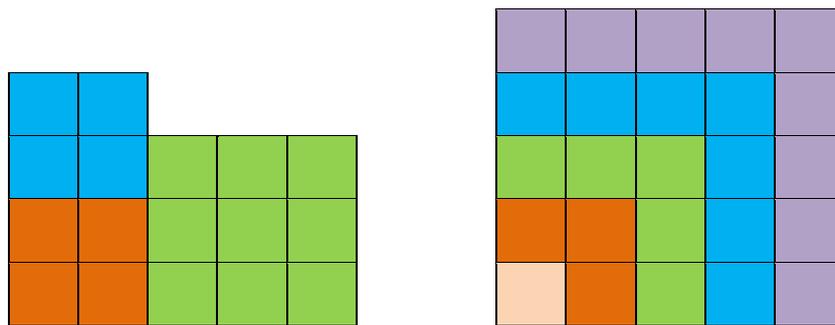


BundleBundle Math

on a BundleBundle Board

$T = 5 = \# = 1B \ 3 \ 2s$	Numeracy = Math with units, where Addition folds while Multiplication holds , using Children's own CountingNumbers with BundleUnits . A paradigm-shift from HalfMath to FullMath
$T = 5 = \# \# = 2B \ 1 \ 2s$	
$T = 5 = \# \# \# = 3B - 1 \ 2s$	
$T = 5 = \# \# = 1BB \ 0B \ 1 \ 2s$	

Existence before **Essence** means **Counting** before **Adding**



$4 \ 2s, \ 2BB \ 2s$ $1BBB \ 2s$ $2 \ 2s, \ 1BB \ 2s$ $1 \ 2s, \ 1B \ 2s$	$3 \ 3s, \ 1BB \ 3s$ $1 \ 3s, \ 1B \ 3s$	$1BB \ 5s = 1BB2B1 \ 4s$ $1BB \ 4s = 1BB2B1 \ 3s$ $1BB \ 3s = 1BB2B1 \ 2s$
--	---	--

4 **2s** plus 3 **3s** add next-to as **3B2 5s** or **3 2/5 5s** or **4B-3 5s**
as an example of Integral Calculus adding areas

	<p>A 10x10 Bundle-Bundle Board, a BBBoard with</p> <ul style="list-style-type: none"> • 6 7s • 4 tens • ten 3s • 4 3s <p> $6*7 = (B-4)*(B-3) (= 6*\frac{1}{2}B \ 2 = 3B12 = 4B2)$ $= 10B - \text{top}4B - \text{side}3B + 4*3$ $= 3B12 = 4B2 = 42$ </p>
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A Reaction to a BBM BundleBundle Math Curriculum

At the 9th ICMI-East Asia Regional Conference on Mathematics Education (EARCOME 9) in 2025 in Korea, the Special Sharing Groups accepted my paper “Can a Decolonized Mathematics Secure Numeracy for All?” with the following announcement:

This proposal tackles an urgently needed conversation in mathematics education by challenging deeply ingrained assumptions about number systems and arithmetic instruction and proposing a truly decolonized approach that foregrounds learners’ intuitive “bundle-number” language.

Its strength lies in weaving together a compelling theoretical critique - drawing on Habermas’s colonization concept and rich philosophical underpinnings - with concrete instructional innovations like the Algebra Square that reframe operations as intuitive spatial and bundling processes.

By aligning this reconceptualization directly with SDG 4’s numeracy targets and illustrating how multiplication-centered reasoning better reflects real-world number use, the paper promises to make a bold and impactful contribution to both research and practice.

We look forward to seeing how this work can reshape numeracy instruction and foster truly inclusive mathematical literacy for all.

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Introduction

Looking at four fingers bundled in twos, an educated person only sees four fingers, the ESSENCE. But, as uneducated before school, children see the EXISTENCE, bundles of twos in space, and two of them when counted in time. This textbook in Numeracy uses the child's own counting-numbers with bundle-units as 2 3s and 4B1 5s visible and tangible on a ten-by-ten BundleBoard. We may now ask how mathematics may be taught to children if using their own two-dimensional numbers with bundle-units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by meeting existence before essence to use the two core concepts of Existentialism holding that existence must precede essence.

Counting outside Totals in Bundles then must precede adding them inside. In this 'Bundle-math' approach, mathematical concepts are re-rooted in outside existing examples instead of being defined as examples itself inside. Now tens, hundreds and thousands become bundles, bundle-bundles, and bundle-bundle-bundles, as does 2, 4 and 8 when counting in twos instead.

Bundle-counting, one-dimensional lines on a ruler are replaced by two-dimensional rectangles on a BBBoard, containing the outside existing subjects that is linked to inside essence predicates in a number-language sentence as in a word-language sentence. Here units are always included in counting sequences as 0Bundle1, 0B2, ..., 1B0. Here digits become icons with as many sticks as they represent. Here also operations become icons created in the counting process. An outside Total may be split by pulling away a Bundle, using a rope as an icon for subtraction. Pulled back, the total is restored in a Reunite-formula, $T = (T - B) + B$, using a double-rope as an icon for pulling back. Or, the outside Total may be counted in Bundles by pushing away Bundles, using a broom as an icon for division. Pushed back, the total is restored in a Recount-formula, $T = (T/B) \times B$, using a lift as an icon for pushing back. The Recount formula is used all over STEM to change units.

Recounting from tens to icons, the equation, $u \times 2 = 8$, together with another basic equation, $u + 2 = 8$, are solved by recounting and reuniting, both moving numbers to 'opposite side with opposite sign': $u \times 2 = 8 = (8/2) \times 2$, so $u = 8/2$, and, $u + 2 = 8 = (8 - 2) + 2$, so $u = 8 - 2$.

Recounting from icons to tens leads to early algebra when including the Bundle-unit. Then 6×7 becomes $(B-4) \times (B-3)$ placed on a BBBoard and found by pulling-away the top 4B and side 3B, and adding the 4×3 pulled away twice. Or by writing $6 \times 7 = 6 \times \frac{1}{2}B^2 = 3B12 = 4B2$. And, bundle-bundles squares allow rectangular stacks to be recounted in squares with the square root as the side.

Recounting in physical units creates per-numbers as $4\$/5\text{kg}$ bridging the units by recounting: $20\text{kg} = (20/5) \times 5\text{kg} = (20/5) \times 4\$ = 16\$$. With like units, per-numbers become fractions: $4\$/5\$ = 4/5$.

Recounting the sides in a stack halved by its diagonal leads to trigonometry before geometry.

Recounting now is followed by reuniting (called 'Algebra' in Arabic). Stacks may add on-top after recounting has made the units like, or next-to as areas, i.e., as integral calculus (or differential calculus if reversed), also used to add per-numbers and fractions that must be multiplied to unit-numbers to add. Squares add as the square formed by their mutual Bottom-Top line.

Units create an 'Algebra Square' to reunite our four number-types. Add and multiply unite unlike and like unit-numbers, where integrate and power unite unlike and like per-numbers. That are split by subtraction, division, differentiation, and the factor-finding root or the factor-counting logarithm.

Counting before adding thus gave us a number-language to tell inside number-tales about outside totals using the same three genres, fact and fiction and fake, as the word-language uses.

Can mathematics be decolonized? Well of course, since mathematics is a socially constructed essence that will always be a colonization of the natural existence it came from and reduces.

Allan.Tarp@gmail.com, Aarhus, Denmark, September 2025

A Future? From STEM to STeN to make all Youth Numerate by 2030

From STEM to STeN, why?

The fourth of the 17 UN Sustainable Development Goals, Quality Education, has as a goal target to “By 2030, ensure that all youth achieve literacy and numeracy”. Replacing STEM with STeN also including economics and Numeracy may reach the UN goal. STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to predict the behavior of predictable physical quantities, and to model unpredictable quantities by scenarios. But economics is missing despite mathematics’ historical root is the marketplace with production, trade and consumption where quantities always carry units. Without units, mathematics must go and be replaced by numeracy that always uses units.

Two different Definitions of ‘Numerate’ exist

The English Oxford Dictionary defines it as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary defines it as “having the ability to understand and work with numbers.” The word ‘competent’ is a predicate, a non-action word, I cannot ‘competent’ something, I can only be judged as competent by someone who is competent. In contrast, ‘work’ is an action word, a verb, since with my body and mind I can work on something and test the result to see if it works. Also, there is a difference between the words ‘mathematics’ and ‘numbers.’ Again, mathematics is a non-action word, I cannot ‘mathematics’ a thing. In contrast, ‘number’ is both a verb and a noun since I can number different degrees of Many to produce a number for later calculations.

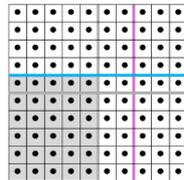
Existentialist philosophy holds that existence must precede essence to prevent colonization. With Numbers as existence and math as essence, numeracy thus should precede math. In fact, numeracy may be defined as ‘math with units where addition folds and multiplication holds’. Which is observed when the foundation of mathematics, ‘ $2+1 = 3$ ’, is falsified by an open and closed V-sign showing that 2 1s plus 1 2s unite as 1 4s and not as 3 3s as mathematics says without units.

Economics gives a Fundamental Understanding of Numbers and Calculations

The basic meanings of geometry and algebra show that they are both rooted in economics. In Greek, geometry means to measure earth, and in Arabic, algebra means to reunite numbers, so they have a common root in the basic economic question “How to divide the earth and what it produces?” Hunter-gatherers need not tell the different degrees of many apart, but farmers do since farmers produce to a market to survive, and here they need to be numerate to answer the question “How many here?”. This immediately leads to the answer “That depends on the unit.” Economics thus begin at once by reusing the number-names when using bundling to count. The Romans unsystematically gave names to the bundles 5s, 10s, 50s, 100s, 500s and 1000s. This worked well for administrative addition and subtraction jobs but not for multiplication. So, when German silver reopened the trade between India and Renaissance Italy, Hindu-Arabic numbers named only the unbundled, the bundles, the bundle of bundles (BB or B^2), the bundle-bundle-bundles (BBB or B^3), etc. With tens, 234 is not 1 but 3 numbers, 2BB 3B 4.

At a market, goods are sold in bundles with units as 2 3s, the same numbers that a 3year old child use when seeing four fingers bundled in twos: “No, that is not four, it is two twos”. But a buyer may want to have 5s, or to trade 4 per 5, or to pay 4\$ per 7. So, changing units is a core job using the proportional recount-formula, $T = (T/B)*B$, saying “From T, push-away B’s to count them”. Thus, recounting from tens gives equations solved by moving to opposite side with opposite sign:

Find u, so $u*2 = 8$. Here, 8 recounts in 2s as $8 = (8/2)*2$, so $u = 8/2$.

<p>Recounting into tens leads to early algebra: $6\ 7s = 6*7 = 6*(\frac{1}{2}B^2) = 3B\ 12 = 4B\ 2 = 42$. Or, $6*7 = (B-4)*(B-3) = BB-3B-4B+4*3 = (10-3-4)B\ 12 = 4B\ 2 = 42$</p> <p>This is seen on a ten-by-ten BundleBundleBoard when pulling away the top and the side and adding what was pulled twice.</p>	
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Likewise, when changing the units for length, weight, and currency. And, when changing from the quantity to the price. Here, Renaissance Italy used ‘regula detri’, the rule of three.

Asking “With the per-number 2\$ per 3kg, what is the price for 9kg?”, first they arranged the three numbers with alternating units: ‘9kg, 2\$, 3kg’. Then the answer comes from multiplying and dividing: $9 \times \frac{2}{3} = 6\$$. Now we recount in the per-number: $9\text{kg} = \frac{9}{3} \times 3\text{kg} = \frac{9}{3} \times 2\$ = 6\$$.

Trigonometry precedes geometry when recounting the height in the base in a stack as 4 5s:

height = (height/base) * base = tan(A) * base, giving $\pi = n * \tan(180/n)$ for n high enough.

Once counted and recounted, 2 3s & 4 5s may add next-to as 8s using calculus to add areas. Or add on-top as 5s using recounting for like units.

In a bill as ‘2kg at 3\$/kg + 4kg at 5\$/kg’, kg’s add directly but per-numbers add by their areas using calculus after being multiplied to \$.

With economics we learn core mathematics by counting in time and space before adding.

Numeracy as Math Counting Totals in Units before Adding them with Units

In a numeracy education using the children’s bundle-numbers with units, Tarp (2018) shows that ‘counting before adding’ give the same concepts but with different identities and order.

Bundle-counting in 3s leads to 9 as a bundle-bundle, a B^2 , which leads on to squares, square roots, and quadratics.

Counting transforms the operations into icons where division and multiplication become a broom and a lift that pushes-away bundles to be stacked as shown when recounting 8 in 2s as $8 = (8/2) \times 2$, which creates per-numbers when recounting $\$ = (\$/\text{kg}) * \text{kg}$.

Subtraction is a rope that pulls-away the stack to find the unbundled that, placed on-top of the stack as part of an extra bundle, become decimals, fractions, or negatives, e.g., $9 = 4B1 = 4\frac{1}{2} = 5B-1$ 2s.

Finally, addition is a cross, showing the two ways to add stacks, on-top using the linearity of recounting to make the units like, or next-to creating integral calculus by adding areas, which is also used when adding per-numbers needed to be multiplied to areas before adding.

Including units provides an ‘Algebra Square’ showing how to unite the four types of numbers:

Multiplication and addition unite like and unlike unit-numbers, and power and integration unite like and unlike per-numbers.

And how to split totals with the opposite operations: division and subtraction, together with the factor-finding root or the factor-counting logarithm and differentiation.

Table 01. The Algebra Square shows the ways to reunite unlike and like unit- and per-numbers

Uniting	Unlike	Like
<i>splitting Totals in</i>		
Unit-numbers	$T = a + n$	$T = a * n$
m, s, kg, \$	$T - n = a$	$T/n = a$
Per-numbers	$T = \int f dx$	$T = a^n$
m/s, \$/100\$ = %	$dT/dx = f$	$\sqrt[n]{T} = a \quad \log_a(T) = n$

References

- Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education* 11(1), 103–117.
- Tarp, A. (2025). Math is fun with bundle-numbers on a bundle-bundle-board. In Kwon, O., Kaur, B., Pang, J., Noh, J., Lee, S., Han, S., Yeo, S., & Lim, M. (Eds.). (2025). *Proceedings of the 9th ICMI-East Asia Regional Conference on Mathematics Education (Vol. 1)*. Seoul National University, Siheung Campus, Korea: EARCOME9, 363-392.

Micro Curricula in BBM BundleMath, Counting before Adding

Children use numbers with Bundle-units. So, we now design micro-curricula, MC, to develop a number-language by working with totals existing as outside things and actions on a BBBoard that may be supplied with centimeter cubes placed on-top of the BBBoard to meet the UN Sustainable Development Goal 4.6 that by 2030, all youth must be Numerate.

MC01. Digits as icons in space, IIII = 5

The total here exists as sticks to be rearranged at a table and reported by a drawing on paper. The ‘T=?’ question is answered in two ways, as a collection of single ones, I I I I, and as one bundle of ones, IIII, that may be rearranged into an icon, 5, called a digit containing the number of sticks that it represents if written in a less sloppy way, and thus similar to the digits on a calculator. Each time a folding ruler is folded to look like the icon.

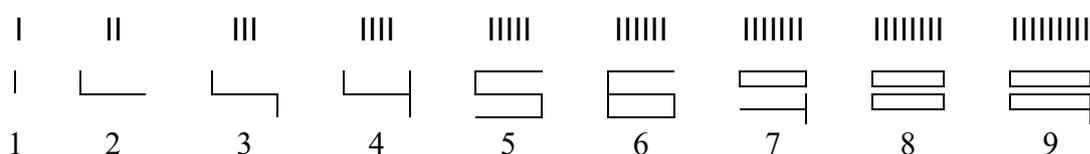


Figure 01. A digit as an icon with the number of sticks as it represents, four in the 4-icon, etc.

If we count in tens, ten sticks are replaced by one stick in a different color or material to allow more bundles to be rearranged as icons also, so that 67 means 6-Bundle-7, 6B7, called 6-ten-7, or more precisely, 6-tens-7, or 6 Bundles with tens per Bundle and 7 unbundled singles. So, 67 is not one number, but three numberings: of the number of Bundles, 6, of the Bundle-size, tens, and of the number of unbundled singles, 7. Ten thus needs no icon since it becomes ‘one bundle and no unbundled’, written as T = 1B0 tens, or T = 1.0 tens, or T = 10 if leaving out the unit and the decimal point.

Example. One stick is one single. An extra stick added to 1 stick gives two singles that may unite to one 2-icon. And so on. An extra stick added to 8 sticks gives nine singles that may unite to one 9-icon. And an extra stick added to 9 sticks gives ten singles that instead of uniting to one ten-icon is bundled together as one bundle replaced by one stick of a different color or a different material and written as 1B0 since there are no singles left.

An extra stick added to ten sticks gives eleven singles that may unite to one bundle and 1 single left, which made the Vikings call eleven ‘one left’ and written as 1B1.

Likewise with twelve that the Vikings called ‘two left’. There is no ‘three left’ because of the ancient counting method “one, two, many”. So, from 3 we specify both the bundles and the singles.

Zero is iconized as a looking glass not finding anything. The name ‘twenty’ comes from the Vikings’ ‘twende ti’.

Skill building. Roll some dice twice (physically or virtually) to get the number of bundles and unbundled singles. Then phrase and report the number.

So, with 3 and 5, say ‘3 Bundles with tens per Bundle and five unBundled’, ‘three-bundle-five’, three-ten-five, and thirty-five; and finally write T = 3B5 = 35.

End test. Roll some dice twice an extra time.

MC02. Tally-counting in time, ***** = IIII I

The total here exists as sticks to be moved one by one on a table and reported by strokes on paper. The ‘T=?’ question is answered by tally-counting the total in time reported as some 5-bundles and some unbundles singles, e.g., T = 2B1 5s.

Example. In a sentence, count the e’s and the a’s.

Skill building. Some dice is rolled a dozen times to show Even (1 2s, 2 2s or 3 2s) or Odd (1, 3, 5). The tally counting in 5s is reported with two totals e.g., W = 1B4, and L = 0B3, giving a total T = 2B2, or 1B7, or 3B-3. And giving the difference D = 1B1.

End test. Roll a dice some extra times.

MC03. Bundle-counting in time with units: 0B1, ..., 0B5 or 1B0, 3 3s = 1BB

<p>The total here exists as lines or stacks on a BBBoard.</p> <p>The ‘T=?’ question is answered in time by moving the finger along the pegs with a counting sequence that by including the bundle as a unit makes the place value system unneeded.</p> <p>First, we count in lines, then in stacks marked by a vertical rubber band.</p>	
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Counting 5 fingers in 3s we ask, “What do we have here?” to emphasize that we focus on existence instead of essence. We cannot count one finger as ‘one’ since 1 3s is 3 1s and we only have one. Instead, we count ‘0 bundle 1, 0B2, 0B3 or 1B0’ since 3 1s is 1 bundle with no unbundled left.

Counting 5 fingers in 2s we notice that four fingers are 1B2, but also 2B0, and 1BB0B0 since 2 2s is a bundle of bundles, a bundle-bundle, a BB, that is a square, as is 3 3s, 4 4s etc.

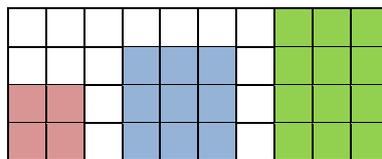


Figure 02. 2 2s, and 3 3s, and 4 4s as bundle-bundle squares

Counting the five fingers on a hand, their essence ‘5’ may exist in various ways:

$$T = 1\ 5s = 1B1\ 4s = 1B2\ 3s = 1B3\ 2s = 2B1\ 2s = 1BB\ 0B\ 1\ 2s = 5\ 1s$$

Counting ten fingers in 3s, we get 0B1, 0B2, 0B3 or 1B0, 1B1, ..., 2B3 or 3B0, 3B1.

T = 3B1 = 1BB0B1 3s, since 3 3s are a bundle of bundles, a bundle-bundle, a BB, a square.

Counting ten fingers in 2s we notice that 8 as 2BB 0B 0 is 1BBB 0BB 0B 0. So, we may also write ten as 1BBB 0BB 1B 0 2s, or as 1010 if leaving out the units. Finally, when counting hundred on the BBBoard we finish with 1BB 0B 0:

0B1, 0B2, ..., 0B9, 0Bten or 1B0, 1B1, ..., 9B8, 9B9, 9Bten or tenB0 or 1BB0B0.

1BB0B0	1BB0B1	1BB0B2	1BB0B3	1BB0B4	1BB0B5	1BB0B6	1BB0B7	1BB0B8	1BB0B9	1BB0B10
10B0	10B1	10B2	10B3	10B4	10B5	10B6	10B7	10B8	10B9	10B10
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	9B10
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	8B10
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	7B10
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	6B10
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	5B10
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	4B10
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	3B10
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	2B10
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	1B10
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	0B10

Figure 03. Counting from zero to 109 with units.

Skill building. Count a dozen and a score in 5s, 4s, 3s, and 2s.

End test. Count 30 in 3s.

MC04. Bundles counted in space with over- and underloads, $5 = 1B3 = 2B1 = 3B-1$ 2s

The total here exists as fingers and sticks. The ‘T=?’ question is answered in space by ‘flexible bundle counting’ that allows unbundled to stay unbundled as an overload, and that allows borrowing extra sticks to fill up an extra bundle with an underload. Using flexible bundle-numbers with units makes carrying and borrowing unneeded.

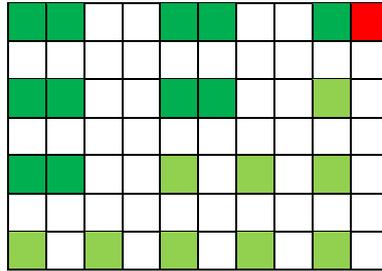


Figure 04. Five fingers may be recounted in 2s as 0B5, or 1B3, or 2B1, or 3B-1

Five fingers may be recounted in 5s as 0B5 (an overload) or 1B0 or 2B-5 (an underload).

Five fingers may be recounted in 4s as 0B5 or 1B1 or 2B-3.

Five fingers may be recounted in 3s as 0B5 or 1B2 or 2B-1.

Ten fingers may be recounted in tens as ½B from 1 to ten:

½B-4, ½B-3, ½B-2, ½B-1, ½B0, ½B1, ½B2, ½B3, ½B4, ½B5 or 1B0 as ten.

This may ease standard calculations.

$$T = 6 + 3 = \frac{1}{2}B1 + \frac{1}{2}B-2 = 1B-1 = 0B9 = 9$$

$$T = 6 + 7 = \frac{1}{2}B1 + \frac{1}{2}B2 = 1B3 = 13$$

$$T = 8 - 3 = \frac{1}{2}B3 - \frac{1}{2}B-2 = 0B5 = 5, \text{ thus showing that } -(-2) = +2$$

$$T = 4 * 7 = 4 * \frac{1}{2}B2 = 2B8 = 28$$

Overload	Underload	Overload	Overload
65 + 27	65 - 27	7 x 48	336 /7
6 B 5 + 2 B 7	6 B 5 - 2 B 7	7 x 4 B 8	33 B 6 /7
8 B12 9 B 2	4 B-2 3 B 8	28 B 56 33 B 6	28 B 56 /7 4 B 8
92	38	336	48

Figure 05. Bundles and over- and underload make place values, carrying and borrowing unneeded

The number 47 is not one number; there are three numberings:

47 = 4 ty 7 = 4 tens 7 = 4 Bundles, with tens per Bundle, and 7 unbundled ones.

Skill building. The action is repeated with nine fingers arranged counted in 5s, 4s, 3s, and 2s. And with 2-digit numbers, e.g., 67 = 6B7 = 7B-3 = 5B17. Then with cubes, and with sticks.

End test. The action is repeated on a BBBoard, or on an abacus.

MC05. Splitting, $8 = (8-2)+2$

The total here exists as a line of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pulling-away’ a bundle hidden under cubes.

A total of 1 8s is split by pulling-away 2. To pull-away just once may be iconized by a rope, -, so that ‘8-2’ means ‘from 8 pull-away 2’ in time, or ‘from 8 pulled-away 2’ in space.

The original 8 now is split in 8-2 and 2 so that $8 = (8-2) + 2$. Here addition is iconized by a cross showing the two directions we can add, next-to or on-top so that '4+2' means '4 with 2 added'. With T for the total and B for the bundle this 'split-formula' may be written as $T = (T-B)+B$.

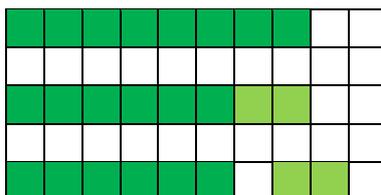


Figure 06. A total of 8 split in two parts by pulling-away 2, so $8 = (8 - 2) + 2$

Splitting may be used to solve equations coming from reversed actions. The question "What is the number that with 2 added gives 8" may be shortened to an equation with a letter for the unknown number, ' $u+2 = 8$ '. Of course, the number is found by reversing the action and pull-away the number that was originally added, so $u = 8-2$, which also comes from splitting 8, ' $u+2 = 8 = (8-2) + 2$ '. So, we see that the solution is found by moving "to the opposite side with the opposite sign". Also, it follows the formal definition of subtraction: 8-2 is the number u that added to 2 gives 8, or if $u+2 = 8$ then $u = 8-2$.

Skill building. The action is repeated with fingers, sticks, cubes, and an abacus. The action is repeated with other numbers, e.g., $9 = (9 - 3) + 3$.

End test. Pick two numbers.

MC06. Recounting, $8 = (8/2) \times 2$

The total here exists as lines of pegs on a BBBoard. The 'T=?' question is answered by 'pushing-away' bundles hidden under cubes.

A total of 1 8s is recounted in 2s by 4 times pushing-away 2s. To pull-away more times may be iconized by a broom, /, so that '8/2' means 'from 8 push-away 2s' in time, and 'from 8 pushed-away 2s' in space.

With the pushed-away 2s arranged in a stack, 8 contains 2s 4 times or 8/2 times so that $8 = 4 \times 2$, or $8 = (8/2) \times 2$. Here multiplication is iconized by a lift so that '4x2' or '4*2' means '4 times stacking 2s'. With T for the total and B for the bundle this 'recount-formula' may be written as $T = (T/B) \times B$.

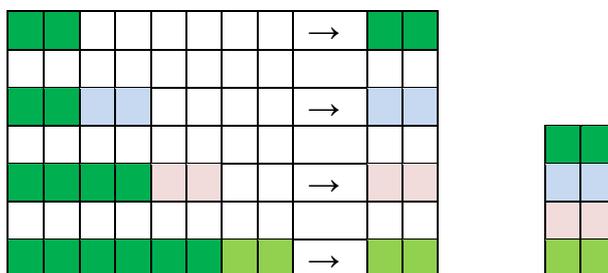


Figure 07. A total of 8 is recounted by pushing-away 2s and lifted into a stack, so $8 = (8/2) \times 2$

Recounting may be used to solve equations coming from reversing actions: The question "What is the number of 2s in 8" may be shortened to an equation with a letter for the unknown number, " $u \times 2 = 8$ ". Of course, the number is found by reversing the action and push-away the 2s that was originally united, so $u = 8/2$, which also comes from recounting 8, ' $u \times 2 = 8 = (8/2) \times 2$ '. So, we see that the solution is found by moving "to the opposite side with the opposite sign". Also, it follows the formal definition of division: 8/2 is the number u that multiplied with 2 gives 8, or if $u \times 2 = 8$ then $u = 8/2$.

Skill building. The action is repeated with 12 counted in 2s and 3s using a finger to hide a bundle.

End test. 18 counted in 2s, and in 3s.

MC07. Including the unbundled, $8 = (8/3)*3 = 2B2 = 2 \frac{2}{3} = 3B-1 \text{ 3s}$

The total here exists as cubes. The ‘T=?’ question is answered by pushing-away bundles to a stack that is then pulled-away to find the unbundles that then are included on-top of the stack.

Recounting 8 in 3s, first 2 times we push-away 3s, then we pull-away the stack of 2 3s and find 2 unbundled that are placed on-top of the stack. Here they may be seen as singles in a bundle described by a decimal number, $8 = 2B2 \text{ 3s}$, or as a fraction part when also recounted in 3s as $2 = (2/3)*3$, $8 = 2 \frac{2}{3} \text{ 3s}$. Or we may write $8 = 3B-1$ to show that in space 1 is missing in the next bundle, or that in time 1 is pulled-away from it.

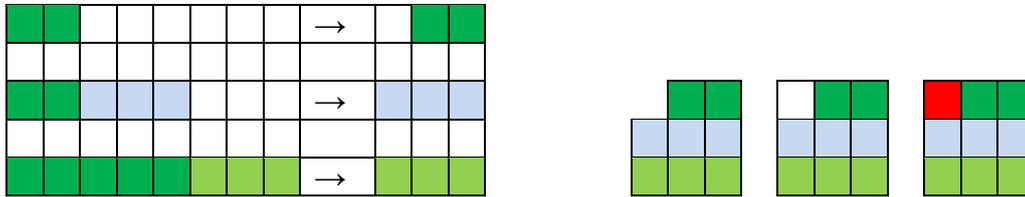


Figure 08. Unbundled become decimals, fractions or less-numbers, $8 = 2B2 = 2 \frac{2}{3} = 3B-1 \text{ 3s}$

With ten as the bundle-number, unbundled occur in the same way: $T = 4B3 = 4 \frac{3}{10} = 5B-7 \text{ tens}$.

Skill building. The action is repeated on a BBBoard with 11 counted in 3s and 4s using cubes or fingers to hide a bundle.

End test. Recount 8 in 5s, and in 3s.

MC08. Recounting in squares, $6 \text{ 4s} = 1 \text{ BB ?s}$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by working on the upper right corner occurring when half the excess is move from the top to the side to give a first guess about the square root. The rectangle may be shown with rubber bands or with cubes.

If we want to square the total $T = 6 \text{ 4s}$ we move half of the excessing 2 4s from the top to the side to get a 5×5 square, and an unfilled square in the upper right corner that we try to fill with a rectangular $4*u$ slice of the top and the side.

Here u is found by the equation $2*u*4 = 1$, or $8*u = 1$, giving $u = 1/8 = 0.125$, and $5-0.125 = 4.875$ as our next guess.

However, now there is too much in the corner, so we repeat the process, or consult a calculator showing that the correct answer is $\sqrt{(6*4)} = 4.90$, which is very close to our third guess.

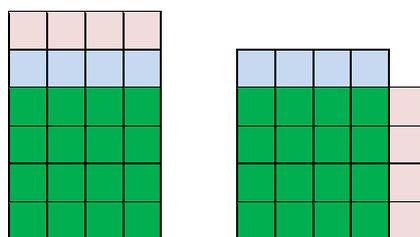


Figure 09. Recounting 6 4s by moving half the excess to the side to try to get a 5 x 5 square

To find the quadratic numbers we see that 5 5s comes from 4 4s by adding 4 twice and 1 for the top right corner. So, with 4 4s as 16, 5 5s is $16 + 4 + 4 + 1 = 25$.

In this way we may predict the square numbers to be 1, 4, 9, 16, 25, 36, 49, 84, 91 and 100.

And we see that a BB square increases with $2B+1$ when B increases with 1.

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. Square 9 5s. Solve the quadratic equation $x^2 + 8*x + 12 = 0$

MC09. Recounting in another icon, $3\ 4s = ?5$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered on a BBBoard and predicted on a calculator.

With rubber bands on a BBBoard we see that 3 4s may be recounted as 2B2 5s. This may be predicted by a calculator. To find how many 5s there is in 3 4s we enter “ $3*4/5$ ”. The answer is ‘2.more’.

To find them we pull-away the stack of 2 5s by entering ‘ $3*4-2*5$ ’ that gives the answer ‘2’. So, the calculator predicts that 3 4s recount as 2B2 5s, which is validated on the BBBoard.



Figure 10. A total of 3 4s recounted in 5s manually. and predicted by a calculator

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. $4\ 5s = ?6s$.

MC10. Recounting from tens to icons, $2\ tens = ?\ 7s$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by recounting.

With rubber bands on a BBBoard we see that 2 tens may be recounted as 2B6 7s. This may be predicted by a calculator. To find how many 7s there is in 2 tens by recounting we enter ‘ $20/7$ ’.

The answer is ‘2.more’ found by pulling away the stack predicted by ‘ $20-2*7$ ’ giving ‘6’. So, the calculator predicts that 2 tens recount as 2B6 7s, which is validated on the BBBoard.

Alternatively, the question “How many 7s in 20?” leads to the equation $u*7 = 20$ that is solved moving to opposite side with opposite sign so that again $u = 20/7$, or $u = 2\ 6/7$.

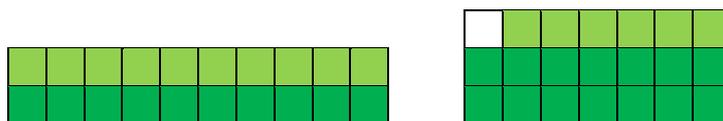


Figure 11. A total of 2 tens recounted in 7s manually, and predicted by a calculator

We notice that decreasing the bundle will increase the height. To study this closer we recount 1 dozen in 6s, 4s, 3s, 2s, and 1s and place a dot in the upper right corner each time.

The points then form a curve called a hyperbola.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. $4\ tens = ?8s$.

MC11. Recounting from icons to tens, $6\ 7s = ?\ tens$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by finding what we must pull-away from the bundle-bundle.

Recounting from icons to tens apparently is another name for the multiplication tables.

With rubber bands on a BBBoard we see that 6 7s is left if from the ten bundles we pull-away 4 top and 3 side bundles, and add the upper right 4 3s that we pull-away twice:

$$T = 6\ 7s = 6*7 = (10 - 4 - 3)*B + 4\ 3s = 3B + 1B2 = 4B2 = 42.$$

This leads to early algebra if instead we write:

$$T = 6\ 7s = 6*7 = (B - 4)*(B - 3) = BB - 4*B - 3*B + 3*4$$

Here we see that minus times minus must be plus.

So, a quick way to find the answer is to add and multiply the less-numbers and subtract the first and add the latter. With 4 and 3 as the less-numbers here, we quickly learn to say:

“Less (4+3) bundle (4*3)”, or “Less 7 bundle 12”, or “3 bundle 12”, or “4 bundle 2”, or “42”.

We may also write $B - 4$ and $B - 3$ on top of each other and then multiply down and across. Or we may use the FOIL method: First, Outside, Inside, Last.

	$T = 6 * 7$ $= (B-4) * (B-3)$ $= BB - 4B - 3B + 4*3$ $= 3B12$ $= 4B2$ $= 42$	$T = \begin{pmatrix} 1B & -4 \\ 1B & -3 \end{pmatrix}$ $= 1BB - 4B - 3B + 4*3$ $= 10B - 7B + 1B2$ $= 3B12 = 42$	$T = \begin{pmatrix} 2B & +3 \\ 4B & +6 \end{pmatrix}$ $= 8BB + 12B + 12B + 18$ $= 8BB + 24B + 18$ $= 10BB 5B 8$ $= 1058$
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Figure 12. $6*7$ is left when pulling-away $4B$ and $3B$ and adding the 4 3 s pulled away twice

Multiplying the two-digit numbers $23*46$ as $2B3$ $4B6$ s, a vertical and a horizontal rubber band between the bundles and the singles allows a BBBoard to show the four stacks $2B$ $4B$ s, and $2B$ 6 s, below the 3 $4B$ s, and the 3 6 s. With overloads, they add up to $8BB$ 24 B 18 , or to $10BB$ $5B$ $8 = 1058$ without.

This process may be reversed when asking ‘ $1058 = ? 46$ s’. First 1058 is written with an overload as $10BB$ $5B$ $8 = 8BB$ $25B$ 8 . Since $4B*2B = 8BB$, the $2B$ contributes $2B*6$ to the 25 B s. The rest $13B$ 8 may be rewritten as $12B$ 18 , which recounted in 3 s gives $4B6$. So, the answer is $1058 = 23$ 46 s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 7 8 s = ? tens.

MC12. Recounting in another physical unit creates per-numbers, $3\$/5\text{kg}$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘ $T=?$ ’ question is answered by changing the unit in the per-number rectangle.

Recounting a physical total T as 3% and 5 kg gives a ‘per-number’ $3\$/5\text{kg}$ called the price and marked as a 3×5 rectangle on a BBBoard.

The question “ $20\text{kg} = ?\%$ ” then is answered by recounting in the per-number:

$$20\text{kg} = (20/5)*5\text{kg} = (20/5)*3\% = 12 \%$$

On a BBBoard the counting sequences now are 5 , 10 , 15 , 20kg , and 3 , 6 , 9 , 12% since the per-number here has changed from $3/5$ to $12/20$.

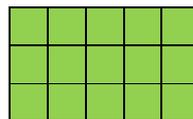


Figure 13. The per-number $3\$/5\text{kg}$ show as 3 5 s, or 6 10 s, or 9 15 s, or 12 20 s, etc.

Or we may introduce a new unit to make the digits like: $3\% = (3/5)*5\% = n*5\% = 5n\%$ with the new $n = 3/5$. So, 20 $\text{kg} = 20n\% = 20*3/5\% = 12\%$.

Alternatively, the units may be recounted:

$$\% = (\%/\text{kg})*\text{kg} = (3/5)*20 = 12$$

Or we may equate the per-numbers: $\%/\text{kg} = u/20 = 3/5$. Moving to opposite side with opposite sign we then get $5*u = 3*20$, or $u = 3*20/5 = 12$.

Skill building. The action is repeated with other numbers.

End test. With $5\$/2\text{kg}$, $12\text{kg} = ?\$,$ and $? \text{kg} = 12\$.$

MC13. With the same unit, per-numbers become fractions, $3\$/5\$ = 3/5$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

If a whole contains a part, they have the same unit. In this case the per-number becomes a fraction without units. Still, we may use the units ‘p’ and ‘w’ for the part and the whole.

To get the fraction $3/5$ of $20\$$ thus means to get $3p/5w$ of a $20\$$ whole. Recounting in the per-number thus gives $20w = (20/5)*5w = (20/5)*3p = 12p,$ or $12\$$ of $20\$.$

To get the fraction $3/5$ of 100 thus means to get $3p/5w$ of a 100 whole. Recounting in the per-number thus gives $100w = (100/5)*5w = (100/5)*3p = 60p,$ or 60 of $100,$ written as $60\%.$

To ask “ $20\$$ is what percentage of $80\$$ ” means asking about the fraction $20/80$ of $100.$ Or we may introduce a new unit $80\$ = 100\%$ to see that $20\$ = (20/80)*80\$ = (20/80)*100\% = 40\%.$

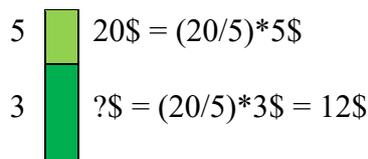


Figure 14. A fraction column with per-numbers to the left and unit-numbers to the right

To add 10% to $200\$$ we introduce the per-number $200\$/100\%.$ After the addition the total is

$$T = 100\% + 10\% = 110\% = (110/100)*100\% = (110/100)*200\$ = 220\$.$$

So, adding 10% means multiplying with $110\%,$ and adding 10% 5 times means multiplying with $110\%^5 = 161.1\%$ thus giving 50% plus 11.1% extra, also called compound interest.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. With $2p/5w,$ $10p = ?w,$ and $?p = 20w,$ and $2p/5w = ?\%.$

MC14. Recounting a stack’s sides gives trigonometry, rise = (rise/run)*run = tanA*run

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

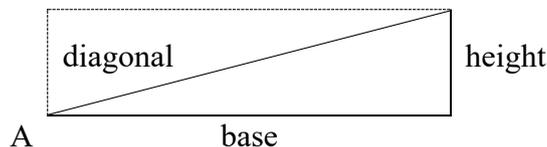


Figure 15. A stack with a base and a height and a diagonal

On a BBBoard, we mark a 3×4 stack as a rectangle with height 3 and base $4.$ If we recount the height and the base in the diagonal, we get the per-numbers sine and cosine:

$$\text{height} = (\text{height/diagonal}) * \text{diagonal} = \text{sine Angle} * \text{diagonal}, \text{ shortened to}$$

$$h = (h/d) * d = \sin A * d = \sin A ds,$$

This gives the formula $\sin A = \text{height} / \text{diagonal},$ or $\sin A = h/d,$ or $\sin A = 3/5$ in our case.

Likewise, $\cos A = \text{base} / \text{diagonal},$ or $\sin A = b/d,$ or $\cos A = 4/5$ in our case.

$$\text{height} = (\text{height/base}) * \text{base} = \text{tangent Angle} * \text{base}, \text{ shortened to}$$

$$h = (h/b) * b = \tan A * b = \tan A bs,$$

This gives the formula $\tan A = \text{height} / \text{base}$, or $\tan A = h/b$, or $\tan A = 5/10$ in our case.

A protractor shows that the angle A is a little above 25 degrees. Testing this we get $\tan 25 = 0.466$. The reverse \tan -button ‘ \tan^{-1} ’ gives the precise result, $\tan^{-1}(0.5) = 26.6$ degrees.

Using the words ‘run’ and ‘rise’ instead of ‘base’ and ‘height’, we get the diagonal’s slope-formula: $\tan A = \text{rise}/\text{run}$. Here the tangent-number describes the steepness of the diagonal.

In a x - y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is a diagonal in a rectangle, and since the run and the rise are changes in x , Δx , and in y , Δy , the tangent-number here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

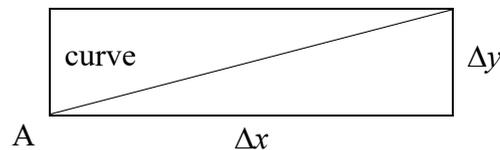


Figure 16. A curving curve is linear locally with small changes in x and y

The word ‘tangent’ is used since the height will be a tangent in a circle with center in A , and with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, $h = \tan A$.

A half circle is 180 degrees that split in 100 small parts as $180 = (180/100) \cdot 100 = 1.8 \cdot 100s = 100 \cdot 1.8s$. With A as 1.8 degrees, the circle and the tangent, h , are almost identical. Half the circumference in a circle with radius 1 is called π , and $\pi = 100 \cdot h = 100 \cdot \tan 1.8 = 100 \cdot \tan (180/100) = 3.1416$. This gives a formula for the number π : $\pi = \tan (180/n) \cdot n$, for n large enough. We also see that in a circle with radius r , the circumference is $2 \cdot \pi \cdot r$, and the area is $\pi \cdot r^2$, or $\pi/4 \cdot d^2$ where d is the circle’s diameter. So, a d -circle takes up almost 80% of the space inside the surrounding d -square.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. Add a 4-square and a 6-square as a square.

MC15. Adding next-to or on-top, $T = 2 \cdot 3s + 4 \cdot 5s = ? \cdot 8s$; $T = ? \cdot 3s$; $T = ? \cdot 5s$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘ $T=?$ ’ question is answered by recounting.

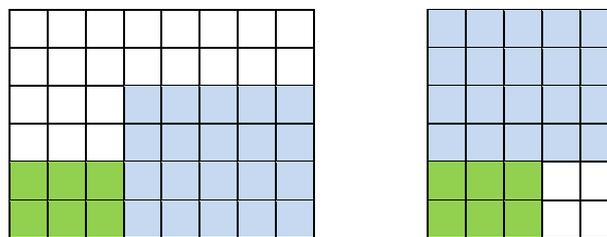


Figure 17. Two totals $2 \cdot 3s$ and $4 \cdot 5s$ added next-to, and added on-top

Adding $2 \cdot 3s$ and $4 \cdot 5s$ as $8s$ means adding areas, which is called integral calculus. Reversing the process by asking ‘ $2 \cdot 3s$ and how many $5s$ total $4 \cdot 8s$ ’ is called differential calculus because you find the difference between the known totals before recounting it in $5s$, $(T_2 - T_1)/5$, or $\Delta T/5$.

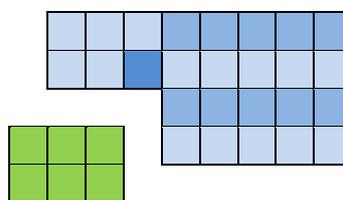


Figure 18. Next-to addition is reversed when asking $2 \cdot 3s + ? \cdot 5s = 4 \cdot 8s$

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like To recount 2 3s in 5s, first we enter '2*3/5' giving '1.more', then we enter '2*3-1*5' giving '1', so 2 3s is 1B1 5s, which give the total 1B1 5s + 4B0 5s = (1B1 + 4B0) 5s = 5B1 5s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 3 4s + 6 5s = ? 9s. And 3 4s + 6 5s = ? 4s. And 3 4s + 6 5s = ? 5s. And 3 2s + ? 5s = 4 6s.

MC16. Subtracting and adding single digit numbers, 8+6 = 1B2 + 1B0 = 2B2 6s

The total here exists as lines of pegs on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

With a subtraction as '8 - 6=?', a rubber band marks 8 on a BBBoard, and fingers hide the 6 that is pulled-away, so 8 - 6 = 2.

With an addition as '8+6=?', two rubber bands marks 8 and 6 on two BBBoard parallel lines to show that the sum may exist in two ways, as 2B2 6s, or as 2B-2 8s.



Figure 19. Adding 6 and 8 as 2B2 6s, or as 2B-2 8s, or as 2*1/2B + 1 + 3

Here, using half bundles, 5s, will easy recounting in tens since 6+8 = 1/2B1 + 1/2B3 = 1B4 = 14.

Multidigit numbers may be added and subtracted with an over- or an under-load, which makes carrying and borrowing unneeded.

T = 36 + 47 = 3B6 + 4B7 = 7B13 = 8B3 = 83

T = 86 - 37 = 8B6 - 3B7 = 5B-1 = 4B9 = 49

T = 4*67 = 4*6B7 = 24B28 = 26B8 = 268

T = 268 /4 = 26B8 /4 = 24B28 /4 = 6B7 = 67

Skill building. The action is repeated with other one-digit and two-digit numbers.

End test. 9 - 7 = ?, 9+7 = ?, T = 38+46 = ?; T = 82 - 54 = ?

MC17. Adding per-numbers and fractions by integral calculus

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

Asking "2kg at 3\$/kg and 4 kg at 5\$/kg total what?" the unit numbers 2kg and 4kg add directly whereas the per-numbers 3\$/kg and 5\$/kg first must be multiplied to unit-numbers before adding, thus added as areas, i.e., as integral calculus. Here the per-numbers are piecewise constant, but they may also be locally constant as in the case with a falling object having an increasing meter/second number.

Before adding, fractions must also be multiplied to unit, numbers. So, with apples, 1red of 2 plus 2red of 3 gives 3red of 5, and of course not 7red of 6 as taught by 'mathematism'.

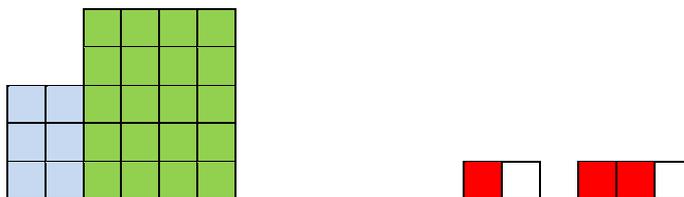


Figure 20. Per-numbers add as areas and fractions add with units, both as integral calculus

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives $106\%^{10}$ or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

	B	ⓑ	
Left	1	2	3
Right	1	1	2
Total	2	3	5

	B	ⓑ	
Left	1/3	2/3	1
Right	1/2	1/2	1
Total	-	-	-

	B	ⓑ	
Left	1/2	2/3	-
Right	1/2	1/3	-
Total	1	1	-

Figure 21. In cross tables, per-numbers must pass through the unit-numbers

Looking at my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bent the two outer fingers. So, $1/3$ of the Ls are bent, and $1/2$ of the Rs. Does ‘ $1/3$ of the Ls are bent’ mean that ‘ $1/3$ of the bent are Ls’? No, $1/2$ is. So, in a cross table we cannot go from the per-numbers in one direction to those in the other direction without going through the unit-number table. This is called the Bayes-principle.

Skill building. The action is repeated with other numbers.

End test. 3kg at 4\$/kg and 5 kg at 6\$/kg total what?”

MC18. Adding and subtracting Bundle-Bundle squares

The total here exists as bundle-bundle number squares on a BBBoard. The ‘T=?’ question is answered by using rubber bands or cubes to mark the bundles.

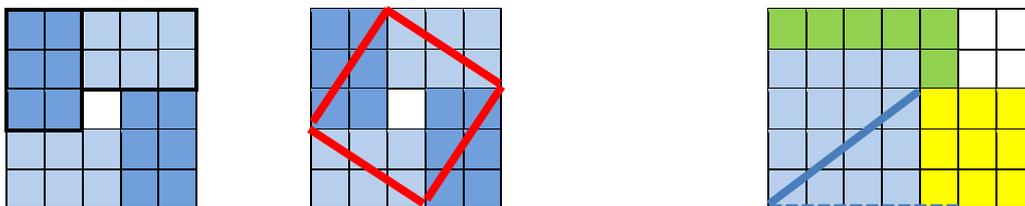


Figure 22. The two squares add as the square formed by the mutual Bottom-Top BT line

On a BBBoard we place four 7 3s so they form a ten-by-ten square that inside contains two squares, 7 7s and 3 3s as well as two stacks. But it also contains one square formed by the diagonals in the stacks as well as four half stacks. So, the two squares add as the square formed by the mutual Bottom-Top BT line thus having the length as the square-root of the sum, i.e., $\sqrt{7^2 + 3^2} = 7.62$. So, in this stack, adding the height and the bundle as squares gives the square of the diagonal. This rule is named by the ancient Greek thinker, Pythagoras.

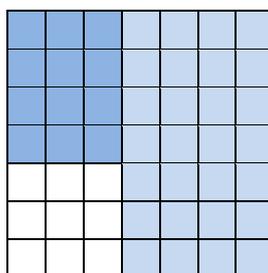


Figure 23. On a BBBoard we see that $7^2 - 3^2 = 7*(7-3) + (7-3)*3 = (7+3)*(7-3)$

On a BBBoard we pull-away the a 3-by-3 square from a 7-by-7 square. This leaves 7 (7-3)s and (7-3) 3s that may be turned to 3 (7-3)s totaling (7+3) (7-3)s. So $7^2 - 3^2 = (7+3)*(7-3)$.

Adding squares may also be involved when solving a quadratic equation. On a BBBoard we see that $T = (x+3)*(x+3)$ is a square with four parts, two squares x^2 and 3^2 , and two stacks $2*3*x$, so that $T = x^2 + 6*x + 9$. The quadratic equation $x^2 + 6*x + 8 = 0$ then makes the whole square go away

except for $9-8 = 1$. So $(x+3)^2 = 1$. This gives two solutions, $x = -2$ and $x = -4$ that hold when tested: $(-2)^2 + 6*(-2) + 8 = 4-12+8 = 0$, and $(-4)^2 + 6*(-4) + 8 = 16-24+8 = 0$.

The quadratic equation ' $x^2 + 6*x + 10 = 0$ ' has no solutions since here ' $(x + 3)^2 = -1$ '.

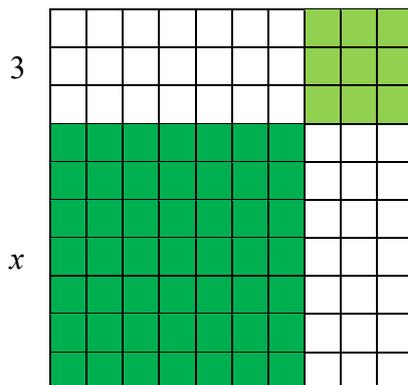


Figure 24. An $(x+3)$ -square contains an x -square and a 3 -square and two $3*x$ stacks

Alternatively, we can rewrite the equation $x^2 + 6*x + 8 = 0$, first as

$$x^2 + 2*3*x + 3^2 - 3^2 + 8 = 0, \text{ then as}$$

$$(x+3)^2 - 9 + 8 = 0, \text{ then as}$$

$(x+3)^2 = 9 - 8 = 1$ again with the solutions $x = -2$ and $x = -4$ that hold when tested:

$$(-2)^2 + 6*(-2) + 8 = 4-12+8 = 0, \text{ and, } (-4)^2 + 6*(-4) + 8 = 16-24+8 = 0$$

Skill building. The action is repeated with other numbers.

End test. Add 3 3s and 4 4s as a bundle-square. Solve the equation $x^2 + 8*x + 12 = 0$

MC19. Adding unspecified letter-numbers

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

In the letter-number $T = 3ab$ the multiplication sign is invisible, and the letters stand for unspecified numbers. Since any factor may be a unit, T may be seen as $3abs$, or as $(3a)bs$, or as $(3b)as$. To avoid being confused by the 's' we will omit it, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$.

Since totals need a common unit to add, this must be first found:

$$T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a$$

$$T = 2ab^2 + 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b$$

Skill building. The action is repeated with other numbers and letters.

End test. $T = 4ab^2d + 8bcd$

MC20. Change in time

The total here exists as dots on a BBBoard. The 'T=?' question is answered by transferring the results to a squared paper and connect the dots with a curve.

In time, a total grows by being added or multiplied by a number, called addition-growth and multiplying-growth, or linear and exponential growth.

Addition-growth:

$$\text{Final number} = \text{Initial number} + \text{growth-number} * \text{growth times}, \text{ or shortly, } T = B + a*n.$$

The number a is also called the slope.

Multiplying-growth:

$$\text{Final number} = \text{Initial number} * \text{growth-factor} ^ \text{growth times}, \text{ or shortly, } T = B * a^n, \text{ since } 200\$ + 5\% = (200*105\%) \$, \text{ so here } a \text{ is } 1 + \text{interest rate} = 100\% + 5\% = 105\%.$$

Combined growth (savings in a bank): Here we have that $A/a = R/r$, where A is end-dollars, a is the period-dollars, R is the end-rate, r is the period-rate, and $1+R = (1+r)^n$, where n is the number of periods.

100% split in n parts will give the Euler number $e = (1+1/n)^n = 2.718$ for n large enough.

Changing the growth-number constantly will give a quadratic growth with a parabola curving upwards or downwards if the number increases or decreases.

Changing the curvature constantly will give cubic growth with a double parabola with curvature and counter-curvature.

Decreasing the growth-factor constantly give logistic saturation growth with a hill-curve in infections. Confusing exponential and saturation growth can cause unnecessary damage.

MC21. Bundle-numbers in a coordinate system

The total here exists as dots on a BBBoard. The ‘ $T=?$ ’ question is answered by rubber bands as lines on the BBBoard. The bundle-number ‘ yxs ’ with a height y and a width x may be called a ‘changing bundle-number’. Here $y = 2*x$ gives a rising and $y = 9-x$ a falling bundle-number.

Marking the top right corners we get two lines. To inside predict the outside intersection point we equate the two heights, $2*x = 9 - x$. Moving to opposite side with opposite sign we get $3*x = 9$, and $x = 9/3 = 3$, which makes $y = 2*3 = 6$.

So, the prediction is that the two bundle-numbers become like as 3 6s, which is validated on the BBBoard where the first dot now is 0 instead of 1.

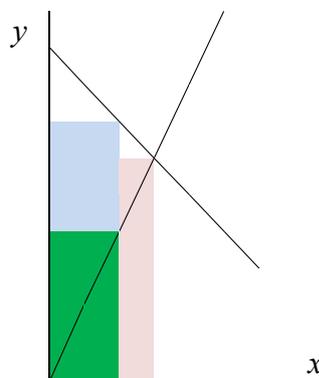


Figure 25. In an x - y coordinate system bundle-number stacks may rise and fall

In a rising bundle-number its total T will increase since here the height increases with increasing width. In the falling bundle-number this is not the case since the height decreases with increasing width. Here the total is $T = y*x = (9-x)*x = 9x - x^2$.

Setting up a table with $x = 0, 1, 2, \dots, 9$ we see that first T increases and then T decreases; and that T tops as 20 for $x = 4$ and $x = 5$; and that $T = 20.25$ for $x = 4.5$.

x	0	1	2	3	4	5	6	7	8	9
y	0	8	14	18	20	20	18	14	8	0

In general, even if a bundle-number is rising, its rise may fall, so its marked corners will lay on a bending line called a parabola where $y = b*x + a*x^2$.

Passing the points $(x,y) = (1,6)$ and $(2,10)$ we find that $10 = b*2 + a*4$, and $6 = b*1 + a*1$, or $12 = b*2 + a*2$. We now equate the two equations for $b*2$: $10 - a*4 = 12 - a*2$. Moving to opposite side with opposite sign we get $10 - 12 = a*4 - a*2$, or $-2 = a*2$, or $-1 = a$. With $6 = b + a$, this gives $b = 7$.

So on the parabola, the points (x,y) are connected by the formula $y = 7*x - x^2$. It thus passes through the points $(0,0)$, $(1,6)$, $(2,10)$, $(3,12)$, $(4,12)$, $(5,10)$, $(6,6)$, and $(7,0)$.

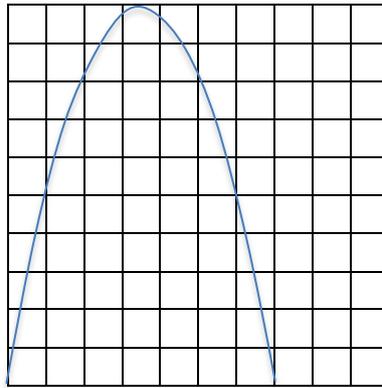


Figure 26. Passing through (0,0) and (1,6) and (2,10), the parabola formula is $y = 7x - x^2$

Can the intersection points be predicted between the parabola and the two line above?

Can it be predicted that a falling bundle-number $b - a * x$ will have its maximum at the width $b / (2 * a)$?

Skill building. The action is repeated with other rising and falling bundle-numbers to find when they are like and when the falling bundle-numbers tops.

End test. $y = 9 - 2 * x$ and $y = x$.

MC22. Games Theory and damage control

The total here exists as towers of cubes and dots on a BBBoard. The 'T=?' question is answered by rubber bands as lines on the BBBoard.

In a Game Theory 2x2 zero-sum game two players A and B each have 2 strategies resulting in four different payments from B to A. It is called a zero-sum game since one player's gain is the other player's loss. In some game, if B chooses strategy B1 then the payment to A is 8\$ or 2\$ if A chooses strategy A1 or A2. And if B chooses strategy B2 then the payment to A is 4\$ or 6\$ and A chooses strategy A1 or A2. We may show this game by building four towers with cubes.

First, we assume that B chooses strategy B1. If A now p times chooses A2 and $n - p$ times A1 then A's outcome after n rounds will total $T = 2 * p + 8 * (n - p) = 8 * n - 6 * p = (8 * n - 6 * p) / n * n = (8 - 6 * p / n) * n$.

Which is $T1 = 8 - 6 * p / n$ per round, shown on a BBBoard as a line connecting 8 to the left where p is 0, to 2 to the right where p is n . Next, we assume that B chooses strategy B2. If A now p times chooses A2 and $n - p$ times A1 then A's outcome after n rounds will total $T = 6 * p + 4 * (n - p) = 4 * n + 2 * p = (4 * n + 2 * p) / n * n = (4 + 2 * p / n) * n$. Which is $T2 = 4 + 2 * p / n$ per round, shown on a BBBoard as a line connecting 4 to the left where p is 0, to 6 to the right where p is n .

With p / n as u we find the intersection point by equating the two totals: $T1 = T2$, or $8 - 6 * u = 4 + 2 * u$, or $8 * u = 4 = (4 / 8) * 8$, or $u = 4 / 8 = 1/2$ giving $T1 = T2 = 5$.$

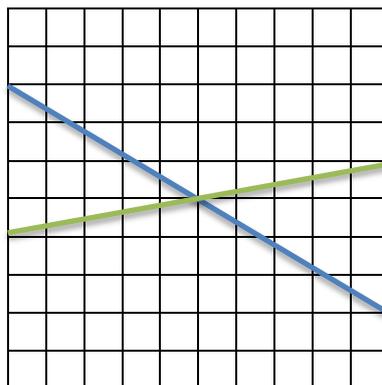


Figure 27. If B choses strategy B1, A will receive between 8 and 2\$, else between 4 and 6\$

So, if A mixes the strategies 1-to-1 by flipping a coin, the average result will be 5\$ per round.

Seen from B's side we also get the two lines $S1 = 4 + 4*u$, and $S2 = 6 - 4*u$ that intersect where $4 + 4*u = 6 - 4*u$, or $8u = 2$, or $u = 2/8$, or $u = 1/4$ giving $S1 = S2 = 5\$$.

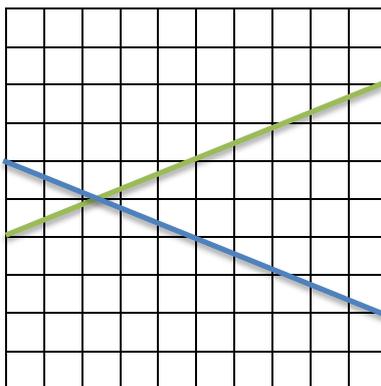


Figure 28. If A chooses strategy A1, A will receive between 4 and 8\$, else between 6 and 4\$

So, if B mixes the strategies 1-to-3 by flipping two coins then the average result again will be 5\$ per round. 5\$ then is called the value of the game, i.e., the amount B must receive per round to make the game fair with no winner or loser in the long run.

From A's side the 5\$ is called the 'maxi-min' value since deviating from it will decrease the value.

From B's side the 5\$ is called the 'mini-max' value since deviating from it will increase the value.

In a similar game 4\$ is changed to 8\$. Here the strategy A1 dominates A2 that will always be lower to A. Likewise, the strategy B1 dominates B2 that will always be higher to B. So, here the value of the game is 6. This point is called a saddle point since the payment goes up one way and down the other.

Skill building. The game is repeated with other payments found, e.g., by rolling some dice.

End test. Replace the four payments 8,2,4,6 with 9,3,5,8.

MC23. Simple board games

The goal is to experience how different games may take place on a BBBoard.

- A racetrack. A 4x4 lawn is placed in the middle of a BBBoard. The start- and end-line goes from (5,0) to (5,3). A trip may change zero or one unit in the horizontal and vertical direction. You may touch but not cross the interior or exterior boundaries. If you do so you restart with 0 speed as close to the crossing point as possible. You may cross your opponent's track, but not end in the same point. The race may be repeated with different lawn shapes.
- Survival. You begin in (5,5). You roll a dice and move the number to the right if the number is even, else to the left. You roll again and now move the number up if the number is even, else down. You may touch but not cross the boundary. How many steps can you survive?
- Vertical race. A rubber band splits a BBBoard vertically in the middle. Two players each have three bricks placed at level one. They roll a dice and pick a brick to move the number upwards, and down again if there is a surplus. The winner is the first to have all three bricks at level ten.

The Algebra Square

There are two kinds of numbers in the world, unit-numbers, and per-numbers, which may be unlike or like and which may be reunited. The aim of mathematics is therefore not to 'math', because you can't do that, but to act: 'Re-Unite Un-like & Like Unit-Numbers & Per-Numbers'.

The four operations that unite unlike and like unit-and per-numbers are: addition, multiplication, integration, and power as shown in the Algebra-square above that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations unite/ <i>split Totals in</i>	Unlike	Like
Unit-numbers m, s, kg, \$	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, \$/100\$ = %	$T = \int f dx$ $dT/dx = f$	$T = a^n$ $n\sqrt{T} = a \quad \log_a(T) = n$

Figure 29. The Algebra square shows how to unite and split our four number-types, and how to solve equations by moving 'to opposite side with opposite sign'.

Fact and fiction and fake, the three genres of number-models

Once we know how to count and recount totals, and how to unite and split the four number-types, unlike and like unit-numbers and per-numbers, we can actively use this number-language to produce inside tales about outside totals existing in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then and what-then models, or room, rate, and risk models.

Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas. Typically, they model the past and the present. They need to be checked for correctness and units.

Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically, they model the future. They need to be supplied with scenarios building on alternative assumptions.

Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas. Typically, they add without units or hide alternatives. Here, number stories need to be replaced by word stories.

Modeling and de-modeling

The goal is to experience how formulas calculating y from x form curves that express change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

Modeling means solving an outside problem inside with four steps. First an outside problem is translated to an inside problem. Then the inside problem leads to an inside solution that then is translated to an outside solution, that finally is evaluated to see if another cycle is needed.

A typical example is mixture problems. the outside problems here may ask "2kg at 3\$ per kg and 4kg at 5\$ per kg total what?" The inside problem places the second information under the first ready to add. The inside solution may then simply add all numbers, which leads to the outside solution "2kg at 3\$ per kg and 4kg at 5\$ per kg total 6 kg at 8\$/kg".

This model is not accepted, so another cycle is needed. This time the per-numbers are multiplied to unit numbers before adding, which leads to the outside solution "2kg at 3\$ per kg and 4kg at 5\$ per kg total 6 kg at 26\$/6kg". This model is accepted.

De-modeling is the opposite process: It means solving an inside problem outside with four steps. First an inside problem is translated to an outside problem, then the outside problem leads to an outside solution that then is translated to an inside solution, that finally is evaluated to see if another cycle is needed.

A typical example is uniting fractions.

Adding fractions as $1/2 + 2/3$ only has meaning when taken of the same unit,

$u = (u/6)*6 = k*6$, where $k = u/6$, and $6 = 2*3$

$T = (1/2 + 2/3)*u = (1/2 + 2/3)*6*k = (3+4)*k = 7*k = 7*u/6 = 7/6*u$,

So, in this case, $1/2 + 2/3 = 7/6$.

- We now model the orbit of a ball sent away with an angle. A constant up-number will give a line that goes up or down or horizontal.

But here gravity makes the up-number decrease so the line curves down as a bent line called a parabola.

We choose the initial angle A determined by $\tan A = 6$.

From (0,0) we assume that the ball takes a '1 out, 5 up' step followed by a '1 out, 3 up' and a '1 out, 1 up', etc., to reach the points (1,5), (2,8), (3, 9), (4,8), (5, 5), (6,0).

Since $y = 0$ for $x = 0$ and for $x = 6$, the formula may contain the two factors $(x-0)$ and $(6-x)$, so a guess could be $y = a*x*(6-x)$.

In the point (1,5) this formula becomes an equation,

$5 = a*1*(6-1)$, or $5 = a*5$, solved by $a = 1$.

So, the parabola formula may be $y = 1*x*(6-x)$, or

$y = -x^2 + 6*x$.

This formula holds when tested on the other points:

$8 = -2^2 + 6*2$, or $8 = -4 + 12$, or $8 = 8$, etc.

We find that with 4 as the first up-number, the orbit formula will be $y = -x^2 + 5*x$, etc.

The height after 5 steps is found by the equation $y = -5^2 + 6*5 = -25 + 30 = 5$.

The height 8 is reached after x steps found by the equation

$8 = -x^2 + 6*x$, or $x^2 - 6*x + 8 = 0$, solved by $x = 2$ and $x = 4$.

The height 10 is never reached since there are no solutions to the equation:

$10 = -x^2 + 6*x$, or $x^2 - 6*x + 10 = 0$.

Instead, that top-point is found in the middle at $x = 6/2 = 3$, giving $y = -3^2 + 6*3$, or $y = 9$.

To see if it breaks through a roof with the formula $y = 12-x$, we equate the two y s and get the equation $12 - x = -x^2 + 6*x$, or $x^2 - 7x + 12 = 0$, that is solved for $x = 3$ and $x = 4$.

- We now model the beginning monthly income of a business trying to establish itself at a market.

We use the formula $y = x^2 - 6x + 9$ where the steps form a parabola curving up when passing the points (0,9), (1,4), (2, 1), (3,0), (4,1), (5,4), and (6,9).

Later the monthly income changes its curvature from up to down until it reaches a maximum level.

So, from $x = 3$ we use a different model that contains the up-numbers 0, 1, 2, 3, 2, 1, 0, 0. This gives a 'logistical' s-shaped curve describing growth with saturation.

When prompted, AI may give a formula for this curve as $y = 9/(1+25*2^(-1.9*x))$

We see that the up-numbers form a hill. When prompted, AI may give a formula for this curve as $y = 3/(2^(0.44*(x-3)^2))$

- A cats and mice cohabitation on an island is an example of a predator-prey model where cats eat mice.

We expect a cycle in time since many cats and many mice leads to many cats and few mice, which leads to few cats and few mice, which leads to few cats and many mice, which leads to many cats and

many mice once again. In a model we assume that a mice-population at 7 and 2 will make the cat-population change with 7-5 and 2-5 respectively.

Also, a cat-population at 7 and 2 will make the mice-population change with 5-7 and 5-2 respectively. We see that initial populations at level 5 will give a stable model. Here we assume that the initial populations for cats and mice are 8 and 1 respectively. The following period the two populations will then be $8 + (1-5) = 4$, and $1 + (5-4) = 2$.

Continuing, we see that the cat population will change as 8, 4, 1, 2, 6, 9, 8; and the mice population will change as 1, 2, 6, 9, 8, 4, 1. This allows the points (8,1), (4,2), etc., to be marked on a BBBoard, showing a cycle continuing again and again. Different initial numbers will give different cycles.

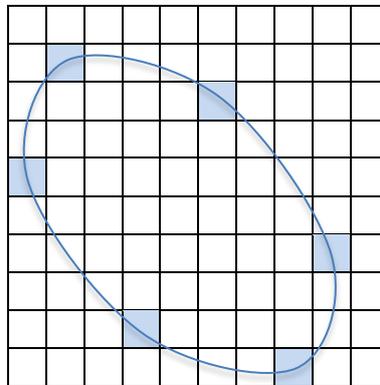


Figure 30. Cats eat mice, so the mice-number decreases, so the cat-number decreases, so the mice-number increases, so the cat number increases, so the mice-number decreases, etc.

- We now model cyclic movements up and down as observed in nature with day and night, with summer and winter, and with tide in an ocean. A cyclic movement may be created by the up-numbers +2, +1, +0, -1, -2, -2, -1, +0, +1 +2.

Beginning at the point (0,5), AI may be prompted to give a formula for this curve as

$$y = 5 + 3 \sin(0.63 \cdot x).$$

- Saving money may take place at home or in a bank. At home the terminal capital c after n months will be $c = b + a \cdot n$, where b is the initial capital, and a is the change-number per month. In a bank, the terminal capital after n months will be $c = b \cdot (1+r)^n$ where r is the change-percent per month. Combining the two in a bank, the terminal capital C may be found by the formula $C/a = R/r$ where R is the total interest rate including the compound interest, $1+R = (1+r)^n$. This capital may be used as an installment plan to pay out a debt D that has grown to $E = D \cdot (1+R)$ in the same period.

- If an interest rate at 100% is split in 12 portions the total interest is found from the equation $1+R = (1+1/12)^{12} = 2.613$ so that $R = 1.613 = 161.3\% = 100\%$ plus 61.3% as additional compound interest. This leads to the Euler number $e = (1+1/n)^n = 2.7183$ for n large, which shows that the additional compound interest cannot surpass 71.8% when splitting up 100%.

- Biological populations typically grow exponentially with a constant periodical rate giving a constant doubling time. This may be shown on a BBBoard where the vertical numbers are in tens. Beginning with $\frac{1}{4}$, a doubling sequence will be $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, 64, 128$.

Alternatively, a capital may decrease by always taking away half of what is left. Beginning with 8 this gives a halving sequence 8, 4, 2, 1, $\frac{1}{2}, \frac{1}{4}$.

This exponential decay may be recognized as a mirror of the exponential growth.

- On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angles and its three sides.

We see that these 7 numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle the angle is predicted by the formula $\tan A = 6/10$, which on a calculator gives $A = 31.0$ degrees. And the area is $\frac{1}{2} * 6 * 10 = 30$. And that the diagonal d is found by squaring: $d^2 = 10^2 + 6^2 = 136$, giving $d = \sqrt{136} = 11.7$.

- On a BBBoard, twice rolling 2 dices may suggest we go to the two points (3,6) and (4,3) that then constitute one side in a square. We now may find the area of the square and the intersection point of the two diagonals. We notice that in the slopes of the sides the out- and up-number change places, and one changes the sign also.

- Optimizing income under constraints (also called 'Linear Programming'). At a fair, a class sells caps and shirts. They may buy at most 6 boxes with caps and 4 boxes with shirts that each cost 1 unit. Their budget is 8 units, and their income is 1 unit per shirt-box and 2 units per cap-box. How can they optimize the income?

On a BBBoard a horizontal and a vertical rubber band shows the limit on the shirts and on the caps. A line connecting (0,8) and (8,0) shows the budget-line not to be passed. A line connecting (0,10) and (5,0) shows the 10 unit income-line that is moved to the right until (6,2) where the first constraint will be violated. So the class should buy 6 boxes with caps and 2 boxes with shirts, which will give them an income at $2 * 6 + 1 * 2$ or 14 units.

Three footnotes

The total here exists as rectangular bundle-number on a BBBoard.

The goal is to experience the content of three calculation laws.

The commutative law: The order does not matter, $a * b = b * a$

The distributive law: When adding, like units may be bracket out, $a * c + b * c = (a + b) * c$

The associative law: Bracket may be moved at will, $a * (b * c) = (a * b) * c$

On a BBBoard two rubber bands mark 6 3s. Turning the board a quarter round we have 3 6s thus illustrating that $6 * 3 = 3 * 6$.

A third rubber band split the 6 3s in 4 3s and 2 3s to illustrate that $4 * 3 + 2 * 3 = (4 + 2) * 3$.

With cubes 2 3s 4 times gives a Total of $(2 * 3) * 4$. Turning it over, twice we have 3 4s, thus illustrating that $(2 * 3) * 4 = 2 * (3 * 4)$.

Teacher education

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating internal predicates by asking the outside subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for the primary school, and C2, A2, T2 and S2 is for the secondary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team.

Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both recognition and new cognition, i.e., both assimilation and accommodation.

The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

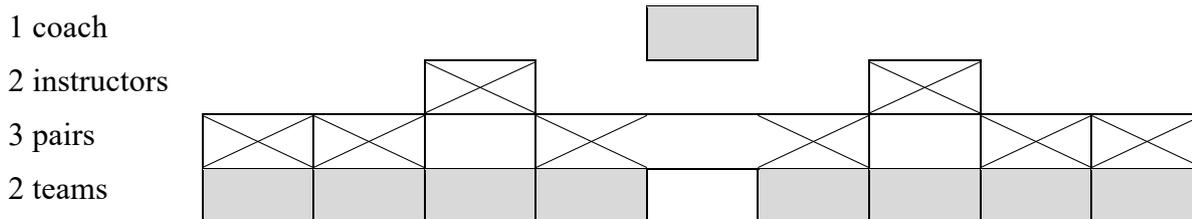


Figure 31. PYRAMIDeDUCATION with 2 teams as 3 pairs and 2 instructors, plus a coach

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles?

The corresponding answers would be: By bundling and stacking the total T predicted by $T = (T/B)*B$. So, $T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2B2 3s$.

Bundling bundles gives a multiple stack, a stock or polynomial: $T = 456 = 4\text{BundleBundle} + 5\text{Bundle} + 6 = 4\text{tente}5\text{ten}6 = 4*B^2 + 5*B + 6*1$.

Additional material may be found as MrAlTarp YouTube videos.

How different is the difference?

Digits now are no longer symbols as letters, but icons with as many sticks as they represent. 3 now is called '1B0 3s' or '0B3 tens'. Ten, eleven and twelve now are also called 'one-bundle-zero', 'one-bundle-one', and 'one-bundle two'. And hundred and thousand are also called 'bundle-bundle' and 'bundle-bundle-bundle'.

Multidigit numbers no longer occur without units since with units, 23 now is 2B3 thus making the place value system unneeded. Calculations with overloads and underloads give bundle-numbers with units a flexibility that makes carrying and borrowing unneeded, e.g., $46+37 = 4B6+3B7 = 7B13 = 8B3 = 83$. And $86 - 37 = 8B6 - 3B7 = 5B-1 = 4B9 = 49$.

Addition now depends on the units so $2+3$ is not 5 by necessity. $2\text{weeks} + 3\text{weeks} = 5\text{weeks}$, but $2\text{weeks} + 3\text{days} = 17\text{days}$. So, without a unit, 3 does not exist, only with a bundle-unit as, e.g., 0B3 tens, or 1B0 3s, or 1B1 2s, or 1B-1 4s, or 1B-2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., ' $2+3$ ' = $(1B0 + 1B1) 2s = 2B1 2s$, or ' $2+3$ ' = $(1B-1 + 1B0) 3s = 2B-1 3s = 1B2 3s$. Likewise with subtraction ' $9-6$ ' = $(1B3 - 1B0) 6s = 0B3 6s = '3'$, or ' $9-6$ ' = $(1B0 - 1B-3) 9s = 0B--3 9s = 0B3 9s$, showing that minus times minus must be plus.

Also, addition now is not well-defined since 2 3s and 4 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and $6*8$ is not 48 by necessity. Instead, $6*8$ exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or tens: 6 8s is 5B3 9s, or 4B8 tens.

Division now is different, since $8/2$ has different meanings in time and space by meaning '8 split in 2 in time', but '8 split in 2s in space' when recounting 8 in 2s.

Solving equations now is different. The equation ' $u*2 = 8$ ' asks "How many 2s in 8?" which of course is found by recounting 8 in 2s as $8 = (8/2)*2$, so that the solution is $u = 8/2$ that is found by 'moving to opposite side with opposite sign', which follows the formal definition: $8/2$ is the number u that multiplied with 2 gives 8, if $u*2 = 8$ then $u = 8/2$. So, the balancing method now is unneeded.

Thus, no longer equations are seen as two equivalent numbers-names that remain equivalent if the same operation is performed on both. And no longer are they transformed by using the communicative, associative, and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements.

And we no longer use the neutralizing ‘do the same to both sides’ weight-method to solve the equation $2*x = 8$ saying: $2*x = 8$; $(2*x)^{1/2} = 8^{1/2}$; $(x*2)^{1/2} = 4$; $x*(2^{1/2}) = 4$; $x*1 = 4$; $x = 4$

The multiple calculation $2+3*4$ no longer is 14 by definition or by the ‘PEMDAS’ rule. With units, $2+3*4$ exits as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation ‘ $2*a + 3*a = (2+3)*a$ ’ no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer ‘go over one’, instead a per-number links the two units by recounting: with 4\$ per 5kg, or 4\$/5kg, $16\$ = (16/4)*4\$ = (16/4)*5\text{kg} = 20 \text{ kg}$.

Fractions are not numbers by themselves, instead they are per-number with like units, $3\text{meter}/4\text{meter} = 3/4$, $3 \text{ meter}/100\text{meter} = 3/100 = 3\%$. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two apples plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., $1/2 + 2/3 = (1+2)/(2+3) = 3/5$ in this case, and not 7 red of 6 apples as mathematism teaches.

Trigonometry no longer must wait to after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal.

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait to secondary school since Bundle-Bundles are squares that lead directly to the question ‘how to square a rectangle’ that provides a double split square containing the three parts of a quadratic equation.

The simplicity of the Algebra Square will no longer be hidden. And no longer will models be seen as mere approximations but as tales with three genres, fact, and fiction, and fake.

Overview of the differences between Essence- math and Existence-math

	Essence-math, mathematism	Existence-math, Many-math
Digits	Symbols	Icons
345	Place value system	T = 3BB 4B 5, BB = B ² , BBB = B ³
Operations	Functions, order: + - x / ^	Icons, opposite order: ^ / x - +
3 + 4	3 + 4 = 7	Meaningless without units
3 * 4	3 * 4 = 12	3*4 = 3 4s, may be recounted to 1.2 tens
9 = ? 2s	Meaningless, only ten-counting	9 = 3B3 = 5B-2 = 4B1 = 4½ 2s
8 = ? 2s	Meaningless, only ten-counting	8 = (8/2)*2, T = (T/B)*B, proportionality
2*u = 8	$2*u = 8$ so, $(2*u)^{1/2} = 8^{1/2}$ so, $(u*2)^{1/2} = 4$ so, $u*(2^{1/2}) = 4$ so, $u*1 = 4$ so, $u = 4$	$2*u = 8 = (8/2)*2$ so, $u = 8/2$
6*7 = ?	eh 44? eh 52? eh 42? OK	$6*7 = (B-4)*(B-3)$ $= (10-4-3)*B + 4*3$ $= 3B12 = 4B2 = 42$ $6*7 = 6 * 1/2B 2 = 3B 12 = 4B2 = 42$

$4\text{kg} = 5\$,$ $6\text{kg} = ?$	$1\text{kg} = 5/4\$,$ $6\text{kg} = 6*5/4\$$	$6\text{kg} = (6/4)*4\text{kg}$ $= (6/4)*5\$$
$1/2 + 2/3 = ?$	$1/2 + 2/3 = 3/6 + 4/6 = 7/6$	$1/2*2 + 2/3*3 = 3/5*5$
$2*3 + 4*5$	$2*3+4*5 = 6+4*5 = 10*5 ?$	$2*3+4*5 = 3\text{B}2\ 8\text{s (next-to)}$ or $5\text{B}1\ 5\text{s, or } 8\text{B}2\ 3\text{s (on-top)}$
$6 + 9 = ?$	$6 + 9 = 15$ $5 - 6$	$2\text{B}3\ 6\text{s or } 2\text{B}-3\ 9\text{s}$ or $2\ 1/2\text{B}(1+4) = 1\text{B}5\ \text{tens}$
Tangent = ?	tan = sine/cosine	raise = (raise/run)*run, tan = raise/run

Reactions to a **BBM Bundle** Math Curriculum

At the 9th ICMI-East Asia Regional Conference on Mathematics Education (EARCOME 9) in 2025 in Korea, the Special Sharing Groups accepted my paper “Can a Decolonized Mathematics Secure Numeracy for All?” with the following announcement:

This proposal tackles an urgently needed conversation in mathematics education by challenging deeply ingrained assumptions about number systems and arithmetic instruction and proposing a truly decolonized approach that foregrounds learners’ intuitive “bundle-number” language. Its strength lies in weaving together a compelling theoretical critique—drawing on Habermas’s colonization concept and rich philosophical underpinnings—with concrete instructional innovations like the Algebra Square that reframe operations as intuitive spatial and bundling processes. By aligning this reconceptualization directly with SDG 4’s numeracy targets and illustrating how multiplication-centered reasoning better reflects real-world number use, the paper promises to make a bold and impactful contribution to both research and practice. We look forward to seeing how this work can reshape numeracy instruction and foster truly inclusive mathematical literacy for all.

In Topic Study Groups, reviewers asked for empirical studies where existence- and essence-math had been compared. My answer was that testing both at the same time is ethically problematic:

Grade one class A has an essence-math curriculum with unit-free 1D line-numbers and the traditional order: addition, subtraction, multiplication, division, power. And only few will hear about calculus. And class B has an existence-math curriculum with 2D Bundle-numbers with units and the opposite order. And, since counting and recounting precedes adding, they will not meet addition until they meet the core of math directly, calculus and linearity. What will happen if a class B student changes to class A, or the other way around? In short, how ethical is it to test essence-math against existence-math in an ordinary school? Or in special education where the students will return far ahead to normal education? Ethical testing is for teacher education and home education.

Grand Theory looks at Mathematics Education

Apparently, two different definitions of ‘numerate’ exist where existence and essence have different order. The English Oxford Dictionary uses the predicate ‘mathematics’ when defining ‘numerate’ as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary uses the verb ‘number’ when defining ‘numerate’ as “having the ability to understand and work with numbers.”

To understand this difference and to enlighten and discuss the core of education formulated as ‘teach learners something’, we now consult the three grand theories, philosophy and sociology and psychology. Philosophy may be able to illuminate the different nature of predicates and verbs and to discuss the ‘something’. Sociology may be able to illuminate the different inter-human power effects coming from using predicates instead of verbs and in the textbook-teacher-learner interaction. And psychology may be able to illuminate the different learning results coming from listening to predicates or practicing verbs.

Within philosophy, a discussion between existentialism (Sartre 2007) and essentialism about the precedence of existence or essence has taken place since philosophy began in ancient Greece (Russell 1945). Here the ‘knowing’ sophists argued that to practice democracy we must tell nature from

choice to avoid being patronized by choice masked as nature. In other words, we must be able to tell outside existence from its many chosen inside essences and especially the ones that have been institutionalized as the tradition and thus colonizes its outside existence (Habermas, 1981). Against this, the 'better knowing' philo-sophists argued that choice is nonexistent since everything physical is only imperfect examples of metaphysical essence as illustrated in Plato's Cave allegory, and that essence is only accessible to philosophers educated at the Plato Academy.

This disagreement between sophists and philosophers about nature versus choice, and existence versus essence, runs through history. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. In the counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe's line-organized office-preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the French Enlightenment republic's post-structuralism inspired by Heidegger (1962) who argued that "P is Q" is a statement judging an outside existence, P, with an inside constructed essence-predicate that may be a preconceived prejudice, e.g., gossip (Gerade), and which therefore should be met with skepticism and be deconstructed. With 'essence' coming from Latin 'esse' meaning 'being', Heidegger gave four answers to the question "What is 'is'?" pointing up, down, over, or nowhere: is an example of, is for example, is like, or is period. In mathematics, a function thus may be an example of a subset in a set-product where first-component identity implies second-component identity, or a name for calculation containing both specified and unspecified numbers, or a number-language sentence including a subject and a verb and a predicate as in a word-language sentence, or simply what it is, a stand-by calculation awaiting a number, e.g., '3 * x' in contrast to '3 * 5'.

Psychology (Skemp, 1971) has a controversy within constructivism where Vygotsky (1986) sees education as adapting to the institutionalized essence-regime by building ladders down from it to the learners' proximal learning zones. Piaget (1969) replaces this top-down view with a bottom-up view inspired by American Grounded Theory (Glaser et al, 1967) allowing categories to emerge from concrete experiences. Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi). Also siding with Piaget, Luhmann (1995) sees the brain as a closed self-referring psychic system inside a world of complexity from which it learns by itself reducing the complexity presented to it by another closed self-referring social system called education with its own reductions that it cannot export. Education thus means disturbing the students' systems with outside complexity or hiding complexity if instead wanting uneducated students not able to reduce this, e.g., hiding that numbers occur differently in time and space. This conflict has sociological consequences. Presented top-down from above as examples of inside abstractions, concepts become hard to learn, which forces many learners to stop learning what is meaningless to them and to accept patronization by those who accept such meaninglessness. In contrast, bottom-up concepts grounded from below in the outside world are natural to learn for children through meeting and acting on the concrete examples that exemplify and root the concepts; and for teenagers since knowing the subject in a sentence creates automatic gossip-like learning.

Within Sociology, reviving the ancient Greek sophist skepticism towards patronization masking choice as nature, the Enlightenment created two democracies, one in North America still having its first republic, and one in France now having its fifth republic. In Germany, Weber (1930) was the first to theorize the increasing social goal-oriented rationalization that disenchant the world and create an 'iron cage' if carried to wide. Later, wanting to establish a third German Enlightenment democracy based upon communicative action and the convincing force of the better argument,

Habermas (1981) accepts the role of science in rationalizing society, but warns against its instrumentalism spreading from the system to the life world to colonize it.

North American showed skepticism towards rationalist philosophy by developing American Pragmatism (Menand, 1997) leading to Symbolic Interactionism (Blumer, 1998) and Grounded Theory (Glaser et al, 1967).

Here, Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now” (p. 16).

Living together involves deciding upon which tasks to do individually and which to do collectively by creating rational institutions, “in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79).” However, Bauman warns against a possible ‘goal displacement’ where “The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).”

An example of a goal displacement may be institutionalizing a knowledge-regime (Foucault 1995) saying ‘The goal of mathematics education is to master mathematics’. By this self-reference, such a goal is meaningless. To master math cannot be the goal of math education, but it can be a means to reach the real end goal, to master Many. Which of course should be reached by different means if a goal displacement has changed institutionalized mathematics from a means to a goal itself colonizing any further road to mastery of Many and numeracy.

The debate on patronization beginning in ancient Greece between the philosophers and the sophists is still with us today between socialist top-down critical theorists and skeptical bottom-up existentialist theorists that are inspired by the French post-structuralist thinker Foucault saying

It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

So, seeing mathematics education as an institution (Freudenthal, 1973), a Weberian viewpoint would ask if the set-concept is a rationalization of Many gone too far thus leaving Many disenchanting and leaving the learners in an iron cage. And a Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. So, as an institution, mathematics is a means, hence the word ‘mathematics’ should go from goal descriptions. If not, there is a risk that mathematics education will practice ‘the banality of evil’ (Arendt, 1963). So, sociology would recommend replacing the truth regime saying ‘the goal of teaching math is to learn mastery of math’ by the real end goal, mastery of Many, e.g., by uncovering and developing the existing mastery of Many children create through adaptation to Many before school. In short, it may be time to show skepticism towards institutionalized mathematics and use difference research (Tarp, 2018) to look for differences that may make a difference.

We therefore ask the Cinderella question: If mastery of Many is the end goal, essence-math may be a means for some but not for all as indicated by the more than fifty years since the first International Congress on Mathematical Education was held in 1969, but are there different means also, e.g., an existence-math, that will make the prince dance? And if so, what is its content and what kind of mathematics does the prince learn during the dance?

Essence-math sees Many as an example of 1D linear cardinality always able to absorb one more by being built on the assumption that ‘ $1+1 = 2$ ’. In contrast, a different existence-math occurs when asking a 3-year-old child “How many years next time?”. The answer is four with four fingers shown. But seeing the four fingers together two by two, the child reacts: “That is not four, that is two 2s”.

This observation shows that numbers occur both as bundle-numbers in space, 2s, and counting-numbers in time, two, that need to be together since asking, “What is 1+1?” leads to asking back “One what + one what?”. And asking, “What is 1s + 1s?” leads to asking back “How many 1s + how many 1s?”. And asking, “What is 1 + 1s” leads to asking back “What??”

We therefore designed a different existence-math curriculum built on numbers with bundle-units as 2 3s and 4 5s that children create before school when adapting to nature’s multiplicity in space and repetition in time.

Conclusion. In Numeracy education, a Luhmann understanding is essential

The United Nations Sustainable Development Goal 4.6 says “By 2030, ensure that all youth achieve Numeracy”. So, numeracy now, but how, and what is it?

We will reach the definition “Numeracy is mathematics without units, where addition folds and multiplication holds”. But first we will look at the two official contradictory definitions of numeracy.

The English Oxford Dictionary defines numerate as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary defines numerate as “having the ability to understand and work with numbers.”

The word ‘competent’ is a predicate, a non-action word, I cannot ‘competent’ something, I can only be judged as competent by someone who is competent. In contrast, ‘work’ is an action word, a verb, since with my body and mind I can work on something and test the result myself to see if it works.

Also, there is a difference between the words ‘mathematics’ and ‘numbers.’ Again, mathematics is a non-action word, I cannot ‘mathematics’ a thing. In contrast, number’ is both a verb and a noun since I can number different degrees of Many to produce numbers for later calculations.

Why this difference? We ask the three Grand Theories.

In Philosophy, Existentialism holds that ‘existence precedes essence’ to quote Sartre, i.e., that, to prevent inside essence from colonizing it, outside existence must come first.

In Sociology, Weber warns that if over-rationalized, society may turn into an iron cage disenchanting the world. Likewise, Habermas warns against a colonization of the life-world by the system-world. And Bauman warns that institutionalizing one essence as the only means may lead to a goal displacement where the means becomes the goal instead by working, not for, but against reaching the original goal, existence. And via Heidegger’s warning against reducing a subject’s being-potential by imposed predicates, Foucault warns that, to stay in power to create docile bodies, an essence needs to install a regime which the original existence has no need for by its mere existence.

In Psychology, Vygotsky sees learning as inside adapting to the inside institutionalized essence, where Piaget sees learning as inside adapting to outside existence. Luhmann instead talks about individual psychic systems that reflect and collective social systems than communicate, where all systems are closed and self-creating and self-referring and construct essence as inside reductions of outside complexity. So, as a self-referring social system, mathematics cannot transfer its essence to self-referring psychic systems that construct their own essence when disturbed by outside existence, e.g., when meeting Many leads to Counting & Adding in Time & Space.

So, to answer the question “math before or after numeracy?” we must first see how numerate children are before school. We therefore ask a 3year-old “How many years next time?” The answers is ‘four’, with four fingers shown. But, held together tow by two, the child says “No, that is not four, that is two 2s.”

We, the educated, see only the essence, four. But the un-educated child sees the existence, two 1s bundled as one 2s in space, and two of them when counted in time.

Children thus understand a number as a two-dimensional combination of a vertical counting-number and a horizontal bundle-unit visible and tangible on a ten-by-ten BundleBoard allowing children to represent outside totals with their own numbers with Bundle-units.

And Totals may be split or counted. Splitting a total by pulling a Bundle away and back will restore the Total as shown by a re-uniting formula, $T = (T - B) + B$. Counting a total by pushing Bundles away and back will restore the Total as shown by a re-counting formula, $T = (T/B) \times B$. Show on fingers, these two core formulas are available to all children learning numeracy. In mathematics education without units, they never occur.

A BBBoard allows children to work with 'Full-matics' using both counting numbers in time and bundle-units in space instead of having their own number-language colonized by 'Half-Matics' using counting-numbers only. And becoming 'Mathema-tism' by claiming that $2+1 = 3$ despite a closed and an open V-sign shows that two 1s and one 2s total one 4s, and not three 3s as claimed by the ruling essence-regime, or essence-paradigm. So, the moment mathematics accepts units, its ' $2+1=3$ ' foundation will collapse. So, maybe the time has come for a Kuhnian paradigm shift in mathematics education where introducing units will create a change from a 'flat-earth math education' to a 'round earth math education'?

Yes, units matter. Therefore, if all youth are to be numerate by 2030 as wished by the United Nations Sustainable Development Goal, then mathematics without units must go to allow numeracy help the children develop their already existing reductions of outside complexity into a number-language used to count and add in time and space in order to create number-tales about the world and to see to which genre they belong, fact or fiction or fake, the same three genres that is used in the word-language.

So, in Numeracy education, a Luhmann understanding is essential. In Math education, it is not.

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