the following announcement:

> This proposal tackles an urgently needed conversation in mathematics education by challenging deeply ingrained assumptions about number systems and arithmetic instruction and proposing a truly decolonized approach that foregrounds learners' intuitive "bundle-number" language. Its strength lies in weaving together a compelling theoretical critique—drawing on Habermas's colonization concept and rich philosophical underpinnings—with concrete instructional innovations like the Algebra Square that reframe operations as intuitive spatial and bundling processes. By aligning this reconceptualization directly with SDG 4's numeracy targets and illustrating how multiplication-centered reasoning better reflects real-world number use, the paper promises to make a bold and impactful contribution to both research and practice. We look forward to seeing how this work can reshape numeracy instruction and foster truly inclusive mathematical literacy for all.

In Topic Study Groups, reviewers asked for empirical studies where existence- and essence-math had been compared. My answer was that testing both at the same time is ethically problematic:

> Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order: addition, subtraction, multiplication, division, power. And only few will hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. And, since counting and recounting precedes adding, they will not meet addition until they meet the core of math directly, calculus and linearity. What will happen if a class B student changes to class A, or the other way around? In short, how ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return far ahead to normal education? Ethical testing is for teacher education and home education.

Conclusion and Recommendations

We now can answer the Cinderella question: If mastery of Many is the end goal, essence-math may be a means for a few, but there is a different means also, an existence-math, which builds upon counting and adding in time and space where counting and recounting to change units come before adding, and which leads to the core of essence-math allowing the rest to be taught as footnotes.

The recommendation therefore is to teach existence-math at least the first three years, and to postpone addition to the second year to allow the children to have fun with counting and recounting the outside world while using a BundleBundleBoard to represent their numbers and a calculator to predict their actions.

Presenting statistically significant testing results would be desirable, or would it because of the ethical challenges in comparing a 'flat-earth' and a 'round-earth' curriculum? As a difference researcher you can hope to find a difference, but you cannot always hope to test it to see if the difference makes a difference. However, in order to test it we first must have a difference to test. So, maybe it is best to let the composer compose, and then leave it to others to play the music?

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REJ.TSG02. FROM ONLY ADDING ESSENCE TO FIRST COUNTING EXISTENCE

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle thus containing both a time-based counting number, 2, and a space-based per-number, 3s. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal.*

Keywords: Early childhood mathematics, numeracy, arithmetic, curriculum, calculus

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world where humans see and name it differently. In contrast, mathematics does not do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions.

It thus seems that with the English definition of numeracy, the assessment must be carried out by persons seen as experts on the predicates competent and mathematics. In contrast, the American definition of numeracy allows laymen to judge themselves the actions carried out when working with numbering and numbers. And, in their common history, England once colonized America, so we may wonder if the two different views are the views of a former colonizer and a former colonized.

To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs. In this way the three grand theories may help to enlighten and discuss the core of education formulated as 'teach learners something'. Here 'something' may be discussed in philosophy. And the power relations in the textbook-teacher-learner interaction may be discussed in sociology. Finally different learning possibilities in working with the subject or hearing about its predicate may be discussed in psychology. Here, discussing may be a better word than enlightening since alle three areas contain conflicting theories.

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that 'existence precedes essence' (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. 'Many' thus should be seen onto-logically in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget (1969) sees learning as adapting to outside existence, whereas a Vygotsky (1986) sees learning as adapting to inside institutionalized socially constructed essence.

Within sociology, a structure-agent debate discusses if institutions should be obeyed or constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. As to the goal of meth education, Bauman (1990) suggests that, by institutionalizing math as the means to reach mastery of Many, 'essence-math' has created a 'goal displacement' making the means a goal instead. Then, with the end goal, mastery of Many, essence-math is a means, but are there other means also?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that '1+1 = 2'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). As to differences, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to be tested with Design Research and building on the observation that when asked "How many years next time?" a 3tear-old child will say "That is not 4, that is 2 2s" if seeing the 4 fingers held 2 by 2.

Counting Many with bundles, children deserve a bundle-number curriculum

A01. Bundle-counting with units and using snap-cubes or a ten-by-ten BundleBundleBoard, 2 3s is 2 bundles with 3s per bundle. So, the per-number 3s exists in space and the counting-number 2 in time. The Algebra square reunites unlike and like counting- & per-numbers (fig. 1). Polynomials use bundle-counting with units. 43 = 4*B + 3*1 = 4B3 tens, and 543 = 5BB4B3 tens. Bundle-numbers falsify '1+1 = 2' with 2 V-signs showing that 1 1s + 1 1s = 1 2s and 2 1s + 1 2s = 1 4s, and not 3 3s.

A02. Flexible Bundle-counting in space. Space-count five and ten fingers in 2s, 3s, 4s and 5s. 5 = 1B3 = 3B-1 = 2B1 = 1BB0B1 2s, and Ten = 2BB0B2 = 1BBB0BB1BB0 2s. And T = 38 = 3B8 = 2B18 = 4B-2. T = 35+46 =3B5+4B6 = 7B11 = 8B1. T = 6*28 = 6*2B8 = 12B48 = 16B8 = 168. And T = 4507 = 4BBB 5BB 0B 7, T = 4*B^3 + 5*B^2 + 0*B + 7*1. Place value and carrying is unneeded.

A03. Add and subtract 1digit numbers counted in half-bundles. T = 6+7 = ½B1 + ½B2 = 1B3 = 13. T = 4+7 = ½B-1 + ½B2 = 1B1 = 11. T = 3+4 = ½B-2 + ½B-1 = 1B-3 = 7. And T = 8-6 = ½B3 - ½B1 = 3-1 = 2. T = 6-4 = ½B1 - ½B-1 = 1 - -1 = 2 (so, - - = +). T = 6-8 = ½B1 - ½B3 = 1-3 = -2

A04. Time-counting fingers in ½B, "1,2,3,4,5,6" no!, "0B1, 0B2, 0B3, 0B4, 0B5 or ½B0, ½B1". Time-count from 88 to 100: "8B8, 8B9, 8Bten or 9B0, …, 9B9, 9Bten or tenB0 or 1BB0B0".

A05. Digits are icons. 4 strokes as a 4-icon: I I I I → IIII → 4. And 5 as a 5-icon: I I I I I → IIIII →5.

A06. Operations are icons also. Push-away and -back to lift to stack, (division & multiplication), 6/2 means 'from 6 push-away 2s to lift', so 6 = 3x2 = (6/2)x2, T = (T/B)xB (the recount-formula). Pull-away and -back (minus and plus) to get decimals, fractions and negatives. 7 = 3B1 = 3½B = 4B-1 2s.

A07. Recounting between icon and tens. "? 5s gives 40": u*5 = 40 = (40/5)*5, so u = 40/5, i.e., "To Opposite Side with Opposite Sign". 6 7s = ? 8s and 6 7s = ? tens leads to division and multiplication tables where 6 7s = 6*7 = (B-4)*(B-3) = From BB, pull-away 3B & 4B and pull-back the 4*3 pulled-away twice = 3B12 = 4B2 = 42. So (B-4)*(B-3) = BB - 3B - 4B + 4*3. Here, minus * minus is +.

A08. Recounting physical units gives per-numbers as 2$/5kg. 20kg = (20/5)*5kg = (20/5)*2$ = 8$. Meter = (meter/second)*second = speed*second. Fractions with like units: 2$/5$ = 2/5. Trigonometry in a stack: height = (height/base)*base = tan(Angle)*base. Pi = n*tan(180/n) for n high enough.

A09. Bundle-bundles are squares. 3 3s= 1BB 3s= 1BB2B1 2s= 1BB-2B1 4s. So, 1BB2B1 = next BB.

A10. Squaring stacks. T = 6 4s = 1BB where B=√(6*4). Guess 1: '(6-1) (4+1)s' or '5 5s', since ½(6-4) = 1. The empty 1-corner needs two 't 4s' stacks, and t*4 = ½ gives t = 1/8. Guess 2: '4.9 4.9s'.

A11. Solving quadratics. A (u+3) square has two squares and stacks: (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6*u + 9. If u^2 + 6*u + 8 = 0, all disappears but 1, so, (u + 3)^2 = 1, so u = -4 or u = -2.

A12. Adding next-to and on-top or reversed. 2 3s + 4 5s = ? 8s. Here integral calculus adds areas, and recounting change units. 2 3s + ? 5s = 3 8s. Here ? = (T2-T1)/5 = ΔT/5 roots differential calculus.

A13. Adding per-numbers & fractions. 2kg at 3$/kg+ 4kg at 5$/kg=(2+4)kg at (3*2+5*4)/(2+4) $/kg. And, 2 apples with ½ red + 3 apples with 2/3 red = (2+4) apples of which (2*½+3*2/3)/(2+3) red. Integral calculus adds piecewise or locally constant per-numbers.

A14. The Algebra square. The 'Algebra Square' reunites unlike and like unit- & per-numbers

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	T = a + n *T – n = a*	T = a * n *T/n = a*
Per-numbers m/s, $/100$ = %	T = ∫ f dx *dT/dx = f*	T = a^b $\sqrt[b]{T}=a$ *loga(T) = b*

Figure 1. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

A15. Fact, fiction & fake models. Fact 'since-then' models quantify and predict predictable quantities by using factual numbers and formulas. Typically modeling the past and the present, they need to be checked for truth and units. Fiction 'if-then' models quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically modeling the future, they need to be supplied with scenarios building on alternative assumptions. Fake 'what-then' models quantify and predict unpredictable qualities by using fake numbers and formulas or by adding without units (Tarp, 2001).

Conclusion and recommendation

"By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". We may reach this UN Development Goal by replacing an essence-based curriculum with an existence-based curriculum answering the question 'how many?' by counting and recounting totals before adding them. Digits then occur as icons with as many strokes as they represent, thus becoming units when counting totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads, which makes place values and carrying unneeded. Bundling bundles also lead to squares and square roots; and to power as the first of the operations. The operations are icons also, but with different meanings and opposite order. Division now means counting iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Thus, both proportionality and trigonometry occur in year one. Once counted and recounted, totals may add on-top, or next-to by areas as with integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own flexible bundle-number with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as self-organized pre-service and in-service teacher training asking the subject on a BundleBundleBoard instead of the instructor as exemplified on the MATHeCADEMY.net website.

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REJ. TSG04. CURE DIAGNOSES - OR HELP THEIR INNATE NUMBER SENSE DEVELOP?

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

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Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that '1+1 = 2'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). And, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to be tested.

Learning from learners

When asked "How many years next time?" a 3year-old child will say "That is not 4, that is 2 2s" if seeing the 4 fingers held 2 by 2. This statement will change math education forever since, as educated, the essence, 4, is all we see. But as uneducated, the child sees what exists, bundles of 2s in space, and 2 of them when counted in time. Counting in tens instead of in 2s, we write 47 and say 'forty-seven' instead of '4 bundles with tens per bundle and 7 unbundled' to respect what exists and call it one number despite it is two numberings. 2 thus exists both in space and in time. In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle.

In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit. And since we count bundles and singles, the unit should be included in the counting sequence: 0B1, 0B2, …, 0B9, 1B0. So, outside in the real world, 2 is a ghost-number. What exists are 0B2 in time and 2s in space.

The school teaches 2+3 = 5' and '2x3 = 6'. Here, 2 3s may always be recounted as 6 1s, but 2weeks + 3days is 17days. So, even if both hold inside the school, outside 'multiplication holds, but addition folds.' Mathematics that adds numbers without units may be called 'mathematism', true inside but seldom outside the school, whereas mathematics that add numbers with units may be called 'Many-math', using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units are called 'calculus' and 'proportionality', the core of mathematics. Where normally they come very late, they here occur in the first lesson.

To get a more precise definition of space-numbers as 2s and time-numbers as 0B2 we observe the following when using the time-number sequence '0B1, 0B2, …, 0B5' to count the fingers on a hand:

After in time saying '0B1' when pulling up one finger we now in space have 1 1s. And, after in time saying '0B2' when pulling up one more finger we now in space have 2 1s that may bundle as 1 2s. Likewise with 0B3, 0B4, and 0B5. Then, after in time saying '0B6' when pulling up one more finger we now in space have 6 1s that may bundle as 1 6s, or that may bundle as ½B1 with ½B as 1 5s. We may repeat the same on a ten-by-ten Bundle-bundle board that contains all the Bundle-numbers.

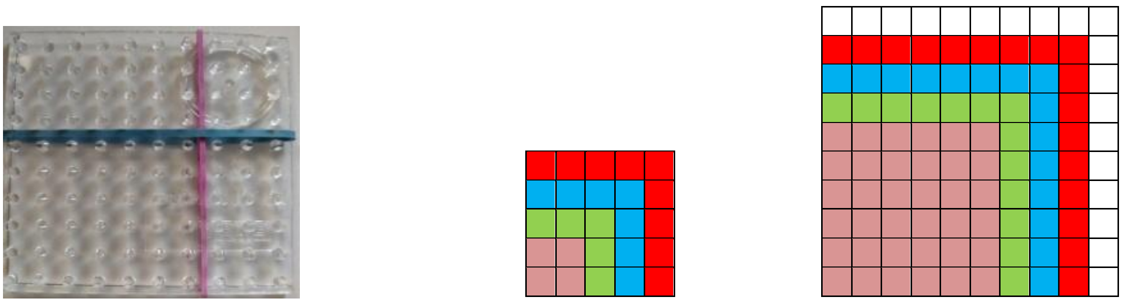

Figure 1. A BundleBundleBoard with 6 7s; and squares with BB2B1 or BB-2B1 as the neighbor

Counting in half-bundles later eases addition, subtraction and multiplication:

6+7 = ½B1 + ½B2 = 1B3 = 13; and 9-7 = ½B4 - ½B2 = 2; and 6*7 = 6* ½B2 = 3B12 = 4B2 = 42.

We may also count in units other than 5s or tens. 5 fingers thus may be counted in 2s in many ways:

5 = 0B5 = 1B3 = 2B1 = 3B-1 = 1BB0B1 2s.

Holding the two hands together we see that ten = 2BB0B2 = 1BBB0BB1B0 = 1010 2s.

And that ten =1B0 = 10, hundred is 1BB0B0 = 100, and thousand = 1BBB0BB0B0 = 1000.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 B 5 + 2 B 7	6 B 5 – 2 B 7	7 x 4 B 8	33 B 6 /7
8 B12 9 B 2	4 B-2 3 B 8	28 B 56 33 B 6	28 B 56 /7 4 B 8
92	38	336	48

Figure 2. Bundles and over- and underload make place values and carrying unneeded

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that we have 8 = (8/2)x2, or T = (T/B)xB with T and B for Total and Bundle. This linear 'proportionality' recounting-formula to shift units now occurs in the first lesson when we bundle-count with units. And, it solves equations where 'u x 2 = 8' is asking 'How many 2s in 8?', which is simply answered by 'u = 8/2' since 8 is recounted in 2s as above, thus moving 'to opposite side with opposite sign'.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, 8 = 2B2 3s, or as a fraction when also counted in 3s as 2 = (2/3)x3, 8 = 2 2/3 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, 8 = 3B-1 3s. Here we may even see the 3 bundles of 3s as one bundle of bundles, one bundle-bundle, 1BB, so that 8 = 1BB 0B -1 3s, where the bundle-bundle is a square.

With BundleBundle-numbers as squares we may wonder if any rectangular bundle-number may be recounted in squares with the square root as its side, and if squares can add as squares, e.g., as the square created by their mutual Bottom-Top line.

Conclusion and recommendations

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equation by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas that again may be added to one square. So, with Many-math's 'counting before adding' we have learned most mathematics almost before we begin. This will please the fourth of the 17 UN Sustainable Development Goals that defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' And where the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women,

To meet this goal, we therefore replace 'mathematism' with 'Many-math'. To make the difference between the two more clear we may use the basic philosophical question: What comes first, existence or essence, what is in the world or what we think about it? Existence, says Existentialism.

The teacher then is left with a choice: Should I cure the diagnosis "You cannot math" or help develop children's innate number sense?

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REJ. TSG07. BUNDLENUMBERS ON A BUNDLEBUNDLEBOARD MAKE LOSERS USERS

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle thus containing both a time-based counting number, 2, and a space-based per-number, 3s. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal.*

Keywords: Early childhood mathematics, numeracy, special education, curriculum, arithmetic

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

The difference in the definitions is interesting. The English uses the passive term 'competent' where the American uses the active term 'work'. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

Finally, Many exists in the outside world where humans see and name it differently. In contrast, mathematics does not do so, it is an institutionalized essence that is socially constructed as inside abstractions from outside examples, or as inside examples from inside abstractions.

To understand these differences, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs. In this way the three grand theories may help

to enlighten and discuss the core of education formulated as ‘teach learners something’. Here ‘something’ may be discussed in philosophy. And the power relations in the textbook-teacher-learner interaction may be discussed in sociology. Finally different learning possibilities in working with the subject or hearing about its predicate may be discussed in psychology. Here, discussing may be a better word than enlightening since alle three areas contain conflicting theories.

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that ‘existence precedes essence’ (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. ‘Many’ thus should be seen onto-logically in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget (1969) sees learning as adapting to outside existence, whereas a Vygotsky (1986) sees learning as adapting to inside institutionalized socially constructed essence.

Within sociology, a structure-agent debate discusses if institutions should be obeyed or constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an ‘iron cage’. As to the goal of meth education, Bauman (1990) suggests that, by institutionalizing math as the means to reach mastery of Many, ‘essence-math’ has created a ‘goal displacement’ making the means a goal instead. Then, with the end goal, mastery of Many, essence-math is a means, but are there other means also?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that ‘1+1 = 2’. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Essence-math could be more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991). And, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula to be tested.

Children use 2D bundle-numbers with units, teachers use 1D line-numbers without units

When asked “How many years next time?” a 3year-old child will say “That is not 4, that is 2 2s” if seeing the 4 fingers held 2 by 2. This statement will change math education forever since, as educated, the essence, 4, is all we see. But as uneducated, the child sees what exists, bundles of 2s in space, and 2 of them when counted in time. Counting in tens instead of in 2s, we write 47 and say ‘forty-seven’ instead of ‘4 bundles with tens per bundle and 7 unbundled’ to respect what exists and call it one number despite it is two numberings. 2 thus exists both in space and in time. In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle.

In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit. And since we count bundles and singles, the unit should be included in the counting sequence: 0B1, 0B2, …, 0B9, 1B0. So, outside in the real world, 2 is a 'ghost-number'. What exists are 0B2 in time and 2s in space.

What kind of knowledge is mathematics when integrating two different phenomena, 0B2 in time and 2s in space, into one phenomenon, 2, defined as 1+1 = 2? Knowledge may be created individually (Piaget, 1969) and collectively (Grounded theory, 1967) both respecting that learning means differentiating a phenomenon into sub-phenomena with different properties. By doing the opposite, mathematics become a self-referring anti-science with inside descriptions of itself and not of the outside world where its foundation, 1+1=2, is falsified by children seeing that with an open and a closed V-sign on a hand, adding the 2 1s and 1 2s gives 1 4s, and not the 3 3s, as mathematics claims.

Likewise, the school teaches both '2+3 = 5' and '2x3 = 6'. Here 2 3s always recounts as 6 1s, but 2weeks + 3days is 17days. So, even if both hold inside the school, outside 'multiplication holds, but addition folds.' Adding numbers without units may be called 'mathematism', true inside but seldom outside the school, whereas mathematics that add numbers with units may be called 'Many-math', using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units is called 'calculus' and 'proportionality', the core of mathematics. And here they occur in the first lessons where normally they enter very late.

To get a more precise definition of space-numbers as 2s and time-numbers as 0B2 we observe the following when using the time-number sequence '0B1, 0B2, …, 0B5' to count the fingers on a hand: After in time saying '0B1' when pulling up one finger we now in space have 1 1s. And, after in time saying '0B2' when pulling up one more finger we now in space have 2 1s that may bundle as 1 2s. Likewise with 0B3, 0B4, and 0B5. Then, after in time saying '0B6' when pulling up one more finger we now in space have 6 1s that may bundle as 1 6s, or that may bundle as ½B1 with ½B as 1 5s. We may repeat the same on a ten-by-ten Bundle-Bundle board that contains all the first Bundle-numbers.

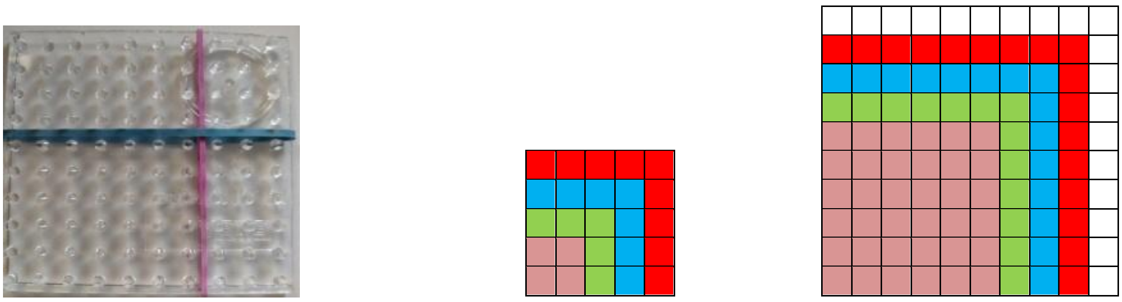

Figure 1. A BundleBundleBoard with 6 7s; and squares with 1BB2B1 or 1BB-2B1 as the neighbor

Counting in half-bundles later eases addition, subtraction and multiplication: 6+7 = ½B1 + ½B2 = 1B3 = 13; and 9-7 = ½B4 - ½B2 = 2; and 6*7 = 6* ½B2 = 3B12 = 4B2 = 42.

To see that also bundles may be bundled we count five fingers in 2s: Five = 0B5 = 1B3 = 2B1 = 3B-1 = 1BB0B1 2s. Holding two hands together we see that ten = 2BB0B2 = 1BBB0BB1B0 = 1010 2s. And that ten =1B0 = 10, hundred is 1BB0B0 = 100, and thousand = 1BBB0BB0B0 = 1000 tens.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 B 5 + 2 B 7	6 B 5 – 2 B 7	7 x 4 B 8	33 B 6 /7
8 B12 9 B 2	4 B-2 3 B 8	28 B 56 33 B 6	28 B 56 /7 4 B 8
92	38	336	48

Figure 2. Bundles and over- and underload make place values and carrying unneeded

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that we have 8 = (8/2)x2, or T = (T/B)xB with T and B for Total and Bundle. This linear 'proportionality' recounting-formula to shift units now occurs in the first lesson when we bundle-count with units. And, it solves equations where '*u* x 2 = 8' is asking 'How many 2s in 8?', which is simply answered by '*u* = 8/2' since 8 is recounted in 2s as above, thus 'moving to opposite side with opposite sign'.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, 8 = 2B2 3s, or as a fraction when also counted in 3s as 2 = (2/3)x3, 8 = 2 2/3 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, 8 = 3B-1 3s = 1BB0B-1 3s since the 3 bundles of 3s as one bundle of bundles.

Conclusion and recommendations

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equations by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas all carried out on a Bundle-Bundle Board. So, with Many-math's 'counting before adding' we have learned most mathematics almost before we begin. This fulfills the UN numeracy goal. And it performs miracles in special education classes where the excluded losers suddenly become users understanding and using numbers to count and calculate things and events in space and time. To reach this goal, we therefore should stop teaching mathematics as 'mathematism' with its ghost-numbers replacing units with a place value system where carrying is needed when adding without units; and built on the rule that '1+1 = 2. Instead, we should use a 'counting before adding' curriculum to make children develop their innate number sense to allow a communicative turn in the number-language as the one that took place in the word-language around 1970 (Widdowson, 1970), so that the children can produce end enjoy number tales with its three genres, fact, fiction and fake (Tarp, 2001).

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REJ. TSG08. CRITICAL VERSUS SKEPTICAL MATHEMATICS, CLIENT VERSUS AGENT

Allan Tarp, MATHeCADEMY.net, Allan.Tarp@gmail.com

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle thus containing both a time-based counting number, 2, and a space-based per-number, 3s. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal.*

Keywords: Early childhood mathematics, numeracy, critical mathematics, curriculum, calculus

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Within sociology, a structure-agent debate discusses if institutions should be obeyed or constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an ‘iron cage’. And, a Baumanian viewpoint (1990) may ask if institutionalizing mastery of ‘essence-math’ as the means to reach mastery of Many has created a ‘goal displacement’ making this means a goal instead. So, if the end goal is mastery of Many we should search for a different ‘existence-math’ instead using, e.g., skeptical Difference Research searching for differences making a difference (Tarp, 2018).

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that ‘1+1 = 2’. Existence-math sees Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. To see more differences between essence- and existence-math we may consult preschool children.

Children use 2D bundle-numbers with units, teachers use 1D line-numbers without units

When asked “How many years next time?” a 3year-old child will say “That is not 4, that is 2 2s” if seeing the 4 fingers held 2 by 2. This surprising statement shows that as educated, the essence, 4, is all we see. But as uneducated, the child sees what exists, bundles of 2s in space, and 2 of them when counted in time. Counting in tens instead of in 2s, we write 47 and say ‘forty-seven’ instead of ‘4 bundles with tens per bundle and 7 unbundled’ to respect what exists; and we call it one number despite with units it is two numberings. 2 thus exists both in space and in time. In space, 2 exists as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united with a 3-bundle. Horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or a stack of 2B-1 3s with B for bundle.

In time, 2 exists together with the unit that was counted, as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit. And since we count bundles and singles, the unit should be included in the counting sequence: 0B1, 0B2, …, 0B9, 1B0. So, outside in the real world, 2 is a 'ghost-number'. What exists are 0B2 in time and 2s in space.

Where essence-math sees mastery of math as its goal, existence-math sees mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018). In education, essence-math then could be made more meaningful by de-modelling it (Tarp, 2020) built on the French poststructuralist version of Existentialism where deconstruction is used to replace predicates with verbs (Derrida, 1991).

Critical and skeptical mathematics education, how do they differ?

Critical thinking and skeptical thinking both question traditional mathematics education, but the former doesn't question the institution, mathematics, only its action, education, while the latter also questions the institution to see if a goal displacement has taken place, making it a goal instead of a means; and shows skepticism towards it predicates to see if a deconstruction will make a difference. And critical is defined circular (I am critical if I criticize), skeptical is not (I am skeptical if I question).

As to critical thinking, Skovsmose & Borba (2000) describes a Brazilian research group that, focusing on issues related to new technologies and mathematics education, has developed software and work with students at different levels and with teachers. The group was approached by a teacher from a nearby school where she had some tough problems to face and hoped that the computers would be able to help her. She was teaching rational numbers to a class of 5th graders, with a mixture of 11-year-old students and 15year old repeaters having given up rational numbers and turning to violence.

> The teacher was enthusiastic about a software, which deals with rational numbers. (..) Both researchers and teacher had the 'intuition' that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (p. 7)

By recommending computers, the researchers showed criticism, not towards fractions as such, but towards how they are taught. Critical thinking thus sees essence-math as an unquestionable goal, only how it is taught can be criticized. Contrary to this, existence-math sees a fraction as a special per-number both coming from recounting in a different unit (Tarp, 2018). That fractions cannot be added without units is shown by the 'fraction-paradox': 1red of 2apples and 2red of 3apples total 3red of 5apples and not 7red of 6apples as says the textbook. With units, two situations should be taught. One finds the weighted average of two fractions, which always is placed between the two: 1/2*A + 2/3*B = p* (A+B) gives p = 1/2*A/(A+B) + 2/3*B/(A+B). The other is finding the total fraction of the same total: 1/2*T + 2/3*T = p* T, which for T = 6 gives 1/2*6 + 2/3*6 = p*6, or 7 = p*6 giving p = 7/6.

Critical and skeptical mathematics education has different philosophical backgrounds

The Enlightenment was received differently in Germany. Asking what it is, Kant said "Sapere Aude", or "Dare to think". In contrast, Hegel and Herder developed a counter-enlightenment theory seeing history guided by a Spirit towards a future controlled by collective institutions, called 'the dictatorship of the proletariat' by Marxism that inspired critical thinking to respect collective institutions and only express criticism as personal opinions without concrete outside objects. In contrast, Kant's skeptical thinking developed into Existentialism through Kierkegaard, Heidegger, Derrida, and Foucault.

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

The formatting power of mathematics

Critical thinking also criticizes mathematical modeling when saying that an adequate understanding of social phenomena "cannot be obtained without an understanding of the formatting power of mathematics" (Skovsmose, 2023). But the criticism again lacks concrete objects and a specification as to what kind of models and what kind of mathematics to criticize.

In contrast, existence-math differentiates between three kinds of models (Tarp, 2001). As the word-language, number-language also comes in three genres: fact, fiction, and fake models, also called since-then, if-then, and what-then models, or room, rate, and risk models. Fact models typically talk about the past and present and they only need to have the units checked. Fiction models typically talk about the future, and they need to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that '2+3 = 5' always despite 2weeks + 3days = 17days, thus transforming mathematics to 'mathematism' (Tarp, 2018).

Conclusion and recommendations

With existence-math, an outside total needs to be counted before being added, which leads to 2D rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers when including the unbundled; and to division and multiplication tables and solving equations when recounting between tans and icon-units as when asking 6 7s is how many tens?; and to per-numbers and fractions when recounting between physical units; and to trigonometry when recounting the height in the base in a number-stack; and to proportionality to make units like when adding on-top; and to calculus when adding next-to as areas all on a Bundle-Bundle Board (Tarp, 2018, 2024, 2025). So, with existence-math's 'counting before adding' we have learned most mathematics almost before we begin. This fulfills the UN numeracy goal. And allows a communicative turn in the number-language as the one that took place in the word-language around 1970 (Widdowson, 1970). And it performs miracles in special education classes where the excluded losers suddenly become users using their bundle-numbers to count and calculate things and events in space and time. In contrast, essence-math teaches 'mathematism' with ghost-numbers replacing units with a place value system using carrying when adding without units; and built on the rule that '1+1 = 2'. So, what to choose?

Skeptical existence-math ensuring the agency that by 2030, "all youth and a substantial proportion of adults, both men and women, achieve numeracy", or critical essence-math as a biopower regime to create docile bodies as clients for institutionalized educational 'pris-pital-baracks' (Foucault, 1995)?

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REJ. TSG09. FROM ADDING PERNUMBERS OVER BAYES THEOREM AND INTEGRAL CALCULUS TO ENJOYING DIFFERENCES' VANISHING MIDDLE TERMS

Allan Tarp, MATHeCADEMY.net, Denmark, Allan.Tarp@gmail.com.

*The fourth United Nations Development Goal wants all youth and most adults to achieve numeracy. In 'essence-based' mathematics using one-dimensional line-numbers without units, numeracy is at the beginning and calculus is at the end. But the opposite is the case within the alternative 'existence-based' paradigm built on the numbers children bring to school, two-dimensional bundle-numbers with units, where 2 3s is short for 2 bundles with 3s per bundle thus containing both a time-based counting number, 2, and a space-based per-number, 3s. Here calculus is needed in grade one for next-to addition of 2 3s and 4 5s as 8s since they add by their areas. Here, adding is preceded by counting leading to a recount-formula, T = (T/B)*B, with T and B for Total and Bundle, used to change units in STEM, and here recounting between icon-numbers and tens leads to a division and a multiplication table as well as to solving equation. Recounting physical units leads to per-numbers becoming fractions with like units and trigonometry when recounting the sides in a bundle-stack. So, changing from essence-based to an existence-based curriculum may allow reaching the UN Development Goal.*

Keywords: Early childhood mathematics, numeracy, limit, curriculum, calculus

Background, the 4th United Nations Sustainable Development Goal

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' Here the subgoal 4.6 wants to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy". However, different definitions of 'numerate' seem to exist. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers."

This difference is interesting. The word 'competent' is a predicate, a non-action word, I cannot 'competent' something, I can only be judged as competent by someone who is competent. In contrast, 'work' is an action word, a verb, since with my body and mind I can work on something.

Also, there is a difference between the words 'mathematics' and 'numbers.' Again, mathematics is a non-action word, I cannot 'mathematics' or even 'math' a thing. In contrast, 'number' is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations. To understand these differences, we now consult the three grand theories.

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that 'existence precedes essence' (Sartre, 2007) so that in a judging is-sentence, an existing subject is being colonized by its predicates. 'Many' thus should be seen onto-logically in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget (1969) sees learning as adapting to outside existence, whereas a Vygotsky (1986) sees learning as adapting to inside institutionalized socially constructed essence.

Within sociology, a structure-agent debate discusses if institutions should be obeyed or constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. As to the goal of meth education, Bauman (1990) suggests that, by institutionalizing math as the means to reach mastery of Many, 'essence-math' has created a 'goal displacement' making the means a goal instead. Then, with the end goal, mastery of Many, essence-math is a means, but are there other means also?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that '1+1 = 2'. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, and bundles of bundles, e.g., T = 345 = 3*BB + 4*B + 5*1. Essence-math sees mastery of math as its goal. A difference could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math automatically (Tarp, 2018, 2024). Built on the observation that when asked "How many years next time?" a 3-year-old child will say "That is not 4, that is 2 2s" if seeing the 4 fingers held 2 by 2, we now use sociological imagination (Bauman, 1990) to guess future education.

Grade one in a decolonized future

The teacher: "Welcome, I am your teacher in math, which is about the numbers that you can see on this number line, and that is built upon the fact that one plus one is two as you can see here. So …"

Showing a V-sign a child says: "Mister teacher, here is one 1s in space, and here is also one 1s. Now we count them in time to see how many 1s we have by saying 'one, two'. So, we have two 1s. But only until we add them as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. And together with this neighbor V-sign the total is one 2s plus two 1s which is one 4s, and not three 3s. And, if I two times show you three 1s I have shown you six 1s. So, the counting-numbers two and three can be multiplied, but they cannot add. Therefore, please help us add the bundle-numbers with units we bring to school, as 2 3s and 4 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on a peg board. If we add them next-to, we add plates, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says it is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school, instead of being colonized with your line-numbers without units."

Reflecting on per-numbers

Per-numbers as 4$ per 5 kg, or 4$/5kg, or 4/5 $/kg, bridge $- and kg-numbers by recounting them in the per-number using the recount-formula T = (T/B)*B: T = 20$ = (20/4)*4$ = (20/4)*5kg = 25kg. (Tarp, 2018). Per-numbers also occur in mixture-problems as 2kg at 3$/kg + 4 kg at 4$/5kg = (2+4)kg at ? $/kg. The unit-numbers add directly. But as operators, the per-numbers are multiplied to become unit-numbers before adding as areas to give (2*3+4*5)$ per (2+4)kg. Here, the per-numbers were piecewise constant changing from 3 to 4 after 2 kg. With a falling object, the per-number is locally constant changing each moment. Here, areas under the per-number graph, $A = \Sigma\ p(x)*\Delta x$ approximates better and better the smaller the moment is chosen, only giving more areas to add. However, since the multiplied per-number is a difference describing a change of the area, $p(x)*\mathrm{d}x = \mathrm{d}A$, addition makes all middle terms disappear leaving only one difference between the terminal and the initial number. This motivates supplementing integration with differentiation solving differential equations as $\mathrm{d}A = x$^2 dx, finding a formula for the area under the per-number graph $p(\mathrm{x}) = x$^2. Looking at the tiny shadows of a rectangular $p*q$ book we find the never falsified formula, $\mathrm{d}(p*q) = \mathrm{d}p*q + p*\mathrm{d}q$, so that d($x$^2) = $\mathrm{d}(x*x) = 2*x*\mathrm{d}x$.

The three kinds of constancy: globally and piecewise and locally

A variable y is globally constant c	$\forall\varepsilon > 0$:	$\lvert y-c \rvert < \varepsilon$ all over
A variable y is piecewise constant c	$\exists\delta > 0\ \forall\varepsilon > 0$:	$\lvert y-c \rvert < \varepsilon$ in the interval δ
A variable y is locally constant c	$\forall\varepsilon > 0\ \exists\delta > 0$:	$\lvert y-c \rvert < \varepsilon$ in the interval δ

Figure 1. The formal definition of the three kinds of constancy. It is strange that the university defines local constancy as piecewise constancy, and not when defining local and piecewise linearity

Reflecting on how to teach calculus in a decolonized future

Using their own bundle-numbers with units, children quickly master ‘primary school calculus’ where integration occur in questions as ‘2 3s + 4 5s = ? 8s’ where multiplication precedes addition. In in the reverse question ‘2 3s + ? 5s = 4 8s’, subtraction precedes division as in differentiation where their hands first pull away the initial stack, 2 3s, before counting the rest by pushing way 5s.

In middle school, per-numbers may be introduced physically as bridging plastic S- and C-letters given that 3 S-letters have the same value as 5 C-letters, which gives the per-number 3S/5C. Recounting in the per-number thus gives T = 12S = (12/3)*3S = (12/3)*5C = 20C. This leads on to traditional proportionality questions with per-numbers as 3\$/5kg, and 3m/5sec, and 3£/5\$. To be followed by fractions introduced as per-numbers with like units, 3\$/5\$ is 3/5. Again we use recounting to see that 3/5 of 20\$ means 3\$/5\$ of 20\$, so 20\$ = (20/5)*5\$ gives (20/5)*3\$ = 12 \$. Adding per-numbers occur in mixture-problems as ‘2kg at 3\$/kg + 4 kg at 4\$/5kg = 6kg at ? \$/kg’ and its reversed version. Likewise with adding fractions in problems as ‘2\$, of which 3/4 + 6\$, of which 4/5 = 8\$, of which ?’ and its reverse. So as operators, per-numbers and fractions add as areas, again needing calculus.

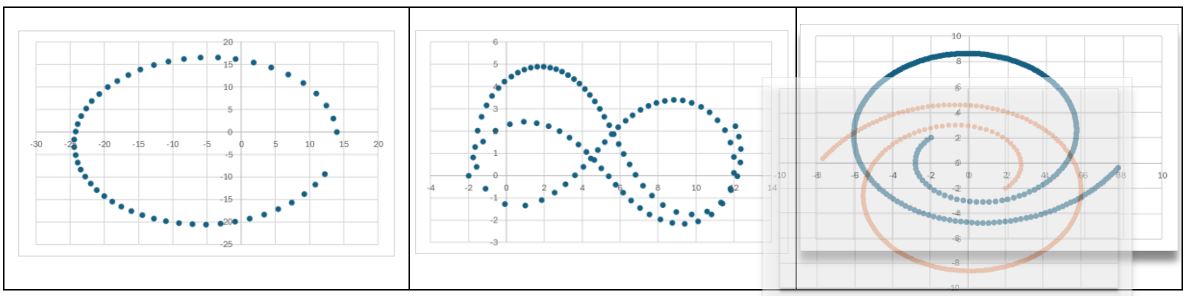

Figure 2. On a ten-by-ten bundle-bundle board, a ‘BBBoard’, difference equations on a spreadsheet can show orbits of a particle around one body, two bodies, as well as of two bodies

In outer space far from earth the acceleration is in inverse proportion to the distance to the particle and the pulling body. Trigonometry then allows splitting the total pulling acceleration into a horizontal and vertical part that changes the horizontal and vertical velocity that again changes the horizontal and vertical position. In the same way it will be possible to study possible a particle's orbit around two fixed bodies, and to study possible orbits of the two bodies if allowed to move freely.

Calculations unite/*split Totals in*	Unlike	Like
Unit-numbers	T = a + n	T = a * n
m, s, kg, $	*T – n = a*	*T/n = a*
Per-numbers	T = ∫ f dx	T = a^b
m/s, $/100$ = %	*dT/dx = f*	$\sqrt[b]{T}=a$ *loga(T) = b*

Figure 3. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

Reflecting on limits

Where a simple interest is added to the initial loan, a compounded interest is also added to former interest amounts. So, a periodical rate, *r*, gives the total simple rate, *R*, where 1+*R* = (1+*r*) ^*n*.

If a yearly rate of 100% split in 12 parts and added 12 times we get *R* = (1+1/12)^12 – 1 = 1.613, showing 61.3% in additional rate. This however has a limit since (1+1/*n*)^*n* can come close to but not exceed 2.718 called the Euler-number *e*. A solver shows that (1+1/*n*)^*n* = 2.712 for *n* = 234.

Inside a circle with radius 1 there are many right triangles with the long side from the center to the circle. Splitting 180 degrees in *n* parts, the height of the triangle is almost tan(180/*n*), so on a half circle, the circumference is close to *n**tan(180/*n*) that is 3.1411 for *n* = 100. Again, we have a limit since *n**tan(180/*n*) can come arbitrarily close to but not exceed 3.1416 called pi, π.

Discussing the tradition

The 2016 ICME-13 Calculus Survey says on its first page that it has "a particular focus on established research topics associated to limit, derivative and integral". Grand theory sees this LDI as a Vygotskian essence-based approach with a goal displacement ensuring that the end goal, per-number addition, is reached by only few, while many give up or are prevented from learning it. The opposite IDL is seen as a Piagetian existence-based approach, learning the goal before a means (Tarp, 2020).

Recommendation

The children's own bundle-numbers with units makes changing units the core job from year one leading to division- and multiplication-tables, solving equations and per-numbers. And to linearity and calculus when adding number-stacks on-top, or next-to by areas. An existence-based curriculum will thus finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). This will show the predicting power of mathematics that also have three genres, fact, fiction and fake (Tarp, 2001). This paradigm shift (Kuhn, 1962) opens new areas for research and innovation where linearity and calculus are used in early childhood mathematics.

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